Teoria quântica de campos em espaco-tempo curvo

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Synopsis of the course

- General structural properties of a quantum theory
- Canonical quantization and all that
- Bogoliubov transformation and construction of the S-matrix
- The Unruh effect
- (The Hawking effect)

I) General Structural Properties of a quantum theory and QFT

A reminder of classical mechanics

- The physical state of a classical system having N degrees of freedom can be completely specified by assigning the values of 2N quantities (the observables) interpreted as the generalized coordinates and velocities of the system

$$(q^1, q^2, \ldots q^N, \dot{q}^1, \dot{q}^2, \ldots \dot{q}^N)$$

- The dynamics of the system is then specified by the knowledge of its Lagrangian function depending on the generalized coordinates and velocities

$$\mathcal{L}(q^1, q^2, \ldots q^N, \dot{q}^1, \dot{q}^2, \ldots \dot{q}^N)$$

- The Lagrangians equation of motion are a system of N second order equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial L}{\partial q_j} \quad j = 1, \ldots, N$$

A reminder of classical mechanics

- The mathematical structure of a classical mechanical system is unveiled in the Hamiltonian formalism (here, an ultra simplified version of it).

- Introduce the momenta canonically conjugate to the generalized coordinates:

$$p_j = \frac{\partial L}{\partial \dot{q}^j}, \quad j = 1, \ldots, N$$

- The collection of all possible values of coordinates and momenta is the phase space of the system; this is the cotangent bundle X=T^*M of the manifold M of all possible configurations

$$(q^1, q^2, \ldots q^N, p_1, p_2, \ldots p_N)$$

- The **physical state** of the same classical system is now completely specified by assigning the values of 2N quantities (the observables) interpreted as the generalized coordinates and momenta of the system

A reminder of classical mechanics

- There is a natural symplectic structure on *T*M* induced by the Liouville differential one-form and the canonical symplectic (invertible) differential two-form:

$$\theta = \sum p_j dq^j; \qquad \omega = -d\theta = \sum dq^j \wedge dp_j$$

- The canonical 2-form gives the Poisson bracket stucture; for any two functions defined on *T*M*:

$$\{f,g\} = \omega(df,dg) = \sum \left[\frac{\partial f}{\partial q^j}\frac{\partial g}{\partial p_j} - \frac{\partial g}{\partial q^j}\frac{\partial f}{\partial p_j} \right]$$

- In particular the canonical variables satisfy

$$\{q^j, p_k\} = \delta^j_k, \ \{q^j, q_k\} = 0, \ \{p_j, p_k\} = 0$$

- The equations of motion are the Poisson brackets of the canonical variables with the Hamiltonian:

$$H(q,p) = \sum p^j \dot{q}_j - L$$

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial \dot{p}_j} = \{q, H\} \qquad \frac{dp_j}{dt} = -\frac{\partial H}{\partial \dot{q}_j} = \{p, H\}$$

Algebra of the observables

- Observables are more generally *measurable* complex functions on the phase space $f(q,p) : X = T^*M \to \mathbf{C}$
- The observables associated to a classical system generate an abelian algebra A of complex continuous function on the (compact) phase space.
- The product is the pointwise composition of functions

$$(fg)(x) = f(x)g(x)$$

- The algebra has an identity element 1, the constant function equal to 1
- There is a natural involution or ★-operation that is simply the complex conjugation

$$f^\star(x) = \bar{f}(x)$$

- Technically the algebra of observables is an Abelian C-star algebra.

States and functionals

- In principle a state determines the value of an observable sharply

$$(q, p) \longrightarrow f(q, p)$$

- Measurements with infinite precision are not possible. The identification of states with points of the phase space relies on such unrealistic idealization
- The standard way to associate a value to an observable *f* in a state ω consists in performing averages of replicated measures of *f* in the given state

$$< f >_n^{(\omega)} \equiv [m_1^{(\omega)}(f) + m_2^{(\omega)}(f) + m_n^{(\omega)}(f)]/n.$$

The limit $n \to \infty$ (whose existence is part of the foundations of experimental physics) defines the *expectation of* f on the state ω

$$\omega(f) \equiv \lim_{n \to \infty} \; < f >_n^{(\omega)}$$

as *average of the results of measurements of* f in the state ω.

