The Particle Spectrum of Strongly Coupled Lattice QCD Models

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CONTEXT: Lattice QCD in $d+1$ EUCLIDEAN Spacetime Dimensions
$d = 1, 2$ (with Pauli 2×2 Spin Matrices)
$d = 3$ (with Dirac 4×4 Spin Matrices)

Framework: UNIT Hypercubic Lattice, 3 Colors , Two/Three **Quark Flavors** and **STRONG COUPLING**.

OBS.: Lattice Spacing is Kept FIXED! NO UV Problem!

GOAL: Apply analytical methods from Constructive Euclidean Quantum Field Theory (QFT) to obtain the Low-Lying Particle Spectrum of Lattice QCD. Rigorous Determination of ONE- and TWO- and, eventually, THREE or $+$-PARTICLE ENERGY-MOMENTUM STATES.

MATH Viewpoint: Determination of the SPECTRUM of the DYNAMICS GENERATOR of an UNDERLYING SEMI-GROUP describing Time-Evolution of a Quantum Physical System.

QFT/Physics Viewpoint: Particle Content. Also, more ambitious: NUCLEAR PHYSICS from QCD. From First Principles: Quarks, Gluons and their QCD Dynamics. Multibody Interactions, Binding Balance & MATTER STABILITY, Binding Dependence on Spin, Isospin, ...

TODAY: Concentrate in $d = 3$, i.e. $3 + 1$ **Spacetime Dimensions,** 4×4 **Dirac Spin Matrices,** 3 **Flavors,** SU(3) **Local Gauge Model** with **STRONG COUPLING**.

For COMPLETENESS: Slides Contain MORE DETAILS THAN WE WILL COVER HERE! (Interested People May Use Them)

THE MODEL and SOME BASICS (PLEASE, BE PATIENT!!!)

Statistical Mechanical **Partition Function**

$$Z = \int e^{-S(\psi,\bar{\psi},g)}\, d\psi\, d\bar{\psi}\, d\mu(g)\,,$$

and for a function $F(\bar{\psi},\psi,g)$, the **Normalized Correlations** are denoted by

$$\langle F\rangle = \frac{1}{Z}\int F(\bar{\psi},\psi,g)\, e^{-S(\psi,\bar{\psi},g)}\, d\psi\, d\bar{\psi}\, d\mu(g)\,.$$

The **Model ACTION** $S \equiv S(\psi,\bar{\psi},g)$ is the **Improved Wilson** action (**no fermion doubling!**)

$$\begin{aligned} S \;=\; & \frac{\kappa}{2}\sum \bar{\psi}_{a,\alpha,f}(u)\Gamma^{\sigma e^{\mu}}_{\alpha\beta}(g_{u,u+\sigma e^{\mu}})_{ab}\psi_{b,\beta,f}(u+\sigma e^{\mu}) \\ & + \sum_{u\in\mathbb{Z}^4_o} \bar{\psi}_{a,\alpha,f}(u)M_{\alpha\beta}\psi_{a,\beta,f}(u) - \frac{1}{g_0^2}\sum_p \chi(g_p)\,, \end{aligned}$$

where, besides the sum over repeated indices α, $\beta = 1,2,3,4$ (spin), $a = 1,2,3$ (color) and $f = 1,2,3 \equiv u,d,s$ (isospin), the first sum runs over $u = (u^0,\vec{u}) = (u^0,u^1,u^2,u^3) \in \mathbb{Z}^4_o \equiv \{\pm 1/2, \pm 3/2, \pm 5/2 ...\} \times \mathbb{Z}^3$, $\sigma = \pm 1$ and $\mu = 0,1,2,3$.

Here:

Label 0 **for the time** direction. Direction 3: also called the z-direction. e^{μ}, $\mu = 0,1,2,3$, unit lattice vector for the μ-direction.

PARAMETERS: Quark-Gauge Coupling or **Hopping Parameter** $\kappa > 0$, the **Pure Gauge Strength** $\beta \equiv (2g_0^2)^{-1} > 0$, **Fermion Masses** $M_{\alpha\beta} \equiv M_{\alpha\beta}(m,\kappa) > 0$, $m > 0$ is the Bare Fermion Mass.

Technical Point:

Shifted lattice for the time direction: avoids the Zero-Time coordinate. In the (formal) continuum limit, two-sided equal time limits of quark, Fermi-field correlations are accommodated.

QUARKS & ANTIQUARKS:

At each site $u \in \mathbb{Z}_o^4$, there are fermionic Grassmann Quark fields $\psi_{a\alpha f}(u)$ and Antiquark fields $\bar{\psi}_{a\alpha f}(u)$, carrying a Dirac Spin index $\alpha = 1,2,3,4$, an SU(3)$_c$ Color index $a = 1,2,3$ and Flavor $f = 1,2,3$.

We refer to $\alpha = 1,2$ as *Upper* Spin indices and $\alpha = 3,4$ or $+$ or $-$ respectively, as *Lower* ones.

SPIN in the Continuum: In a Poincaré invariant theory, recall the Fields and Physical States verify definite transformation laws under the Poincaré Group.

Lorentz Boost: Pass to Improper States in the rest frame. Fields transform according to a (Irreducible) Representation of SU(2) - the Continuum Spatial Rotation Subgroup.

The Infinitesimal Generators are the **Spin Operators** and Satisfy the Usual **Angular Momentum Algebra**.

Here: Only Discrete $\pi/2$ Rotations about the Spatial Axes are SYMMETRIES, which we use to define the Components of our Lattice Total Angular Momentum.

Other Rotational Symmetries for a tridimensional cube (e.g. rotations about an axis passing by the cube diagonal) are also Rotation Symmetries Here!

For Improper Zero-Momentum Particle States: Obtained with LOCAL COMPOSITE Fields, which are expected to have zero spatial angular momentum, **Define Rectangular Components of Total Spin:** which Agree with the Infinitesimal Generators of Rotations of the Continuum.

REMARK: for $\kappa = 0$, there is an SU(4) symmetry which includes the SU(2) $\oplus$ SU(2) Symmetry in the Spin Space of the model; for $\kappa > 0$ of course only the discrete rotation subgroup survive.

GAUGE FIELDS: For each nearest neighbor oriented lattice bond $< u, u \pm e^{\mu} >$ there is an $\mathrm{SU}(3)_c$ matrix $U(g_{u,u \pm e^{\mu}})$ parametrized by the gauge group element $g_{u,u \pm e^{\mu}}$ and satisfying $U(g_{u,u+e^{\mu}})^{-1} = U(g_{u+e^{\mu},u})$.

PLAQUETTES: At each lattice oriented **plaquette** p (**Single Square Circuit**) there is a **Plaquette Variable** $\chi(U(g_p))$ where $U(g_p)$ is the orientation ordered product of matrices of $\mathrm{SU}(3)_c$ of the plaquette oriented bonds, and χ **is the Real Part of the Trace** (Character).

For simplicity, we sometimes drop U from $U(g)$.

STRONG COUPLING REGIME: The Hopping Parameter $\kappa > 0$ and the Plaquette Coupling $\beta \equiv (2g_0^2)^{-1} > 0$ satisfy

$$0 < \beta \ll \kappa \ll 1 \, .$$

FERMION MASSES: $M \equiv M(m,\kappa) = (m+2\kappa)I_4$. Given κ, for simplicity and without loss of generality, $m > 0$ is chosen such that $M_{\alpha\beta} = \delta_{\alpha\beta}$, meaning that $m + 2\kappa = 1$.

Strong Coupling: $m = 1 - 2\kappa \lesssim 1$ so that **Quarks are Heavy** here, i.e. STATIC! Also: **COVARIANT Dirac term in the Action DOMINATES the PLAQUETTE (Pure Gauge) Term**.

DIRAC-GAUGE MATRICES: $\Gamma^{\pm e^{\mu}} = -I_4 \pm \gamma^{\mu}$, where the γ_{μ} are the 4×4 Hermitian traceless anti-commuting Dirac matrices satisfying $\{\gamma_{\mu},\gamma_{\nu}\} = 2\delta_{\mu\nu}I_4$, and

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \quad ; \quad \gamma^j = \begin{pmatrix} 0 & i\sigma^j \\ -i\sigma^j & 0 \end{pmatrix},$$

σ^j, $j = 1,2,3$ are the Hermitian traceless anti-commuting 2×2 Pauli matrices.

GAUGE FIELD MEASURE: The measure $d\mu(g)$ is the product measure over non-oriented bonds of normalized $\mathrm{SU}(3)_c$ Haar measures. There is only one integration variable per bond, so that g_{uv} and g_{vu}^{-1} are not treated as distinct integration variables.

GRASSMANN INTEGRALS: Grassmann Integrals are **Berezin**!
For a polynomial in the Grassmann variables with coefficients depending on the gauge variables, it is the coefficient of the monomial of maximum degree, i.e. of $\prod_{u,\ell} \bar{\psi}_\ell(u)\psi_\ell(u)$, $\ell \equiv (\alpha, a, f)$.

FERMION INTEGRATION ELEMENTS: $d\psi\, d\bar{\psi}$ is a product measure

$$\prod_{u,\ell} d\psi_\ell(u)\, d\bar{\psi}_\ell(u)\,.$$

<u>GAUSSIAN FERMION INTEGRATION</u>: With a normalization $\mathcal{N}_1 = \langle 1 \rangle$, we have

$$\begin{aligned} \langle \psi_{\ell_1}(x)\,\bar\psi_{\ell_2}(y)\rangle &= \tfrac{1}{\mathcal{N}_1}\int \psi_{\ell_1}(x)\,\bar\psi_{\ell_2}(y)\,e^{-\sum_{u,\ell_3,\ell_4}\bar\psi_{\ell_3}(u)O_{\ell_3\ell_4}\psi_{\ell_4}(u)}\,d\psi\,d\bar\psi \\ &= O^{-1}_{\alpha_1,\alpha_2}\delta_{a_1a_2}\delta_{f_1f_2}\delta(x-y) \end{aligned},$$

with a Kronecker delta for space-time coordinates, and where O -the bilinear form operator- is diagonal in the color and isospin indices.

Setting $\kappa = 0$, we have **UNIT COVARIANCE**: the two-fermion correlation yields

$$\langle \psi_{\ell_1}(x)\,\bar\psi_{\ell_2}(y)\rangle = \delta_{\alpha_1,\alpha_2}\delta_{a_1a_2}\delta_{f_1f_2}\delta(x-y)\,.$$

The integral over a string with a different number of ψ and $\bar\psi$ vanishes, and the integral of monomials is given by Wick's theorem.

GAUGE INVARIANCE:

The Wilson action is **invariant by the LOCAL GAUGE transformations** (for $x \in \mathbb{Z}_o^4$ and $h(x) \in \mathrm{SU}(3)_c$)

$$\begin{aligned} \psi(x) &\mapsto h(x)\,\psi(x)\,, \\ \bar\psi(x) &\mapsto \bar\psi(x)\,[h(x)]^{-1}\,, \\ U(g_{x+e^\mu,x}) &\mapsto h(x+e^\mu)\,U(g_{x+e^\mu,x})\,[h(x)]^{-1}\,. \end{aligned}$$

FLAVOR SYMMETRY: There is also a $\mathrm{SU}(3)_f$ Global FLAVOR symmetry of the action.

$[\mathrm{SU}(2)\times\mathrm{U}(1)] \subset \mathrm{SU}(3)_f$ is the **ISOSPIN** & **HYPERCHARGE** Subgroup

LATTICE: Today Main Tool to Deal With Non-PERTURBATIVE Regime. Many Calculations Rely on NUMERICAL SIMULATIONS!????!

STRONG COUPLING Regime $0 < \beta \ll \kappa \ll 1$ **is GOOD for Doing Analysis.**

Plaquette Term in the action can be 'NEGLECTED'! Many Old Results Available!

CLAIM: NOT ruled out as a Good Regime to treat the Physics we are interested in here!

BESIDES: Strong Coupling Regime has been shown to **PRESERVE IMPORTANT QUALITATIVE FEATURES** of the Model!

PHYSICALLY DESIRABLE: To Have a Picture of LQCD for ALL POINTS in the (κ, β) plane!

HERE: **NOT Claiming to be Uncovering the Whole Story!**

INFINITE-VOLUME RESULTS:

EXISTENCE: Start with the Strongly Coupled LQCD Model in a FINITE BOX. By Polymer Expansion, the THERMODYNAMIC Limit of Correlations Exists. (Check Books by E. Seiler & B. Simon)

IMPORTANT PROPERTIES:

Limiting Correlations: Correlations are **Lattice Translational Invariant**.

Truncated Correlations: have **Exponential TREE-DECAY**.

Analyticities:

1) Correlations **Extend to COMPLEX ANALYTIC FUNCTIONS** functions in the Global Couplings κ and β.

2) Also, taking the various κ, β to be DISTINCT: Analyticity in any Finite Number of Local Couplings

HILBERT SPACE & ENERGY-MOMENTUM OPERATORS:

The Underlying **Quantum Mechanical Physical HILBERT SPACE** $\mathcal{H}$ as well as the **ENERGY** H and **MOMENTA** P^j, $j = 1,2,3$ (E-M) Operators are defined using the **Osterwalder-Schrader construction**.

Start from Gauge-Invariant correlations supported on $u^0 = 1/2$.
Let $T_0^{x^0}$, $T_i^{x^i}$, $i = 1,2,3$, denote **Translation of the Functions** of Grassmann and gauge variables by $x^0 \geq 0$, $\vec{x} = (x^1, x^2, x^3) \in \mathbb{Z}^3$.

For F and G only depending on coordinates with $u^0 = 1/2$, with an abuse of notation, we have the **FEYNMAN-KAC** (F-K) formula

$$(G, \check{T}_0^{x^0} \check{T}_1^{x^1} \check{T}_2^{x^2} \check{T}_3^{x^3} F)_{\mathcal{H}} = \langle [T_0^{x^0} \vec{T}^{\vec{x}} F] \Theta G \rangle \ ,$$

where $T^{\vec{x}} = T_1^{x^1} T_2^{x^2} T_3^{x^3}$.

We do not distinguish between Grassmann, gauge variables and their associated Hilbert space vectors in our notation.

1) LHS involves 'INNER PRODUCTS' in $\mathcal{H}$.
2) RHS only Functional Integrals, i.e. Statistical Mechanical Correlations.
3) The Equality comes by employing a Lie-Trotter Product Formula, Random Processes, Wiener Paths, ... (See books by Glimm-Jaffe and Simon.)

Θ is an **anti-linear operator and involves <u>Reflection in Time</u>**. Actions:

$$\Theta\bar{\psi}_{a\alpha}(u) = (\gamma^0)_{\alpha\beta}\psi_{a\beta}(tu)\,,$$
$$\Theta\psi_{a\alpha}(u) = \bar{\psi}_{a\beta}(tu)(\gamma_0)_{\beta\alpha}\,;$$

where $t(u^0,\vec{u}) = (-u^0,\vec{u})$, for A and B monomials, $\Theta(AB) = \Theta(B)\Theta(A)$; and for a function of the gauge fields $\Theta f(\{g_{uv}\}) = f^*(\{g_{(tu)(tv)}\})$, $u,v \in \mathbb{Z}_o^{d+1}$, where $*$ means complex conjugate.

Θ extends anti-linearly to the algebra of fields.

As Linear Operators in $\mathcal{H}$, $\check{T}_\mu$, $\mu = 0, 1, 2, 3$, **are Mutually Commuting**.

$\check{T}_0$ is **Self-Adjoint**, with $-1 \leq \check{T}_0 \leq 1$.

$\check{T}_{j=1,2,3}$ are **Unitary**.

We write $\check{T}_j = e^{iP^j}$ and $\vec{P} = (P^1, P^2, P^3)$ is the Self-Adjoint **Momentum Operator** with Spectral Points $\vec{p} \in \mathbf{T}^3 \equiv (-\pi, \pi]^3$.

Since $\check{T}_0^2 \geq 0$, the **Energy Operator** $H \geq 0$ is defined by $\check{T}_0^2 = e^{-2H}$.

H, P^j **Commute!** Joint spectrum: **E-M Spectrum**

A Point in the E-M spectrum with spatial momentum $\vec{p} = \vec{0}$ is a MASS.

Consider the **Spectral Families** of $\check{T}_0$, P^1, P^2 and P^3. For Real κ, by the Spectral Theorem, have the Spectral Representations

$$\check{T}_0 = \int_{-1}^{1} \lambda^0 dE_0(\lambda^0) \quad , \quad \check{T}_{j=1,2,3} = \int_{-\pi}^{\pi} e^{i\lambda^j} dF_j(\lambda^j) \, ,$$

For future use: Let $\mathcal{E}(\lambda^0, \vec{\lambda}) = E_0(\lambda^0) \prod_1^3 F_j(\lambda^j)$.

For Real $\kappa > 0$, the **O-S POSITIVITY Condition** $\langle F \Theta F \rangle \geq 0$ can be Established.

But there may be vectors $F \neq 0$ such that $\langle F \Theta F \rangle = 0$.

If the set of such vectors F's is denoted by $\mathcal{N}$, a pre-Hilbert space $\mathcal{H}'$ can be constructed from the inner product $\langle G \Theta F \rangle$ and the physical Hilbert space $\mathcal{H}$ is the completion of the quotient space $\mathcal{H}'/\mathcal{N}$, including also the cartesian product of the inner space sectors, the color space $\mathbb{C}^3$, the spin space $\mathbb{C}^4$ and the isospin space $\mathbb{C}^3$.

MORE SYMMETRIES: Besides **Spatial $\pi/2$ & Diagonal Rotations**

The Usual Spacetime Symmetries of **Parity** $\mathcal{P}$, **Charge Conjugation** $\mathcal{C}$, **Time-Reversal** $\mathcal{T}$ and **Reflection in z-coordinate** Hold and are Implemented by Operators in $\mathcal{H}$.

A *new* **Time-Reflection** T is found! Used to define a **Local Spin Flip** Symmetry.

- **Time Reversal** $\mathcal{T}$: $\psi_\alpha(x) \to \bar{\psi}_\beta(x^t)A_{\beta\alpha}$, $x^t \equiv (-x^0, \vec{x})$, $\bar{\psi}_\alpha(x) \to B_{\alpha\beta}\psi_\beta(x^t)$, $A = B = B^{-1} = \gamma^0$, $f(g_{xy}) \to [f(g_{x^t y^t})]^*$;

- **Charge Conjugation** $\mathcal{C}$: $\psi_\alpha(x) \to \bar{\psi}_\beta(x)A_{\beta\alpha}$, $\bar{\psi}_\alpha(x) \to B_{\alpha\beta}\psi_\beta(x)$, with $A = -B = B^{-1} = \begin{pmatrix} 0 & i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix}$, $f(g_{xy}) \to f(g^*_{xy})$;

- **Parity** $\mathcal{P}$: $\psi_\alpha(x) \to A_{\alpha\beta}\psi_\beta(x^0, -\vec{x})$, $\bar{\psi}_\alpha(x) \to \bar{\psi}_\beta(x^0, -\vec{x})B_{\beta\alpha}$ where $A = B = A^{-1} = \gamma^0$, $f(g_{xy}) \to f(g_{\bar{x}\bar{y}})$, with $\bar{z} = (z^0, -\vec{z})$.

Time Reflection T: NOT to be Confused with Time Reversal symmetry

- **Time Reflection** T: $\psi_\alpha(x) \to A_{\alpha\beta}\psi_\beta(-x^0, \vec{x})$, $\bar{\psi}_\alpha(x) \to \bar{\psi}_\beta(-x^0, \vec{x})B_{\beta\alpha}$ where $A = B = A^{-1} = \begin{pmatrix} 0 & -iI_2 \\ iI_2 & 0 \end{pmatrix}$, $f(g_{xy}) \to f(g_{\bar{x}\bar{y}})$, with $\bar{z} = (-z^0, \vec{z})$.

These Operations extend to monomials and are taken to be order preserving, except for $\mathcal{C}$ and $\mathcal{T}$ which are order reversing. They extend linearly to polynomials and their limits, except for time reversal which is anti-linear. For all of them, except time reversal, the field average equals the transformed field average; for time reversal the transformed field average is the complex conjugate of the field average.