Classical systems: summary

- A classical system is defined by the Abelian C-star algebra A of its observables
- A state of a classical system i.e. normalized positive linear functional on the algebra of A of its observables:

$$\omega(\lambda f + \mu g) = \lambda\,\omega(f) + \mu\,\omega(g)$$

$$\omega(f f^*) \geq 0$$

The Riesz-Markov theorem then guarantees that there exists a unique Borel **probability measure** such that

$$\omega(f) = \int_X f\, d\mu_\omega, \quad \mu_\omega(X) = \omega(\mathbf{1}) = 1,$$

$$\omega_{q_0,p_0}(f) = \int_\Gamma f(q,p)\delta_{(q_0,p_0)} = f(q_0,p_0)$$

Quantum observables

- The great Heisenberg's discovery : there is an intrinsinc limitation in the relative precision by which *q* and *p* can be measured independently of the state ω

$$(\Delta_\omega q_j)\,(\Delta_\omega p_j) \;\geq h/4\pi \equiv \hbar/2. \qquad (\Delta_\omega f)^2 \equiv \omega((f-\omega(f))^2)$$

- Given a state and any two observables

$$A = A^*,\; B = B^* \quad (A - i\lambda B)(A + i\lambda B) \geq 0,\; \forall \lambda \in \mathbf{R}$$

$$\omega\,(A^2) + |\lambda|^2 \omega\,(B^2) + i\lambda\omega\,([A,B]) \geq 0$$

$$4\,\omega\,(A^2)\,\omega\,(B^2) \geq \;|\omega\,(i[A,B])|^2$$

$$\Delta_\omega\,(A)\;\Delta_\omega\,(B)\;\geq\;\frac{1}{2}\,|\omega\,([A,B])|$$

The quantum algebra is non abelian

$$(\Delta_\omega \, q_j) \, (\Delta_\omega \, p_j) \; \geq h/4\pi \equiv \hbar/2.$$

$$\Delta_\omega \, (A) \; \Delta_\omega \, (B) \; \geq \; \frac{1}{2} \, |\omega \, ([A, B])|$$

Heisenberg idea is that the uncertainty relations arise as direct consequences of the following *Heisenberg commutation relations*

$$q_j \, p_k - p_k \, q_j = \; i\hbar\delta_{jk}\mathbf{1} \, ,$$

Thus observables (say the position and momentum variable) of an atomic particle cannot be described by an abelian algebra.
The algebra of quantum observables must be non commutative.

Representing the CCR

- Consider a one-dim quantum system: $QP - PQ = i\hbar$

- To find a quantization is to realize or "represent" these algebraic rules by concrete operators in a Hilbert space H. This amount to "choose" a particular state

$$Q \to \widehat{Q} : \mathcal{H} \to \mathcal{H}, \qquad P \to \widehat{P} : \mathcal{H} \to \mathcal{H}$$

$$\widehat{Q}\widehat{P} - \widehat{P}\widehat{Q} = i\hbar$$

- Equivalently $\quad a = \frac{Q+iP}{\sqrt{2}}, \quad a^\dagger = \frac{Q-iP}{\sqrt{2}}, \quad [a, a^\dagger] = \hbar\mathbf{1}$

$$a \to \widehat{a} : \mathcal{H} \to \mathcal{H}, \qquad a^\dagger \to \widehat{a}^\dagger : \mathcal{H} \to \mathcal{H}$$

$$\widehat{a}\widehat{a}^\dagger - \widehat{a}^\dagger\widehat{a} = \hbar\mathbf{1}$$

- Can $\widehat{Q}$ and $\widehat{P}$, or $\widehat{a}$ and $\widehat{a}^\dagger$ be matrices? Or bounded operators?