Remark 1 *An important (NEW!) result is that the composed operation*

$$\mathcal{F}_s \equiv -i\mathcal{TC}\mathsf{T}$$

gives the **Spin Flip Transformation***.*

$$\begin{array}{rcl} \psi_\alpha(x) & \to & A_{\alpha\rho}\psi_\rho(x) \\ \bar{\psi}_\beta(x) & \to & \bar{\psi}_\gamma(x)B_{\gamma\beta}\,, \end{array}$$

where $A = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}$ *is real anti-symmetric and* $B = A^{-1} = -A$*.*

It is local and is also a symmetry of the system.

In contrast to the above transformations, it surprisingly leaves invariant each individual term in the Wilson action.

SCENARIO FIGURE

CONCENTRATE ON STATES INVOLVING ONLY BARYON PARTICLES.

SIMILAR for MESONS.

In Strong Coupling: Glueballs Masses are MUCH HIGHER in Spectrum and Do Not Intervene in Our PRESENT ANALYSIS.

ONE-PARTICLE SPECTRUM:

HERE: **One-Baryon** Sector $\mathcal{H}_b \subset \mathcal{H}_o \subset \mathcal{H}$.

$\mathcal{H}_o$ is the Subspace with an **ODD NUMBER of Fermions** (quarks).

Points in the E-M spectrum are **Detected as Singularities in Momentum Space Spectral Representations** of Suitable Two-Baryon Correlations.

PHYSICAL PARTICLES: Associated with **ISOLATED (DISPERSION) CURVES** in the E-M SPECTRUM. NEED TO SHOW **LOWER and UPPER** SPECTRAL GAP PROPERTIES!

CONTINUOUS SPECTRUM (BANDS) Are Related to the Spectrum Corresponding to certain Group of Particles.

SINGULAR SPECTRUM usually has no physical interpretation (except for quasi-crystals).

RECURRENT in LATTICE COMMUNITY: NOT BASED ON SPECTRAL REPRESENTATIONS.

SIMULATIONS: DEAL ONLY WITH OBTAINING EXPONENTIAL DECAY RATES OF CORRELATIONS!!!

WARNING: WILL **NEGLECT PLAQUETTE Term** in the Wilson Action as $\beta \ll \kappa$.

Using β ANALYTICITY: RESULTS for SMALL $|\beta|$ Are Stated Afterwards!

DYNAMICAL BARYON FIELDS:

Baryon Fields of the form (Hat Means All Barred or All Unbarred!)

$$\widehat{b}_{\vec{\alpha}\vec{f}} = \epsilon_{abc} \widehat{\psi}_{a\alpha_1 f_1} \widehat{\psi}_{b\alpha_2 f_2} \widehat{\psi}_{c\alpha_3 f_3} \,,$$

with only upper u ($\alpha_i = 1, 2$) or lower ℓ ($\alpha_i = 3, 4$) Spin Components **EMERGE NATURALLY From The LQCD DYNAMICS**.

RECALL: Each κ derivative acting on exp[HOPPING Term] Gives a Multiplicative Factor

$$-\frac{1}{2} \sum \bar{\psi}_{a,\alpha,f}(u) \Gamma^{\sigma e^\mu}_{\alpha\beta} (g_{u,u+\sigma e^\mu})_{ab} \psi_{b,\beta,f}(u + \sigma e^\mu)$$

Using the COFACTOR METHOD (M. Creutz) only the gauge field integrals with a String with a **MULTIPLE of 3** Matrix Elements of U DO NOT VANISH!

The Elements of $U^\dagger = U^{-1}$ count as PRODUCTS of 2 Matrix Elements of U as, for $U \in \mathrm{SU(3)}$, $\det U = 1$ and

$$U^{-1} = [\mathrm{cof}\ U]^t \qquad (\mathrm{cof} = \mathrm{COFACTOR})\,.$$

Using a Generating Functional for the String of U's, we Compute Some Useful Nonvanishing Gauge Integrals (recall $U(g) \equiv g$):

$$\mathcal{I}_2 \equiv \int g_{a_1 b_1} g^{-1}_{a_2 b_2} \, d\mu(g) = \frac{1}{3} \delta_{a_1 b_2} \delta_{a_2 b_1} \,, \tag{1}$$

$$\mathcal{I}_3 \equiv \int g_{a_1 b_1} g_{a_2 b_2} g_{a_3 b_3} d\mu(g) = \frac{1}{6} \epsilon_{a_1 a_2 a_3} \epsilon_{b_1 b_2 b_3}, \tag{2}$$

$$\begin{aligned}\mathcal{I}_4 &\equiv \int g_{a_1 b_1} g^{-1}_{a_2 b_2} g_{a_3 b_3} g^{-1}_{a_4 b_4} \, d\mu(g) \\ &= \frac{1}{8} [\delta_{a_1 b_2} \delta_{a_3 b_4} \delta_{b_1 a_2} \delta_{b_3 a_4} + (a_2 \rightleftarrows a_4, b_2 \rightleftarrows b_4)] \\ &\quad - \frac{1}{24} [\delta_{a_1 b_2} \delta_{a_3 b_4} \delta_{b_1 a_4} \delta_{b_3 a_2} + (a_2 \rightleftarrows a_4, b_2 \rightleftarrows b_4)]\end{aligned} \tag{3}$$

$$\begin{aligned}\mathcal{I}_6 &\equiv \int g_{a_1 b_1} g_{a_2 b_2} g_{a_3 b_3} g_{a_4 b_4} g_{a_5 b_5} g_{a_6 b_6} \, d\mu(g) \\ &= \frac{2}{3!4!} \Big[\epsilon_{a_1 a_2 a_3} \epsilon_{b_1 b_2 b_3} \epsilon_{a_4 a_5 a_6} \epsilon_{b_4 b_5 b_6} + \epsilon_{a_1 a_2 a_4} \epsilon_{b_1 b_2 b_4} \epsilon_{a_3 a_5 a_6} \epsilon_{b_3 b_5 b_6} + \\ &\quad \epsilon_{a_1 a_2 a_5} \epsilon_{b_1 b_2 b_5} \epsilon_{a_3 a_4 a_6} \epsilon_{b_3 b_4 b_6} + \epsilon_{a_1 a_2 a_6} \epsilon_{b_1 b_2 b_6} \epsilon_{a_3 a_4 a_5} \epsilon_{b_3 b_3 b_5} + \\ &\quad \epsilon_{a_1 a_3 a_4} \epsilon_{b_1 b_3 b_3} \epsilon_{a_2 a_5 a_6} \epsilon_{b_2 b_5 b_6} + \epsilon_{a_1 a_3 a_5} \epsilon_{b_1 b_3 b_5} \epsilon_{a_2 a_4 a_6} \epsilon_{b_2 b_4 b_6} + \\ &\quad \epsilon_{a_1 a_3 a_6} \epsilon_{b_1 b_3 b_6} \epsilon_{a_2 a_4 a_5} \epsilon_{b_2 b_4 b_5} + \epsilon_{a_1 a_4 a_5} \epsilon_{b_1 b_4 b_5} \epsilon_{a_2 a_3 a_6} \epsilon_{b_2 b_3 b_6} + \\ &\quad \epsilon_{a_1 a_4 a_6} \epsilon_{b_1 b_4 b_6} \epsilon_{a_2 a_3 a_5} \epsilon_{b_2 b_3 b_5} + \epsilon_{a_1 a_5 a_6} \epsilon_{b_1 b_5 b_6} \epsilon_{a_2 a_3 a_4} \epsilon_{b_2 b_3 b_4} \Big] \,.\end{aligned} \tag{4}$$

Using all this and considering a general correlation $\langle F(x)G(y)\rangle$, with F and G with an ODD NUMBER of Barred $\bar{\psi}$ PLUS Unbarred ψ the lowest κ order F and G admissible are the BARYON and ANTIBARYON Fields.

EXPLANATION:

ZERO DERIVATIVES: $\langle F\rangle = 0 = \langle G\rangle$ by Fermion Integration.

ONE DERIVATIVE: One $U(g) = g$ is produced and has ZERO Integral.

TWO DERIVATIVES: EITHER Two U Are PRODUCED and have ZERO Integral, OR ONE U **and** ONE U^{-1} Are produced, Which Gives a Non-vanishing Gauge Integral, BUT Has ZERO FERMION Integral.

THREE DERIVATIVES: Three U Are Produced and Gauge Integral Gives Levi-Civita $\epsilon_{a_1 a_2 a_3}$. Together With the Three Accompanying Fermion Fields, gives the Baryon!

To **Classify and Label the Baryon States**, USE:

1) Flavor Symmetry SU(3)$_f$ (which holds for any κ!): **Total Isospin** I, Third Component of Total Isospin I_3 and **Total Hypercharge** Y Operators (related to the SU(2) and U(1) Subgroups).

2) Value of the Quadratic Casimir C_2.

3) Total Spin and its z-component, J and J_z.

Enough: No need for the cubic Casimir C_3!

The **Baryon Fields**

$$\hat{b}_{\vec{\alpha}\vec{f}} = \epsilon_{abc}\,\hat{\psi}_{a\alpha_1 f_1}\hat{\psi}_{b\alpha_2 f_2}\hat{\psi}_{c\alpha_3 f_3}\,,$$

comprise a **set of** $216 = 6^3 = (2\ spins\ for\ a\ Particle\ State\ \times\ 3\ flavors)^3$ **fields**.

NOT ALL ARE Linearly Independent: **Totally Symmetric Property** (tsp), i.e. these are symmetric under the Interchange of Any Two Pairs $\alpha_i f_i \leftrightarrow \alpha_j f_j$, there are equalities among them.

ONLY 56 **distinct, linearly independent, states**.

Normalized Baryon fields

$$\hat{B}_{\vec{\alpha}\vec{f}} = \frac{1}{n_{\vec{\alpha}\vec{f}}}\hat{b}_{\vec{\alpha}\vec{f}}$$

with $n_{\vec{\alpha}\vec{f}}$ may assume the values 6, $2\sqrt{3}$ and $\sqrt{6}$ (depending on the number of repeated pairs $\alpha_i\,,\ f_i$).