  Answer: No they cannot (take the trace)

Representing the CCR

- Abstract CCR's: $QP - PQ = i\hbar$
- CCR's represented: $\widehat{Q}\widehat{P} - \widehat{P}\widehat{Q} = i\hbar$
- Example I: x-space representation:

$$\mathcal{H} = L^2(\mathbf{R}_x) = \{\psi(x) : \int |\psi(x)|^2 dx < \infty\}$$

$$Q \rightarrow \widehat{Q}_x = x, \qquad P \rightarrow \widehat{P}_x = -ih\frac{d}{dx}$$

- Example II: momentum-space representation

$$\mathcal{H} = L^2(\mathbf{R}_p) = \{\phi(p) : \int |\phi(p)|^2 dx < \infty\}$$

$$Q \rightarrow \widehat{Q}_p = ih\frac{d}{dp}, \qquad P_p \rightarrow \widehat{P} = p$$

Representing the CCR ©

$$QP - PQ = i\hbar$$

- Representation I: $L^2(\mathbf{R}_x) = \{\psi(x) : \int |\psi(x)|^2 dx < \infty\}$

$$Q \to \widehat{Q}_x = x, \qquad P \to \widehat{P}_x = -ih\frac{d}{dx}$$

- Representation II: $L^2(\mathbf{R}_p) = \{\phi(p) : \int |\phi(p)|^2 dx < \infty\}$

$$Q \to \widehat{Q}_p = ih\frac{d}{dp}, \qquad P_p \to \widehat{P} = p$$

- The two representations are unitarily equivalent

$$U : L^2(\mathbf{R}_x) \to L^2(\mathbf{R}_p)$$

$$U\widehat{Q}_x U^{-1} = \widehat{Q}_p \qquad U\widehat{P}_x U^{-1} = \widehat{P}_p$$

Stone-Von Neumann Uniqueness

- All the representations of the CCR's by (essentially) self-adjoint operators are unitarily equivalent.
- This is true also for finite-dimensional systems: the CCR algebra can be written in terms of position and momentum "operators"

$$[Q_i, Q_j] = 0, \quad [P_i, P_j] = 0,$$
$$[Q_i, P_j] = i\hbar\delta_{ij} \quad i, j = 1, \ldots N$$

- Under suitable technical assumptions there exists only one Hilbert space representation of the CCR's,

$$Q \to \hat{Q} \quad P \to \hat{P}$$

- i.e. all the representations are unitarily equivalent.

Representing the CCR

$$QP - PQ = i\hbar$$

- Representation III: $L^2(\mathbf{R}_x^+) = \{\psi(x) : \int_0^\infty |\psi(x)|^2 dx < \infty\}$

$$Q \to \widehat{Q}_x^+ = x, \qquad P \to \widehat{P}_x^+ = -ih\frac{d}{dx}$$

- At variance with reps I and II is that $\widehat{P}_x^+$ is not essentially self-adjoint

- It follows that rep III cannot be unitarily equivalent to reps I and II because it cannot exist any unitary operator such that

$$U\widehat{P}_x U^{-1} = \widehat{P}_x^+$$

Classical Relativistic Field Theory

- A classical field is a quantity defined at each point of the spacetime manifold, necessary to describe interactions in a local way (vs action at a distance):

$$x = (t, \mathbf{x}) \in \mathbf{M}^4 \qquad \mathbf{M}^4 \ni x \to \phi(x) \in \mathbf{V}$$

- The function $\phi(t, \mathbf{x})$ may take real values, complex values, or values in some finite dimensional vector space.

- Magnetic field lines were introduced by Michael Faraday (1791-1867) who named them "lines of force." Faraday understood the physical reality of fields: space containing magnetic "lines of force" was no longer empty!

- Fields however may or may not be observable or physical (a spinor field is not, the Electromagnetic four-potential potential is not.)

- A field encodes an **infinite number of degrees of freedom**: the "values" of the field at each spacetime point.

- A relativistic field has suitable tensorial transformation properties under the appropriate relativity group.