TWO-BARYON CORRELATIONS

$$G_{\ell\ell'}(u,v) = \langle B_\ell(u)\bar{B}_{\ell'}(v)\rangle\,\chi_{u^0\leq v^0} - \langle \bar{B}_\ell(u)B_{\ell'}(v)\rangle^*\,\chi_{u^0>v^0}\,,$$

Here, ℓ and ℓ' are Collective Indices and χ denotes the INDICATOR.

By Time Reversal Symmetry: $\langle B_\ell(u)\bar{B}_{\ell'}(v)\rangle = -\langle \bar{B}_\ell(-u^0,\vec{u})B_{\ell'}(-v^0,\vec{v})\rangle^*$,

$$G_{\ell\ell'}(u,v) = G_{\ell\ell'}((-u^0,\vec{u}),(-v^0,\vec{v}))\,.$$

Using the Previous Spectral Reps for the E-M Operators in the F-K Formula, we have (NOTICE: NO Zero-Time Fields!):

Theorem 1 *For $x \equiv u - v$, and recalling -by translation invariance- that $G(u,v) = G(u-v)$ with $\bar{B}_\ell \equiv \bar{B}_\ell(1/2,\vec{0})$ we have*

$$G_{\ell_1\ell_2}(x) \;=\; -\int_{-1}^{1}\int_{\mathbb{T}^3}(\lambda^0)^{|x^0|-1}e^{-i\vec{\lambda}.\vec{x}}\,d_\lambda(\bar{B}_{\ell_1},\mathcal{E}(\lambda^0,\vec{\lambda})\bar{B}_{\ell_2})_{\mathcal{H}}\,;$$

for $x \in \mathbb{Z}^4$, $x^0 \neq 0$, and is an even function of $\vec{x}$ by parity symmetry.

For $p \equiv (p^0, \vec{p})$, the lattice Fourier Transform of $G_{\ell_1\ell_2}(x)$ is

$$\tilde{G}_{\ell_1\ell_2}(p) = \sum_{x \in \mathbb{Z}^4} G_{\ell_1\ell_2}(x) e^{-ip.x} \,.$$

Combining with the last result, after separating the **Zero-Time** Contribution, $\tilde{G}_{\ell_1\ell_2}(p)$ admits the Spectral Representation:

Theorem 2

$$\tilde{G}_{\ell_1\ell_2}(p) \;=\; \tilde{G}_{\ell_1\ell_2}(\vec{p}) - (2\pi)^3 \int_{-1}^{1} f(p^0, \lambda^0) d_{\lambda^0} \alpha_{\vec{p},\ell_1\ell_2}(\lambda^0) \,,$$

where

$$d_{\lambda^0}\alpha_{\vec{p},\ell_1\ell_2}(\lambda^0) = \int_{\mathbb{T}^3} \delta(\vec{p} - \vec{\lambda}) d_{\lambda^0} d_{\vec{\lambda}} (\bar{B}_{\ell_1}, \mathcal{E}(\lambda^0, \vec{\lambda}) \bar{B}_{\ell_2})_{\mathcal{H}} \,,$$

with

$$f(p^0, \lambda^0) \equiv (e^{ip^0} - \lambda^0)^{-1} + (e^{-ip^0} - \lambda^0)^{-1} \,,$$

and we set $\tilde{G}(\vec{p}) = \sum_{\vec{x}} e^{-i\vec{p}.\vec{x}} G(x^0 = 0, \vec{x})$.

USE $\tilde{G}(p)$ to DETECT Baryon Particles in the E-M Spectrum.

Complex p^0 Singularities of $\tilde{G}(p)$ Are Points in the E-M spectrum!
Particles correspond to Singularities which form <u>*Isolated*</u> **Dispersion Curves** $w(\vec{p})$ associated with $p^0 = iw(\vec{p})$.

ISOLATED: **Need to show UPPER and LOWER Spectral Gaps**

LOWER Spectral Gap: OK since G has exponential decay, as follows from the Polymer Expansion Method.

MORE PRECISE BOUND: Lattice Adaptation of Spencer's **Hyperplane Decoupling Method**. The Above κ expansion is a Good Guide to Understand this!

Theorem 3 *The two-point function kernel* $G_{\ell_1\ell_2}(u,v,\kappa) \equiv G_{\ell_1\ell_2}(u,v) \equiv G_{\ell_1\ell_2}(u-v)$ *verifies the following global bound, with* $|x| = |x^0| + \sum_{j=1}^{3} |x^j|$,

$$|G_{\ell_1\ell_2}(u,v)| \le \mathcal{O}(1)\, |\kappa|^{3|u-v|} = \mathcal{O}(1)\, e^{-(-3\ln|\kappa|)|u-v|}\,,$$

for some positive constant $\mathcal{O}(1)$ *uniform in* κ *and in the multindices* ℓ_1 *and* ℓ_2.

By the Payley-Wiener theorem:

Corollary 1 *$\tilde{G}(p)$ is analytic in the polystrip $|\mathcal{I}\mathrm{m}\, p^\mu| \leq -(3-\epsilon)\ln\kappa$, $\mu = 0,1,2,3$, $0 < \epsilon \ll 1$, and we have a spectral mass gap of at least $-(3-\epsilon)\ln\kappa$, $0 < \epsilon \ll 1$.*

TO GO HIGHER in SPECTRUM and Obtain an UPPER GAP: **USE a Meromorphic Extension** of $\tilde{G}(p)$ in p^0. For fixed $\vec{p}$ and $\kappa \neq 0$, it is provided by

$$\tilde{\Gamma}^{-1}(p) = \frac{\{\mathrm{cof}[\tilde{\Gamma}(p)]\}^t}{\det\tilde{\Gamma}(p)},$$

where $\tilde{\Gamma}(p)\tilde{G}(p) = 1$, such as Γ is the **convolution inverse of** G.

Γ is Defined by a **Neumann series** in κ. Series Converges by the global bound on G, for small $|\kappa| > 0$.

That $\tilde{\Gamma}^{-1}(p)$ provides a Suitable Extension of $\tilde{G}(p)$ since Γ has a **Faster Falloff** Than G: From **COMPENSATIONS** in Neumann Series.

Faster Falloff Also Ensures: **Singularities** of $\tilde{G}(p)$ are **contained in the zeroes of** $\det\tilde{\Gamma}(p)$. CAN CHECK: **NO** $\frac{0}{0}$ **Cancellation in Quotient!**

In the above discussion, we used the result from the Spencer's Hyperplane Decoupling Expansion:

Theorem 4 *The convolution inverse kernel* $\Gamma_{\ell_1\ell_2}(u,v,\kappa) \equiv \Gamma_{\ell_1\ell_2}(u,v) \equiv \Gamma_{\ell_1\ell_2}(u-v)$ *is bounded and satisfies*

$$|\Gamma_{\ell_1\ell_2}(u,v)| \leq \mathcal{O}(1)\,|\kappa|^{3|\vec{u}-\vec{v}|}\,|\kappa|^{3+5(|u^0-v^0|-1)},\ |u^0-v^0| \neq 0\,,$$

for some constant $\mathcal{O}(1) > 0$*, uniform in* κ*,* ℓ_1 *and* ℓ_2*. The rhs is replaced by* const $\kappa^{3|\vec{u}-\vec{v}|}$*, if* $u^0 = v^0$*.*

Again, by Payley-Wiener theorem, we obtain:

Corollary 2 $\tilde{\Gamma}(p)$ *is analytic in the polystrip* $|\mathcal{I}\mathrm{m} p^0| \leq -(5-\epsilon)\ln\kappa$*,* $|\mathcal{I}\mathrm{m} p^i| \leq -(3-\epsilon)\ln\kappa$*,* $i = 1,2,3$*,* $0 < \epsilon \ll 1$*.*

RESTRICTION to Baryons (lower indices in B fields!). Charge Conjugation Ensures Similar Results for Antibaryons (upper indices).

The **Baryon DISPERSION CURVES** $w(\kappa,\vec{p})$ are defined by

$$\det\tilde{\Gamma}\left(p^0 = iw(\kappa,\vec{p}),\vec{p}\right) = 0\,.$$

It follows that, with fixed $\vec{p}$, the Curves $w(\kappa,\vec{p})$ are **ISOLATED**.

We still do not know the number and form of the $w(\kappa, \vec{p})$.

INTUITIVE ARGUMENT: Retaining only terms to order κ^3, i.e. using only the values for distance zero and one (see below):

$$\tilde{G}_{\ell_1 \ell_2}(p) = [-1 - 2\kappa^3 \cos p^0 - \tfrac{\kappa^3}{4} \textstyle\sum_{j=1,2,3} \cos p^j]\, \delta_{\ell_1 \ell_2} + \mathcal{O}(\kappa^4)\,,$$

$$\tilde{\Gamma}_{\ell_1 \ell_2}(p) = [-1 + 2\kappa^3 \cos p^0 + \tfrac{\kappa^3}{4} \textstyle\sum_{j=1,2,3} \cos p^j]\, \delta_{\ell_1 \ell_2} + \mathcal{O}(\kappa^4)\,.$$

Dropping the $\mathcal{O}(\kappa^4)$ terms in $\tilde{\Gamma}(p)$, $\det \tilde{\Gamma}(p)$ factorizes into 56 identical factors. Under the above approximation, with $p_\ell^2 \equiv 2 \sum_{i=1}^{3} (1 - \cos p^i)$, for each factor, we get identical dispersion curves

$$w(\vec{p}) \equiv w(\vec{p}, \kappa) = \left[-3 \ln \kappa - \frac{3\kappa^3}{4} + \frac{\kappa^3}{8} p_\ell^2 \right] + \mathcal{O}(\kappa^4)\,.$$

Particle Mass: $M \equiv w(\vec{0}, \kappa) = \left[-3 \ln \kappa - \frac{3\kappa^3}{4} \right] + \mathcal{O}(\kappa^4)\,,$

SOLVING THE IMPLICIT EQUATION for $w(\vec{p}, \kappa)$

NEW BASIS: The Two-Baryon Matrix is 'more diagonal'.

SEEN using the SU(3) flavor and other symmetries like time reversal, charge conjugation, parity and time reflection, and the analysis becomes much simpler.

PARTICLE BASIS: Related to the (original) Individual Basis by a **Real Orthogonal Transformation**.

LABELS: Eightfold Way quantum numbers of I_3, Y, C_2, coming from SU(3)$_f$. Also, J and J_z.

The 56-Dimensional One-Baryon vector subspace $\mathcal{B}$ of the Hilbert space $\mathcal{H}_\wr$ admits an orthogonal direct sum decomposition:

$$\mathcal{B} = \left(\oplus_{i=1,\ldots,10}\, \mathcal{D}_i\right) \oplus \left(\oplus_{i=1,\ldots,8}\, \mathcal{O}_i\right) . \tag{5}$$

Here, each $\mathcal{D}$ is 4-Dimensional and each $\mathcal{O}$ is 2-Dimensional.

$G(\kappa, x)$ presents a compatible block decomposition, with eight *identical* 2×2 blocks associated with Total Spin 1/2 OCTET BARYONS and ten *identical* 4×4 blocks, associated with Total Spin 3/2 DECUPLET BARYONS.

G is **diagonal in all of the above quantum numbers, except for the SPIN**, for all $\kappa > 0$.

$\Gamma(\kappa, x)$ inherits the same block decomposition as $G(\kappa, x)$.

Also their Fourier transforms $\tilde{G}(\kappa, p)$ and $\tilde{\Gamma}(\kappa, p)$.

The decays of G and Γ are preserved as well as Analyticity Properties of $\tilde{G}(\kappa, p)$ and $\tilde{\Gamma}(\kappa, p)$.

OCTET FIELDS: (± denote the $J_z = \pm 1/2$.)

$$
\begin{cases}
p_{\pm} = \frac{\epsilon_{abc}}{3\sqrt{2}} (\bar{\psi}_{a+u}\bar{\psi}_{b-d} - \bar{\psi}_{a+d}\bar{\psi}_{b-u})\bar{\psi}_{c\pm u}, \\
n_{\pm} = \frac{\epsilon_{abc}}{3\sqrt{2}} (\bar{\psi}_{a+u}\bar{\psi}_{b-d} - \bar{\psi}_{a+d}\bar{\psi}_{b-u})\bar{\psi}_{c\pm d}, \\
\Xi^0_{\pm} = \frac{\epsilon_{abc}}{3\sqrt{2}} (\bar{\psi}_{a+u}\bar{\psi}_{b-s} - \bar{\psi}_{a+s}\bar{\psi}_{b-u})\bar{\psi}_{c\pm s}, \\
\Xi^-_{\pm} = \frac{\epsilon_{abc}}{3\sqrt{2}} (\bar{\psi}_{a+d}\bar{\psi}_{b-s} - \bar{\psi}_{a+s}\bar{\psi}_{b-d})\bar{\psi}_{c\pm s}, \\
\Sigma^+_{\pm} = \frac{\epsilon_{abc}}{3\sqrt{2}} (\bar{\psi}_{a+u}\bar{\psi}_{b-s} - \bar{\psi}_{a+s}\bar{\psi}_{b-u})\bar{\psi}_{c\pm u}, \\
\Sigma^0_{\pm} = \frac{\epsilon_{abc}}{6} (2\bar{\psi}_{a\pm u}\bar{\psi}_{b\pm d}\bar{\psi}_{c\mp s} - \bar{\psi}_{a-u}\bar{\psi}_{b+d}\bar{\psi}_{c\pm s} - \bar{\psi}_{a+u}\bar{\psi}_{b-d}\bar{\psi}_{c\pm s}), \\
\Sigma^-_{\pm} = \frac{\epsilon_{abc}}{3\sqrt{2}} (\bar{\psi}_{a+d}\bar{\psi}_{b-s} - \bar{\psi}_{a+s}\bar{\psi}_{b-d})\bar{\psi}_{c\pm d}, \\
\Lambda_{\pm} = \frac{\epsilon_{abc}}{2\sqrt{3}} (\bar{\psi}_{a+u}\bar{\psi}_{b-d} - \bar{\psi}_{a+d}\bar{\psi}_{b-u})\bar{\psi}_{c\pm s} \, ,
\end{cases}
$$

n, p, Ξ^- and Ξ^0 have $I = 1/2$; Σ^+, Σ^0 and Σ^- have $I = 1$ and Λ has $I = 0$.

DECUPLET FIELDS:

$$
\left\{
\begin{array}{l}
\Delta^{+}_{\pm\frac{1}{2}} = \frac{\epsilon_{abc}}{6}(\bar\psi_{a\pm u}\bar\psi_{b\pm u}\bar\psi_{c\mp d} + 2\bar\psi_{a\pm u}\bar\psi_{b\mp u}\bar\psi_{c\pm d}), \\
\Delta^{+}_{\pm\frac{3}{2}} = \frac{\epsilon_{abc}}{2\sqrt{3}}\,\bar\psi_{a\pm u}\bar\psi_{b\pm u}\bar\psi_{c\pm d}, \\
\Delta^{0}_{\pm\frac{1}{2}} = \frac{\epsilon_{abc}}{6}(2\bar\psi_{a\pm u}\bar\psi_{b\pm d}\bar\psi_{c\mp d} + \bar\psi_{a\mp u}\bar\psi_{b\pm d}\bar\psi_{c\pm d}), \\
\Delta^{0}_{\pm\frac{3}{2}} = \frac{\epsilon_{abc}}{2\sqrt{3}}\,\bar\psi_{a\pm u}\bar\psi_{b\pm d}\bar\psi_{c\pm d}, \\
\Delta^{-}_{\pm\frac{1}{2}} = \frac{\epsilon_{abc}}{2\sqrt{3}}\,\bar\psi_{a\pm d}\bar\psi_{b\pm d}\bar\psi_{c\mp d}\,, \\
\Delta^{-}_{\pm\frac{3}{2}} = \frac{\epsilon_{abc}}{6}\,\bar\psi_{a\pm d}\bar\psi_{b\pm d}\bar\psi_{c\pm d}\,, \\
\Delta^{++}_{\pm\frac{1}{2}} = \frac{\epsilon_{abc}}{2\sqrt{3}}\,\bar\psi_{a\pm u}\bar\psi_{b\pm u}\bar\psi_{c\mp u}\,, \\
\Delta^{++}_{\pm\frac{3}{2}} = \frac{\epsilon_{abc}}{6}\,\bar\psi_{a\pm u}\bar\psi_{b\pm u}\bar\psi_{c\pm u}\,, \\
\Sigma^{*+}_{\pm\frac{3}{2}} = \frac{\epsilon_{abc}}{2\sqrt{3}}\bar\psi_{a\pm u}\bar\psi_{b\pm u}\bar\psi_{c\pm s}, \\
\Sigma^{*+}_{\pm\frac{1}{2}} = \frac{\epsilon_{abc}}{6}\,(\bar\psi_{a\pm u}\bar\psi_{b\pm u}\bar\psi_{c\mp s} + 2\bar\psi_{a\pm u}\bar\psi_{b\mp u}\bar\psi_{c\pm s})\,, \\
\Sigma^{*0}_{\pm\frac{3}{2}} = \frac{\epsilon_{abc}}{6}\,\bar\psi_{a\pm u}\bar\psi_{b\pm d}\bar\psi_{c\pm s}\,,
\end{array}
\right.
$$

Continuing...

$$
\begin{cases}
\Sigma^{*0}_{\frac{\pm 1}{2}} = \frac{\epsilon_{abc}}{3\sqrt{2}}(\bar{\psi}_{a\pm u}\bar{\psi}_{b\pm d}\bar{\psi}_{c\mp s} + \bar{\psi}_{a\pm u}\bar{\psi}_{b\mp d}\bar{\psi}_{c\pm s} + \bar{\psi}_{a\mp u}\bar{\psi}_{b\pm d}\bar{\psi}_{c\pm s}), \\
\Sigma^{*-}_{\frac{\pm 3}{2}} = \frac{\epsilon_{abc}}{2\sqrt{3}}\bar{\psi}_{a\pm d}\bar{\psi}_{b\pm d}\bar{\psi}_{c\pm s}, \\
\Sigma^{*-}_{\frac{\pm 1}{2}} = \frac{\epsilon_{abc}}{6}(\bar{\psi}_{a\pm d}\bar{\psi}_{b\pm d}\bar{\psi}_{c\mp s} + 2\bar{\psi}_{a\pm d}\bar{\psi}_{b\mp d}\bar{\psi}_{c\pm s}), \\
\Xi^{*0}_{\frac{\pm 3}{2}} = \frac{\epsilon_{abc}}{2\sqrt{3}}\bar{\psi}_{a\pm u}\bar{\psi}_{b\pm s}\bar{\psi}_{c\pm s}, \\
\Xi^{*0}_{\frac{\pm 1}{2}} = \frac{\epsilon_{abc}}{6}(\bar{\psi}_{a\mp u}\bar{\psi}_{b\pm s} + 2\bar{\psi}_{a\pm u}\bar{\psi}_{b\mp s})\bar{\psi}_{c\pm s}, \\
\Xi^{*-}_{\frac{\pm 3}{2}} = \frac{\epsilon_{abc}}{2\sqrt{3}}\bar{\psi}_{a\pm d}\bar{\psi}_{b\pm s}\bar{\psi}_{c\pm s}, \\
\Xi^{*-}_{\frac{\pm 1}{2}} = \frac{\epsilon_{abc}}{6}(\bar{\psi}_{a\mp d}\bar{\psi}_{b\pm s} + 2\bar{\psi}_{a\pm d}\bar{\psi}_{b\mp s})\bar{\psi}_{c\pm s}, \\
\Omega^{-}_{\frac{\pm 3}{2}} = \frac{\epsilon_{abc}}{6}\bar{\psi}_{a\pm s}\bar{\psi}_{b\pm s}\bar{\psi}_{c\pm s}, \\
\Omega^{-}_{\frac{\pm 1}{2}} = \frac{\epsilon_{abc}}{2\sqrt{3}}\bar{\psi}_{a\pm s}\bar{\psi}_{b\pm s}\bar{\psi}_{c\mp s}\,,
\end{cases}
$$

Using the SPIN FLIP Symmetry $\mathcal{F}_s = -i\mathcal{T}\mathcal{C}\mathrm{T}$, together with parity $\mathcal{P}$, and time reversal $\mathcal{T}$, conclude that:

1) Each spin 1/2, OCTET 2×2 block of $\tilde{\Gamma}(p)$ is **diagonal, real and multiple of the identity**

2) For each spin 3/2, DECUPLET 4×4 block of $\tilde{\Gamma}(p)$, we obtain some zeroes and relations between the elements. Labelling $J_z = 3/2, 1/2, -1/2, -3/2$ by the matrix indices $1, 2, 3, 4$, respectively, this yields to a **Hermitian** matrix (the bar here denotes complex conjugation)

$$\mathcal{M} = \begin{pmatrix} a & 0 & c & d \\ 0 & a & \bar{d} & -\bar{c} \\ \bar{c} & d & b & 0 \\ \bar{d} & -c & 0 & b \end{pmatrix} \quad , \quad a, b \in \mathbb{R} \, ,$$

i.e. with 2×2 block diagonal elements which are multiple of the identity I_2, and a normal 2×2 block matrix in the off-diagonal entries.