Classical Klein-Gordon Fields

- Consider the Minkowski manifold

$$\mathbf{M}^4, \quad \eta_{\mu\nu} = diag(1,-1,-1,-1)$$

- The model is based on the quadratic Lagrangian

$$S[\phi] = \int \mathcal{L}(\phi, \partial_\mu \phi) d^4x = \frac{1}{2}\int d^4x \left[\eta^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi) - m^2\phi^2\right]$$

$$\phi(t,\mathbf{x}), \qquad \pi(t,\mathbf{x}) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \dot{\phi}(t,\mathbf{x})$$

- Generalized coordinates and their canonically conjugated momenta are

$$\phi(t_0,\mathbf{x}),\ \pi(t_0,\mathbf{x})$$

- The assignment of a pair of functions $\phi(t_0,\mathbf{x}),\ \pi(t_0,\mathbf{x})$ at $t = t_0$ completely determines the solution of the Klein-Gordon (classical) field equation

$$(\Box + m^2)\phi = 0$$

$$\Omega(\phi_1,\phi_2) = -\int_{t=t_0} [\phi_1(t,\mathbf{x})\pi_2(t,\mathbf{x}) - \pi_1(t,\mathbf{x})\phi_2(t,\mathbf{x})] d^3\mathbf{x}$$

The Pauli Jordan function

The Pauli-Jordan function

$$\begin{cases} (\Box + m^2)D(x) = 0 \\ D(0,\vec{x}) = 0 \\ \partial_t D(x)|_{t=0} = \delta(\vec{x}) \end{cases}$$

$$D(x) = \frac{i}{(2\pi)^3}\int e^{-ipx}(\theta(p^0) - \theta(-p^0))\delta(p^2 - m^2)d^4p$$

a): Uniqueness.

D(x) is the unique solution of the above Cauchy problem

b): Antisymmetry: *D(x) = -D(-x)*

c): Locality. *D(x)* vanishes for $x^2 < 0$

The retarded and advanced propagators

$$(\Box + m^2)G(x) = \delta^4(x)$$

Solve by Fourier transform $\quad (-p^2 + m^2)\tilde{G}(p) = 1$

$$G(x) = -\frac{1}{(2\pi)^4}\int \frac{e^{-ipx}}{p^2 - m^2} d^4p$$

Define two regularizations as follows

$$D^{ret}(x) = -\frac{1}{(2\pi)^4}\int \frac{e^{-ipx}}{(p^0 + i\epsilon)^2 - \vec{p}^2 - m^2} d^4p$$

$$D^{adv}(x) = -\frac{1}{(2\pi)^4}\int \frac{e^{-ipx}}{(p^0 - i\epsilon)^2 - \vec{p}^2 - m^2} d^4p$$

The retarded and advanced propagators

$$D^{ret}(x) = -\frac{1}{(2\pi)^4}\int \frac{e^{-ipx}}{(p^0+i\epsilon)^2-\vec{p}^2-m^2}d^4p$$

$$D^{adv}(x) = -\frac{1}{(2\pi)^4}\int \frac{e^{-ipx}}{(p^0-i\epsilon)^2-\vec{p}^2-m^2}d^4p$$

$$(p^0 \pm i\epsilon)^2 = (p^0)^2 \pm i\epsilon p^0 = (p^0)^2 \pm i\epsilon\,\mathrm{sgn}(p^0)$$

$$\mathrm{sgn}(p^0) = \epsilon(p^0) = \theta(p^0) - \theta(-p^0)$$

$$\text{Since} \quad \frac{1}{x+i\epsilon} - \frac{1}{x-i\epsilon} = -2\pi i\delta(x)$$

$$D^{ret}(x) - D^{adv}(x) = \frac{i}{(2\pi)^3}\int e^{-ipx}\epsilon(p^0)\delta(p^2-m^2)d^4p = D(x)$$

Support of the propagators

$$D^{ret}(x) = -\frac{1}{(2\pi)^4}\int \frac{e^{-ipx}}{(p^0+i\epsilon+\omega)(p^0+i\epsilon-\omega)}d^4p$$

Poles are located at $p^0 = -i\epsilon \pm \omega, \quad \omega = \sqrt{\vec{p}^2+m^2}$