Hence,

$$\det\mathcal{M} = (\mu_+\mu_-)^2\,,$$

where $\mu_\pm$ are the MULTIPLICITY TWO Eigenvalues

$$\mu_\pm = \frac{1}{2}\left\{(b+a) \pm \sqrt{(b-a)^2 + 4(|c|^2 + |d|^2)}\right\}\,.$$

The mutually orthogonal $\mathcal{M}$ eigenvectors for μ_+ are $(a-\mu_-, 0, \bar{c}, \bar{d})$ and $(0, a-\mu_-, d, -c)$; for μ_- we have $(c, \bar{d}, b-\mu_+, 0)$ and $(d, -\bar{c}, 0, b-\mu_+)$.

The preceding normality property holds because:

$$N = \begin{pmatrix} c & d \\ \bar{d} & -\bar{c} \end{pmatrix} = (\mathcal{R}\mathrm{e}\, c)\,\sigma_3 + (\mathcal{R}\mathrm{e}\, d)\,\sigma_1 + i(\mathcal{I}\mathrm{m}\, c)\,I_2 - (\mathcal{I}\mathrm{m}\, d)\,\sigma_2\,,$$

where the σ's are Pauli matrices, and verifies $N^\dagger = N - 2(\mathcal{I}\mathrm{m} c)\,I_2$. Furthermore, $N^\dagger N = (|c|^2 + |d|^2)\,I_2$.

To proceed, we need to control $\tilde{\Gamma}(p)$. For this, we need a refined control of the short-distance behaviors of $G(x)$ and $\Gamma(x)$. NEGATIVE POWERS OF κ INTERVENE in FOURIER Transform!

Theorem 5 *The short-distance behaviors appearing below hold for G and Γ. We start with the DECUPLET two-point functions. In this case, we have*

$$G_{r_1r_2}(x)=\begin{cases}
[-1+c_8\kappa^8+\mathcal{O}(\kappa^9)]\delta_{r_1r_2} & , \ x=0\,; \\
(-\kappa^3+c_9\kappa^9)\delta_{r_1r_2}+\mathcal{O}(\kappa^{11}) & , \ x=\epsilon e^0; \\
-\frac{1}{8}\kappa^3\delta_{r_1r_2}+\mathcal{O}(\kappa^9) & , \ x=\epsilon e^j; \\
\left[(-\kappa^6+c_{12}\kappa^{12})\delta_{r_1r_2}+\mathcal{O}(\kappa^{13})\right]\delta_{\mu\,0}+\left[-\frac{1}{8}\kappa^6\delta_{r_1r_2}+\mathcal{O}(\kappa^{10})\right]\delta_{\mu\,j} & , \ x=2\epsilon e^\mu\,; \\
(\frac{1}{16}\delta_{r_1\frac{3}{2}}-\frac{1}{16}\delta_{r_1\frac{1}{2}})\kappa^6\delta_{r_1r_2}+\mathcal{O}(\kappa^{10}) & , \ x=\epsilon e^1+\epsilon' e^2; \\
(-\frac{1}{32}\delta_{r_1\frac{3}{2}}+\frac{1}{32}\delta_{r_1\frac{1}{2}})\kappa^6\delta_{r_1r_2}+\mathcal{O}(\kappa^{10}) & , \ x=\epsilon e^1+\epsilon' e^3, \epsilon e^2+\epsilon' e^3; \\
-\frac{1}{4}\kappa^6\delta_{r_1r_2}+\mathcal{O}(\kappa^{10}) & , \ x=\epsilon e^0+\epsilon' e^j; \\
\frac{17}{64}\kappa^9\delta_{r_1r_2}+\mathcal{O}(\kappa^{10}) & , \ x=\epsilon e^0+2\epsilon' e^j; \\
-\frac{3}{8}\kappa^9\delta_{r_1r_2}+\mathcal{O}(\kappa^{13}) & , \ x=2\epsilon e^0+\epsilon' e^j; \\
(\frac{3}{32}\delta_{s\frac{3}{2}}-\frac{5}{32}\delta_{s\frac{1}{2}})\kappa^9\delta_{r_1r_2}+\mathcal{O}(\kappa^{10}) & , \ x=\epsilon e^0+\epsilon' e^1+\epsilon'' e^2; \\
(-\frac{3}{32}\delta_{s\frac{3}{2}}+\frac{1}{32}\delta_{s\frac{1}{2}})\kappa^9\delta_{r_1r_2}+\mathcal{O}(\kappa^{10}) & , \ x=\epsilon e^0+\epsilon' e^i+\epsilon'' e^3, i=1,2; \\
(-\kappa^9+c_{15}\kappa^{15})\delta_{r_1r_2}+\mathcal{O}(\kappa^{16}) & , \ x=3\epsilon e^0; \\
d_{12}\kappa^{12}\delta_{r_1r_2}+\mathcal{O}(\kappa^{13}) & , \ x=3\epsilon e^0+\epsilon' e^j; \\
(-\kappa^{12}+c_{18}\kappa^{18})\delta_{r_1r_2}+\mathcal{O}(\kappa^{19}) & , \ x=4\epsilon e^0; \\
d_{15}\kappa^{15}\delta_{r_1r_2}+\mathcal{O}(\kappa^{16}) & , \ x=4\epsilon e^0+\epsilon' e^j;
\end{cases}$$

$$
\Gamma_{r_1 r_2}(x) = \begin{cases}
[-1 - \frac{67}{32}\kappa^6 + \mathcal{O}(\kappa^8)]\delta_{r_1 r_2} & , \quad x = 0; \\
(\kappa^3 + c_9'\kappa^9)\delta_{r_1 r_2} + \mathcal{O}(\kappa^{10}) & , \quad x = \epsilon e^0; \\
\frac{1}{8}\kappa^3\delta_{r_1 r_2} + \mathcal{O}(\kappa^9) & , \quad x = \epsilon e^j; \\
\left[c_{12}'\kappa^{12}\delta_{r_1 r_2} + \mathcal{O}(\kappa^{13})\right]\delta_{\mu\,0} + \left[\frac{7}{64}\kappa^6\delta_{r_1 r_2} + \mathcal{O}(\kappa^{10})\right]\delta_{\mu\,j} & , \quad x = 2\epsilon e^\mu; \\
(-\frac{3}{32}\delta_{r_1 \frac{3}{2}} + \frac{1}{32}\delta_{r_1 \frac{1}{2}})\kappa^6\delta_{r_1 r_2} + \mathcal{O}(\kappa^{10}) & , \quad x = \epsilon e^1 + \epsilon' e^2; \\
(0\,\delta_{r_1 \frac{3}{2}} - \frac{1}{16}\delta_{r_1 \frac{1}{2}})\kappa^6\delta_{r_1 r_2} + \mathcal{O}(\kappa^{10}) & , \quad x = \epsilon e^1 + \epsilon' e^3, \epsilon e^2 + \epsilon' e^3; \\
\mathcal{O}(\kappa^{10}) & , \quad x = \epsilon e^0 + \epsilon' e^j; \\
\mathcal{O}(\kappa^{10}) & , \quad x = \epsilon e^0 + 2\epsilon' e^j; \\
\mathcal{O}(\kappa^{13}) & , \quad x = 2\epsilon e^0 + \epsilon' e^j; \\
\mathcal{O}(\kappa^{10}) & , \quad x = \epsilon e^0 + \epsilon' e^i + \epsilon'' e^{j>i}; \\
c_{15}'\kappa^{15}\delta_{r_1 r_2} + \mathcal{O}(\kappa^{16}) & , \quad x = 3\epsilon e^0; \\
c_{18}'\kappa^{18}\delta_{r_1 r_2} + \mathcal{O}(\kappa^{19}) & , \quad x = 4\epsilon e^0;
\end{cases}
$$

NOTE THE COMPUTATIONS UP TO κ^{18} !!!

For the OCTET two-point function, the short distance behaviors for $x = r\epsilon e^\mu$, $r = 1, 2, 3, 4$, $\epsilon e^0 + \epsilon' e^j$, $\epsilon e^0 + 2\epsilon' e^j$, $2\epsilon e^0 + \epsilon' e^j$, $3\epsilon e^0 + \epsilon' e^j$, $4\epsilon e^0 + \epsilon' e^j$ are the same as for the decuplet, up to the considered order in each case. However, for $x = \epsilon e^i + \epsilon' e^j$, $x = \epsilon e^0 + \epsilon' e^i + \epsilon'' e^j$, for $ij = 12, 13, 23$, the results are different and given respectively by

$$G_{r_1 r_2} = -\frac{1}{16}\kappa^6 \delta_{r_1 r_2} + \mathcal{O}(\kappa^7) \quad , \quad \Gamma_{r_1 r_2} = \frac{1}{32}\kappa^6 \delta_{r_1 r_2} + \mathcal{O}(\kappa^7)\,,$$

and

$$G_{r_1 r_2} = -\frac{5}{32}\kappa^9 \delta_{r_1 r_2} + \mathcal{O}(\kappa^{10}) \quad , \quad \Gamma_{r_1 r_2} = \mathcal{O}(\kappa^{10})\,.$$

In the above, the c's, c''s, d's and d''s are computable κ and spin independent constants.