For $x^0 < 0$ the contour is closed in the upper half-plane

there are no poles and the integral vanishes

For $x^0 > 0$ the contour can be closed in the lower half-plane and get

$$D^{ret}(x) = \frac{i}{(2\pi)^3}\theta(x^0)\int \frac{e^{-i\omega x^0+i\vec{p}\vec{x}} - e^{i\omega x^0 - i\vec{p}\vec{x}}}{(2\omega)}d^3p = \theta(x^0)D(x)$$

$$D^{adv}(x) = -\theta(-x^0)D(x)$$

$$D^{ret}(x) - D^{adv}(x) = (\theta(x^0)+\theta(-x^0))D(x) = D(x)$$

A formula of fundamental importance

$$D^{ret}(x) - D^{adv}(x) = D(x)$$

This formula proves the key property c)
the locality property (also called microcausality):
D(x) vanishes for $x^2 < 0$.

The Pauli-Jordan function and the Cauchy problem

The Pauli- Jordan allows to solve the Cauchy problem

$$\begin{cases} (\Box + m^2)\phi(x) = 0 \\ \phi(0,\vec{x}) = g(\vec{x}) \\ \partial_t \phi(x)|_{t=0} = h(\vec{x}) \end{cases}$$

The solution is given by the formula:

$$\phi(x) = \int D(t,\vec{x}-\vec{y})h(\vec{y})d^3y + \int \partial_t D(t,\vec{x}-\vec{y})g(\vec{y})d^3y$$

- There is a second possibility to build a solution of the KG equation using the Pauli Jordan commutator. One such solution is

$$\phi(x) = \int D(x-y)g(y)d^4y, \quad g \in \mathcal{C}_0^\infty(\mathbf{R}^4)$$

- Consider now the inhomogeneous problem

$$(\Box + m^2)\phi(x) = f(x) \qquad\qquad f \in \mathcal{C}_0^\infty(\mathbf{R}^4)$$

$$\phi_1(x) = Rf = \int D^{ret}(x-y)f(y)d^4y$$

$$\phi_2(x) = Af = \int D^{adv}(x-y)f(y)d^4y$$

$$\phi = \phi_1 - \phi_2, \quad (\Box + m^2)\phi = 0,$$

$$supp\ \phi = supp f + V$$

A fundamental lemma

- $\psi \in \mathcal{S}$ *the space of solutions of the KG eq. with compact initial data. There exists* $f \in \mathcal{C}_0^\infty(\mathbf{R}^4)$

$$\psi(x) = Ef = Rf - Af = \int D(x-y)f(y)d^4y$$

Let χ be a smoothed step function: $\chi \in \mathcal{C}^\infty$

$\chi(t) = 0, \quad t < 0, \quad \chi(t) = 1, \quad t > 1.$

$$\boxed{f = (\Box + m^2)\chi\psi, \quad f \text{ has compact support}}$$

$$(\Box + m^2)Af = f = (\Box + m^2)\chi\psi, \quad Af = \chi\psi + h, \quad (\Box + m^2)h = 0$$

$$Af = \chi\psi - \psi \text{ vanishes in the future shadow of } f$$

$$\text{Similarly } Rf = \chi\psi. \quad \text{Thus } (R-A)f = \psi$$

A fundamental lemma

- Ef *vanishes if and only if*

$$f = (\Box + m^2)g(y), \quad supp\, g \ compact$$

$$\int D(x-y)(\Box_y + m^2)g(y)d^4y = 0 \ \text{ obvious}$$

$$\text{Conversely} \quad Ef = 0 \quad \to \quad Rf = Af \in \mathcal{C}_0^\infty(\mathbf{R}^4)$$

$$\text{But} \ \ (\Box + m^2)Af = f \quad \text{i.e.} \ \ g = Af$$

A fundamental lemma

- *For all* $\psi \in \mathcal{S}$ *and all* $f \in \mathcal{C}_0^\infty(\mathbf{R}^4)$

$$\int \psi(x) f(x) d^4x = \Omega(\psi, Ef)$$

Chose $[t_1, t_2]$ such that $supp\, f \subset [t_1, t_2] \times \mathbf{R}^3$

$$\int_{t_1}^{t_2} dt \int_{\mathbf{R}^3} \psi(x) f(x) d^3x = \int_{t_1}^{t_2} dt \int_{\mathbf{R}^3} \psi(x)(\Box + m^2) Af(x) d^3x$$

$$\psi \,\Box\, Af = \partial^\mu \left(\psi\, \partial_\mu Af\right) - (\partial^\mu \psi)(\partial_\mu Af)$$
$$= \partial^\mu \left(\psi\, \partial_\mu Af\right) - \partial_\mu(\partial^\mu \psi\, Af) + \Box\psi\, Af$$

$$\int \psi(x) f(x) d^4x = \int_{t_1}^{t_2} dt \int_{\mathbf{R}^3} \partial^\mu \, \left(\psi\, \partial_\mu Af - (\partial_\mu \psi)\, Af\right) d^3x$$

A fundamental lemma

$$\int \psi(x) f(x) d^4x = \int_{t_1}^{t_2} dt \int_{\mathbf{R}^3} \partial^\mu \; (\psi\, \partial_\mu A f - (\partial_\mu \psi)\, A f)\, d^3x$$

$$= - \int_{\mathbf{R}^3} (\psi\, \partial_t A f - (\partial_t \psi)\, A f)\, |_{t=t_1} d^3x$$

$$= \int_{\mathbf{R}^3} (\psi\, \partial_t E f - (\partial_t \psi)\, E f)\, |_{t=t_1} d^3x$$

because $Af = -Ef = -(Rf - Af)$ on the surface $t = t_1$

$$\int f(x) D(x-y) g(y) d^4y = -\Omega(Ef, Eg)\,, \quad f, g \in \mathcal{C}_0^\infty(\mathbf{R}^4)$$

Peierls symplectic form

$$\int f(x)D(x-y)g(y)d^4y = -\Omega(Ef, Eg)\,, \quad f, g \in C_0^\infty(\mathbf{R}^4)$$

$$\Omega(\phi_1, \phi_2) = -\int_{t=t_0} [\phi_1(t,\mathbf{x})\pi_2(t,\mathbf{x}) - \pi_1(t,\mathbf{x})\phi_2(t,\mathbf{x})] d^3\mathbf{x}$$

- Stokes theorem: The Peierls symplectic form is conserved. Given any two solutions the current is conserved

$$j_\mu = \phi_1 \overleftrightarrow{\partial}_\mu \phi_2 = \phi_1 \partial_\mu \phi_2 - (\partial_\mu \phi_1)\phi_2, \quad \partial^\mu j_\mu = 0$$

- Covariant Poisson Brackets

$$\{\phi(x), \phi(y)\} = -D(x,y) = -D(x-y)$$

Quantizing fields

- Replace the Peierls brackets by the canonical commutation relation (here written covariantly)

$$\{\phi(x), \phi(y)\} = -D(x, y) = -D(x - y)$$

$$[\phi(x), \phi(y)] = -i\hbar D(x - y)$$

- Construct a Hilbert space representation of the CCR

$$\phi(x) \to \hat{\phi}(x)$$
$$[\hat{\phi}(x), \hat{\phi}(y)] = -i\hbar D(x - y)$$

- **Uniqueness fails!**
- In quantum field theory the Lagrangian is not enough.

Infinitely many dof: uniqueness fails!

- Formal CCR algebra written in terms of fixed time fieds "operators"

$$[\phi(t_0,\vec{x}),\phi(t_0,\vec{y})] = [\pi(t_0,\vec{x}),\pi(t_0,\vec{y})] = 0$$

$$[\phi(t_0,\vec{x}),\pi(t_0,\vec{y})] = i\hbar\delta(\vec{x}-\vec{y})$$

- Or in terms of creation and annihilation operators (discrete normalization)

$$[a_i, a_j] = 0, \quad [a_i^\dagger, a_j^\dagger] = 0,$$
$$[a_i, a_j^\dagger] = \hbar\delta_{ij} \qquad i,j = 1,\ldots\infty$$