For $p^0 = i\chi = iw(\kappa, \vec{p})$ and $\vec{p} = 0$, it turns out that $\tilde{G}_{ss'}(p^0 = i\chi, \vec{p} = 0)$ and $\tilde{\Gamma}_{ss'}(p^0 = i\chi, \vec{p} = 0)$ **are diagonal**. The whole baryon mass spectrum is easily determined exactly using the **Analytic Implicit Function** Theorem.

Theorem 6 *Concerning the One-Baryon Mass Spectrum, we have that all the 56 Baryons (Antibaryons) have the mass satisfying*

$$M \equiv M(\kappa) = -3\ln\kappa - 3\kappa^3/4 + \kappa^6 r(\kappa)\,,$$

where $r(\kappa)$ is analytic and $r(0) \neq 0$. $r(\kappa)$ is the same for all members of the octets, and we let $r_o(0)$ denote $r(0)$ for this case. For the decuplets, $r(\kappa)$ only depends on $|J_z|$. For all members of the decuplets, $r(0) = r_d(0)$. There is a Mass Splitting between the Octets and the Decuplets given by $[r_d(0) - r_o(0)]\kappa^6 = 3\kappa^6/4$. If there is eventually mass splitting within the decuplets, it is of order κ^7 or higher.

For $\vec{p} \neq \vec{0}$, to solve $\det \tilde{\Gamma}(p^0 = iw(\kappa, \vec{p}), \vec{p}) = 0$ we pass to the variable

$$w(\kappa, \vec{p}) = -3 \ln \kappa + r(\kappa, \vec{p}) \, .$$

With the new variable r, we avoid the $(-3 \ln \kappa)$ singularity. Besides, using this new variable, we get solutions close to zero as $\kappa \searrow 0$, and can exhibit jointly analyticity in κ, $\vec{p}$, depending on the considered case.

For $\vec{p} \neq \vec{0}$ and considering the **OCTET**, because of the **DIAGONAL STRUCTURE** of $\tilde{\Gamma}(p)$, we can again apply the **Analytic Implicit Function** Theorem to obtain:

Theorem 7 *The sixteen OCTET BARYONS (Antibaryons) dispersion curves are all equal and have the form*

$$w(\vec{p}) \equiv w(\kappa, \vec{p}) = \left[-3 \ln \kappa - 3\kappa^3/4 + p_\ell^2 \kappa^3/8 \right] + r(\kappa, \vec{p}) \, ,$$

where $p_\ell^2 \equiv 2 \sum_{i=1}^3 (1 - \cos p^i)$, *and* $p^{i=1,2,3} \in [-\pi, \pi)$ *are the spatial momentum components. Furthermore,* $r(\kappa, \vec{p}) = \kappa^6 r_0(\kappa, \vec{p})$, *where* $r_0(\kappa, \vec{p})$ *is jointly analytic in* κ *and in each* p^i, $i = 1, 2, 3$, *for* $|\mathcal{I}\mathrm{m}\, p^j|$ *small. Also, the OCTET dispersion curves* $w(\vec{p})$ *are convex for small* $|\vec{p}|$*.*

REMARK: As seen above, by direct calculation, the OCTET dispersion curves are real. That this is true, in ANY CASE, is guaranteed by the Spectral Thm! (Recall: requires κ real.)

For $\vec{p} \neq \vec{0}$, and considering the **DECUPLETS** the situation is more complex. The fact we are dealing with a **DEGENERATE CASE** (multiplicity TWO) and the PRESENCE of the SQUARE ROOTS Does Not Allow us to apply the Analytic Implicit Function Thm.

For a **DECUPLET** block, using the normality condition given above we have that the **meromorphic extension** $\tilde{G}(\kappa,p) \equiv [\tilde{\Gamma}(\kappa,p)]^{-1}$ is given by (as if we had a matrix of numbers!)

$$\tilde{G}(\kappa,p) = \frac{1}{\Phi} \begin{pmatrix} \tilde{\Gamma}_{33}(\kappa,p)\, I_2 & \begin{pmatrix} -\tilde{\Gamma}_{13}(\kappa,p) & -\tilde{\Gamma}_{14}(\kappa,p) \\ -\overline{\tilde{\Gamma}_{14}(\kappa,p)} & \overline{\tilde{\Gamma}_{13}(\kappa,p)} \end{pmatrix} \\ \begin{pmatrix} -\overline{\tilde{\Gamma}_{13}(\kappa,p)} & -\tilde{\Gamma}_{14}(\kappa,p) \\ -\overline{\tilde{\Gamma}_{14}(\kappa,p)} & \tilde{\Gamma}_{13}(\kappa,p) \end{pmatrix} & \tilde{\Gamma}_{11}(\kappa,p)\, I_2 \end{pmatrix},$$

Here,

$$\Phi^2 \equiv \det \tilde{\Gamma}(\kappa, p)\,,$$

and the **Two Multiplicity-Two Solutions** $r_\pm \equiv r_\pm(\kappa, \vec{p})$ satisfy the implicit equation

$$\Phi(\kappa, r, \vec{p}) \equiv \tilde{\Gamma}_{11}(\kappa, r, \vec{p})\,\tilde{\Gamma}_{33}(\kappa, r, \vec{p}) - \left|\tilde{\Gamma}_{31}(\kappa, r, \vec{p})\right|^2 - \left|\tilde{\Gamma}_{41}(\kappa, r, \vec{p})\right|^2 = 0\,, \quad (6)$$

Writing,

$$\Phi^2 = \left[\lambda_+ \lambda_-\right]^2\,, \quad (7)$$

the solutions are determined by the **zeroes** of λ_+ and λ_-, where

$$\lambda_\pm \equiv \frac{1}{2}\left[\tilde{\Gamma}_{11} + \tilde{\Gamma}_{33}\right] \pm \sqrt{\frac{1}{4}\left[\tilde{\Gamma}_{11} - \tilde{\Gamma}_{33}\right]^2 + \left|\tilde{\Gamma}_{13}\right|^2 + \left|\tilde{\Gamma}_{14}\right|^2}\,. \quad (8)$$

All the matrix elements above are analytic functions and originally depend on κ, p^0 and $\vec{p}$.

ABOVE: we introduced $p^0 = i(-3\ln\kappa + r)$. We also consider an extension of all the functions, depending originally on the real variables κ, r, $\vec{p}$, to the complex plane.

PRESCRIPTION: As proved before, the $\tilde{\Gamma}_{ss'}$ are complex joint analytic functions in the real variables κ, p^0 and the components of $\vec{p}$. Hence, they admit a power series expansion in a real domain in real κ, p^0 and $\vec{p}$. The coefficients may be complex, because of Fourier Transform. These power series are used to write a power series for the above quantities with real κ, p^0 and $\vec{p}$. These power series expansions provide an analytic extension for κ, r and $\vec{p}$ complex. Whenever a Square Root shows up, we assume we **take the positive** one!

With this, we have the Lemma:

Lemma 2 *Let* $\Sigma(\vec{p}) = \sum_{i=1,2,3} \cos p^i$. *We have the following behaviors for the matrix elements* $\tilde{\Gamma}_{ij}(\kappa, r, \vec{p})$

$$\tilde{\Gamma}_{jj}(\kappa, r, \vec{p}) = (e^r - 1) + \tfrac{1}{4}\Sigma(\vec{p})\kappa^3 + \kappa^6 \left[\ell(r) + H_{jj}(\vec{p})\right] + A_{jj}(\kappa, r, \vec{p})\ ,\ j = 1, 3$$
$$\tilde{\Gamma}_{31,41}(\kappa, r, \vec{p}) = \kappa^7 A_{31,41}(\kappa, r, \vec{p})\,,$$

giving

$$\begin{aligned} \Phi(\kappa, r, \vec{p}) \ &= \ (e^r - 1)^2 + \tfrac{1}{2}\Sigma(\vec{p})(e^r - 1)\kappa^3 + (e^r - 1)\left[2\ell(r) + H_{11}(\vec{p})\right. \\ &\quad \left. + H_{33}(\vec{p})\right]\kappa^6 + \tfrac{1}{16}\Sigma(\vec{p})^2\kappa^6 + \kappa^7 A(\kappa, r, \vec{p})\,, \end{aligned}$$

where the $A_{ij}(\kappa, r, \vec{p})$ *are jointly analytic in* (κ, r), *uniformly in* $\vec{p}$, *and the diagonal elements* $A_{jj}(\kappa, r, \vec{p})$ *are of order* $\kappa^7 r^{k\geq 0}$, $A(\kappa, r, \vec{p})$ *is analytic in* $(\kappa, r, \vec{p})$, *with* $\vec{p} \in \mathbb{T}^3$. *Furthermore,*

$$\begin{aligned} H_{11}(\vec{p}) \ &= \ \tfrac{1}{32}\left[7(\cos p_1 + \cos p_2 + \cos p_3) - 12 \cos p_1 \cos p_2\right]\ , \\ H_{33}(\vec{p}) \ &= \ \tfrac{1}{32}[7(\cos p_1 + \cos p_2 + \cos p_3) + 4 \cos p_1 \cos p_2 \\ &\quad -8(\cos p_1 \cos p_3 + \cos p_2 \cos p_3)]\,, \\ \ell(r) \ &= \ -\tfrac{67}{32} + c'_9 e^r + c'_{12} e^{2r} + c'_{15} e^{3r} + c'_{18} e^{4r}\,, \end{aligned}$$

where the c' coefficients are computable (the same in the short-distance behavior thm). Last, for any fixed $\vec{p}$, we obtain

$$\Phi(0,0,\vec{p}) = 0 \quad , \quad \frac{\partial \Phi}{\partial r}(0,0,\vec{p}) = 0 \quad , \quad \frac{\partial^2 \Phi}{\partial r^2}(0,0,\vec{p}) = 2\,.$$

For real $\kappa, r, \vec{p}$ and $\kappa > 0$, $\tilde{\Gamma}_{11}(\kappa,\vec{p})$, $\tilde{\Gamma}_{33}(\kappa,\vec{p})$ and $\Phi(\kappa,\vec{p})$ are real.

REMARK: ϕ is Analytic But Has ZERO First-Derivative! We cannot apply the Analytic Implicit Function Thm to solve the implicit eqn $\phi(\kappa, r, \vec{p}) = 0$. (A DEGENERACY is PRESENT, Multiplicity Two Eigenvalues!)

An intuitive picture for the solutions is obtained by using the leading behavior of $\Phi(\kappa, r, \vec{p})$

$$\Phi(\kappa, r, \vec{p}) \approx (r + \frac{1}{4}\Sigma(\vec{p})\kappa^3)^2 = 0\,,$$

with the two degenerate explicit solutions $r(\kappa,\vec{p}) = -\frac{1}{4}\Sigma(\vec{p})\kappa^3$.

The existence of the exact solutions for the DECUPLET DISPERSION CURVES emerge from the application of the **Weierstrass Preparation Thm**.

The WPT reduces the implicit equation to the zeroes of a quadratic polynomial in r, namely

$$r^2 + b(\kappa, \vec{p})r + c(\kappa, \vec{p}) = 0\,,$$

where $b(\kappa, \vec{p})$ and $c(\kappa, \vec{p})$ are jointly analytic in $\kappa, \vec{p}$ and are given recursively in terms of κ, r partial derivatives of $\tilde{\Gamma}_{ij}$. Thus, we obtain the explicit solutions

$$r_\pm(\kappa, \vec{p}) = \frac{1}{2}\left[-b(\kappa, \vec{p}) \pm \sqrt{\Delta(\kappa, \vec{p})}\right]\,, \tag{9}$$

where $\Delta(\kappa, \vec{p}) = b(\kappa, \vec{p})^2 - 4c(\kappa, \vec{p})$ is the discriminant.

The WPT Only Ensures CONTINUITY of the solutions $r_\pm$ (Square Root!). They can then be analyzed for Additional Smoothness Properties.

Theorem 8 *Consider the analytic function $\Phi(\kappa, r, \vec{p})$. There exist a function $Q(\kappa, r, \vec{p})$ and a Weierstrass polynomial $W(\kappa, r, \vec{p})$, both jointly analytic in κ, r in a neighborhood $\mathcal{N}$ of $(\kappa, r) = (0, 0) \in \mathbb{C}^2$, independent of $\vec{p}$ (and in each component of $\vec{p}$), such that we have $Q(\kappa, r, \vec{p}) \neq 0$ in $\mathcal{N}$ and also*

$$Q(\kappa, r, \vec{p})\Phi(\kappa, r, \vec{p}) = W(\kappa, r, \vec{p})\,,$$

in $\mathcal{N}$. Furthermore the polynomial is quadratic in r and is given by

$$W(\kappa, r, \vec{p}) = r^2 + b(\kappa, \vec{p})r + c(\kappa, \vec{p}) \,,$$

where the coefficients are given by

$$\begin{aligned}
b(\kappa, \vec{p}) \quad &= \tfrac{1}{2}\Sigma(\vec{p})\kappa^3 + [2c'_9 + 2c'_{12} + 2c'_{15} + 2c'_{18} - \tfrac{67}{16} + H_{11}(\vec{p}) + H_{33}(\vec{p}) \\
&\quad + \tfrac{1}{16}\,(\Sigma(\vec{p}))^2]\kappa^6 + \mathcal{O}(\kappa^7)\,, \\
c(\kappa, \vec{p}) \quad &= \tfrac{1}{16}\left(\textstyle\sum_{j=1}^{3} \cos p_j\right)^2 \kappa^6 + \mathcal{O}(\kappa^7)\,, \\
Q(\kappa, r, \vec{p}) &= 1 - r + \tfrac{5}{12}r^2 - \tfrac{1}{12}r^3 + \tfrac{1}{4}\Sigma(\kappa, \vec{p})\kappa^3 + \mathcal{O}(\kappa^\alpha\, r^j)\,,
\end{aligned}$$

with $\alpha + j \geq 4$; $\alpha \geq 6$, for $j = 0$. Moreover, $b(\kappa, \vec{p})$, $c(\kappa, \vec{p})$ and $Q(\kappa, r, \vec{p})$ are bounded functions for $(\kappa, r) \in \mathcal{N}$, $\vec{p} \in \mathbb{T}^3$.

OBSERVATION: We have a bunch of results on Smoothness Properties for the DECUPLET Dispersion Curves.

ALSO: Need a SLIGHTLY SOPHISTICATED SUBTRACTION PROCEDURE To EXTEND RESULTS From The BARYONIC SUBSPACE $\mathcal{H}_b$ To The ODD SUBSPACE $\mathcal{H}_o$.

The Gell'Mann-Ne'eman Baryonic Eightfold Way is then VALIDATED RIGOROUSLY!

The 36 Eightfoldway MESON States at $\vec{p} = 0$ MASSES are Determined (Splittings, ...). MASSES are $\simeq -2\ln\kappa$!

The COMPLETE ANALYSIS: For the **NONET of PSEUDO-SCALAR** Mesons is OK, for all $\vec{p}$.

Weierstrass Preparation Theorem: Should Do the Job for the 3 NONETS of VECTOR Mesons, with $\vec{p} \neq 0$.

With this: the Whole EIGHTFOLD WAY PICTURE is VALIDATED. CONFINEMENT of STATES (Without Gluons!) is CHECKED up to near the MESON-BARYON Energy Threshold.

TWO-BARYON SPECTRUM: BARYON-BARYON BOUND STATES

Like the TWO-POINT CORRELATION G Above: HERE We Use a PARTIALLY TRUNCATED FOUR-POINT Correlation D.

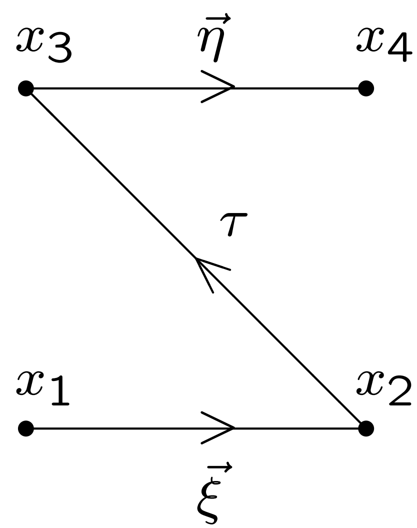
LATTICE RELATIVE COORDINATES

(FOURIER Transform ONLY in τ!)

$$\vec{\eta} = x_4 - x_3$$

$$\tau = x_3 - x_2$$

$$\vec{\xi} = x_2 - x_1$$



In Fourier Transform: Establish SPECTRAL REPRESENTATION for $\tilde{D}$

SINGULARITIES of $\tilde{D}$ Determine the Two-Particle Spectrum.

SINGULARITY DETECTION: Bethe-Salpeter (B-S) Equation

$$D = D_0 + D_0 \, K \, D \, ,$$

where D_0 is Known and Corresponds to the GAUSSIAN RESTRICTION of D (Wick Thm, Product of TWO G's).

B-S EQN Defines the Bethe-Salpeter Operator K.

K: Formally Satisfies

$$K = D_0^{-1} - D^{-1}\,,$$

and PLAYS a ROLE Analogous to the Above Γ.

<u>By Hyperplane Method:</u> Distance Decay for D. The Decay for D_0 follows from the decay of G.

K Decays FASTER Than D: Analyticity in a WIDER STRIP in Fourier Space.

The ANALYSIS Goes On in TWO-STEPS:

1. LADDER APPROXIMATION: OBTAIN K in the Lower, Leading κ Order, denoted by L

USING L instead of K in the B-S Eqn, obtain

$$D = (1 - L D_0)^{-1}\, D_0\,,$$

Using Regularity of D_0 ABOVE the ONE-PARTICLE DISPERSION CURVE (meromorphic extension for G), FIND SINGULARITIES of $\tilde{D}$.

2. Complete Model: Deep KNOWLEDGE of ONE-PARTICLE SPECTRUM (Spectral Measure!) and K FAST Decay CONTROLS Perturbation and ALLOW TO GO BEYOND the LADDER APPROXIMATION.

USEFUL: B-S Eqn is SIMILAR to a RESOLVENT SCHRÖDINGER EQN

$$(H - z)^{-1} = (H_0 - z)^{-1} - \lambda (H_0 - z)^{-1} V (H - z)^{-1},$$

where $H = H_0 + \lambda V$ and $H_0 = -a\Delta/2$, $a > 0$ (Δ is the Laplacian).

SUMMARY OF MAIN RESULTS:

VARIOUS BARYON-BARYON BOUND STATES (INCLUDING A SPIN 1, DEUTERON-LIKE BS. NO SPIN 0 'Deuteron' as Expected)

DIPROTON and DINEUTRON SHOW UP!

EXPONENTIALLY DECAYING YUKAWA INTERACTION: Easily Obtained! (Polynomial Correction is HARD!)

NO MESON-BARYON PENTAQUARKS (Only 3 Flavors. No Contradiction up to now!)

MESON-MESON BS: Various in $2+1$ Dimensions

BINDING STRUCTURES: STILL TOO COMPLICATED To Be DECIPHERED!

REFERENCES: MANY Publications GIVEN in My SITE

www.icmc.sc.usp.br/∼veiga/pdvpubl.html

Criticism to Usual THEORETICAL PHYSICS & SIMULATION LATTICE WORKS: Absence of Spectral Representations DOES NOT ENSURE That People Determine Spectrum. May Determine SOMETHING ELSE. May Fall in BETWEEN TWO STATES WITH TINY SPLITTING. MAY BE IN THE MIDDLE OF A BAND.

TO MY KNOWLEDGE: NO DISPERSION CURVES! ONLY MASSES.

LACK of **ISOLATED** STATES DOES NOT GUARANTEE PARTICLES!

TO MY KNOWLEDGE: BS ONLY WITH EFFECTIVE FIELDS Like in Yukawa Theory, instead of FUNDAMENTAL COMPOSED FIELDS.

WE WORK IN STRONG COUPLING: WE SEE NO REASON FOR IT TO BE RULED OUT FOR THE ENERGY SCALE WE ARE INTERESTED IN.

STRONG COUPLING IS ENOUGH TO PROVE QUARK CONFINEMENT UP TO AN ENERGY THRESHOLD.

OUR ANALYSIS IS MEANINGFUL.

WE HOPE SOMEONE CAN DO IT BETTER!