- Or in terms of creation and annihilation operators (continuous normalization)

$$[a(\mathbf{k}), a(\mathbf{k}')] = \delta(\mathbf{k}-\mathbf{k}'), \quad [a^\dagger(\mathbf{k}), a^\dagger(\mathbf{k}')] = 0,$$
$$[a(\mathbf{k}), a^\dagger(\mathbf{k})] = \hbar\delta(\mathbf{k}-\mathbf{k}')$$

- All the above algebras have uncountably many Hilbert space representations

Curved spacetime: global hyperbolicity

- All the above construction can be extended literally to the class of globally hyperbolic manifolds (this is of course a little tautological)
- A Lorentzian manifold is globally hyperbolic if it admits a Cauchy Surface: a 3-dim closed achronal set whose domain of dependence is the manifold itself.
- A globally hyperbolic spacetime admits a global time coordinate and a foliation:

$$\mathcal{M} = \bigcup_t \Sigma_t$$

- Each surface of constant time is a smooth Cauchy surface: smooth spacelike 3-surface cut exactly once by each inextendible causal curve
- As a consequence of global hyperbolicity there are global solution of the Cauchy problem (Leray-Lichnerowicz-Choquet Bruhat)

$$(\Box + m^2)E^{\pm} = \mathbb{I}$$

$$E^{\pm} : \mathcal{C}_0^{\infty}(\mathcal{M}) \to \mathcal{C}^{\infty}(\mathcal{M})$$

$$\mathrm{supp}\ E^{\pm}(f) = J^{\pm}(\mathrm{supp}\ f)$$

Curved spacetime: global hyperbolicity

- Define the analogous of the Pauli-Jordan function

$$E(x,y) = E^{+}(x,y) - E^{-}(x,y)$$

- And the Peierls brackets (Klein-Gordon inner product)

$$\int f(x)E(x-y)g(y)d^4x d^4y = -\Omega(Ef, Eg)\,, \quad f,g \in \mathcal{C}_0^{\infty}(\mathcal{M}^4)$$

$$\Omega(\phi_1,\phi_2) = -\int_{\Sigma_t}(\phi_1\pi_2 - \pi_1\phi_2)d^3x = -\int_{\Sigma_t}\phi_1 \overleftrightarrow{\nabla}_{\mu}\,\phi_2\, d\sigma^{\mu}$$

$$\pi(x) = \frac{\delta L}{\delta\dot{\phi}} = |h|^{1/2} n^a \nabla_a \phi \qquad \Gamma = \{(\phi_0,\pi_0)|\phi_0,\pi_0 \in C_0^{\infty}(\Sigma_0)\}$$

$h_{ij} \rightarrow$ induced metric on Σ_t

$n^a \rightarrow$ future-directed unit normal vector field on Σ_t.

Quantizing free fields

- Replace the Peierls brackets by the canonical commutation relation (here written covariantly)

$$\{\phi(x), \phi(y)\} = -E(x,y)$$
$$C(x,y) = [\phi(x), \phi(y)] = -i\hbar E(x,y)$$

- Construct a Hilbert space representation of the CCR

$$\phi(x) \to \hat{\phi}(x)$$
$$[\hat{\phi}(x), \hat{\phi}(y)] = -i\hbar E(x,y) = C(x,y)$$

- **<u>Uniqueness fails again!</u>** (this has nothing to do with the fact that the spacetime is curved. It is the fact that the system is infinite-dimensional)

Commutator: standard construction

Let (M,g) a globally hyperbolic manifold. Consider the KG equation on the manifold M

$$\Box_g \phi + V(x)\phi = 0.$$

Introduce the invariant Peierls aka Klein-Gordon inner product in the space of complex solution of the KG equation

$$(f,g) = -i \int_{\Sigma_t} \overline{f}\, \overleftrightarrow{\nabla}_\mu \, g \, d\sigma^\mu$$

Find a basis $\{u_i\}$ so that $(u_i, u_j) = \delta_{ij}, \quad (\bar{u}_i, \bar{u}_j) = -\delta_{ij}, \quad (u_i, \bar{u}_j) = 0$

The unequal time commutator admits the following expansion

$$C(x,y) = \sum [u_i(x)\overline{u_i(y)} - u_i(y)\overline{u_i(x)}]$$

It is basis independent (uniqueness)