FUNDAMENTAL PROBLEMS IN QFT

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AS EVERYONE HERE KNOWS...

Historical Level: QFT has its origin by $\approx$ 1930 incorporating both Quantum Mechanics + Special Relativity.

Employed in the Formulation of Physics Models of Particles and Their Interactions (Electrons, Protons, Photons, Pions, Neutrinos, ...)

QUITE SUCCESSFUL! In QED: PERTURBATION Leads to Precision of One Part in 10^8: Lambshift, ... **Theoretical Precision is WORSE THAN the EXPERIMENTAL ONE!**

FROM a MATHEMATICAL VIEWPOINT: **DIFFICULT CHALLENGE!** (UNCOUNTABLE) INFINITE NUMBER OF DEGREES OF FREEDOM. **HARD MATHEMATICAL ANALYSIS!**

MY GOAL: In the MATHPHYS, Constructive QFT Context
REVIEW SOME of the **MAIN STRUCTURAL PROBLEMS** FOUND
If We TRY TO GO **BEYOND the Beginning of PERTURBATION!**

CLASSIFICATION of the ANALYTICAL PROBLEMS

TWO TYPES:
CONSTRUCTION OF MEASURES and DETERMINATION OF THE PARTICLE CONTENTS of MODELS

Determination of the PARTICLES and BOUND STATES: Subject of My Second Talk (Seminar 5, Later), in the Special Case of **LQCD**

TODAY: CONCENTRATE on the First Problem

TWO EXAMPLE CASES: the **SCALAR** $\lambda\,\phi^4_d$ **and the GROSS-NEVEU MODELS** in d SPACETIME EUCLIDEAN DIMENSIONS.

In EUCLIDEAN SPACETIME: the SCALAR $\lambda\,\phi^4_d$ Model is a CLASSICAL STATISTICAL MECHANICS Model of CONTINUOUS SPINS.

Osterwalder-Schrader: O-S Axioms (Euclidean Invariance, **Reflection Positivity**, Ergodicity, Analyticity, Regularity) WAY to GO **BACK to MINKOWSKI** with PHYSICAL WIGHTMAN-HAAG-RUELLE AXIOMS (Lorentz Covariance, Observables,...). See Glimm-Jaffe's book.

I will try to give a relatively complete scenario, but avoiding technicalities as much as I can ...

ORGANIZATION

1. Introduction: Quick REVIEW of QUANTUM MECHANICS, SEMI-GROUPS & Feynman FUNCTIONAL INTEGRALS.

2. Definition of a QFT (Constructive Context)

3. MAIN PROBLEMS Related to the Construction of Physical Measures: Existence of a Functional Integral

1) QUANTUM Mechanics (in unities where $\hbar = 1$)

$t \in \mathbb{R}$ denotes TIME
$\vec{r} \equiv (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ denotes POSITION. HERE, $d = 3$ is the *SPATIAL Dimension*

PHYSICS: Described by Complex Vectors $\Psi \equiv \Psi(t, \vec{r})$ in a HILBERT Space $\mathcal{H}$ (Complete Normed Space with Inner Product $(\,.\,,\,.\,)_{\mathcal{H}}$)

PHYSICAL OBSERVABLES: Associated with SELF-ADJOINT (Hermitian) OPERATORS in $\mathcal{H}$.

EXPECTED VALUE of Observable $\mathcal{O}$ on *STATE* Ψ is

$$E_\Psi(\mathcal{O}) = \frac{(\Psi, \mathcal{O}\Psi)_{\mathcal{H}}}{(\Psi, \Psi)_{\mathcal{H}}} \quad , \quad \left(\mathcal{O}^*\Psi\,,\,\Psi'\right)_{\mathcal{H}} = \left(\Psi\,,\,\mathcal{O}\Psi'\right)_{\mathcal{H}}$$

SPECTRUM (Eigenvalues,...): $\operatorname{spec}\mathcal{O} \subset \mathbb{R}$, MEASURABLE in EXPERIMENTS !

OBS: IF $[\mathcal{O}_1, \mathcal{O}_2] \equiv \mathcal{O}_1\mathcal{O}_2 - \mathcal{O}_2\mathcal{O}_1 = 0$ there exists a basis B of $\mathcal{H}$ in which $\mathcal{O}_1$ e $\mathcal{O}_2$ are Simultaneously Diagonal.

Particle Moving UNDER the ACTION of a POTENTIAL V $(\hbar = 1)$:

ENERGY OPERATOR: $H = -\frac{1}{2m}\Delta + V(\vec{r})$, $m > 0$ is mass

$-\Delta$ is minus the Laplacian: $-\frac{\partial^2}{\partial x_1^2} - \ldots - \frac{\partial^2}{\partial x_d^2} \geq 0$

$H_0 \equiv -\Delta/2m$ describes the KINETIC ENERGY $p^2/2m$

$V(\vec{r})$ Multiplication Operator by the Physical Potential

TIME EVOLUTION (Schrödinger's Eqn): DETERMINISTIC!

$$i\,\frac{\partial}{\partial t}\Psi = H\Psi$$

PROBABILISTIC: is the COPENHAGEN Interpretation

Basic Physical Choice: Square Integrable Function Space $\mathcal{H} = \mathcal{L}^2(d^d\vec{x}, \mathbb{R}^d)$

Heisenberg Uncertainty: Cauchy-Schwarz Inequality in $\mathcal{H}$

Ψ measures **Probability Amplitude** to find the particle in $(t, \vec{r})$

$\|\Psi\|^2 \equiv \langle \Psi \,,\, \Psi \rangle \equiv \int \bar{\Psi}(t, \vec{r}) \Psi(t, \vec{r}) \; d^d\vec{x}$ gives the PROBABILITY to Find the Particle in $(t, \vec{r})$.

UNDERLYING UNITARY TIME EVOLUTION SEMI-GROUP:

$$\Psi(t, \vec{r}) \;=\; e^{-iH(t-t_0)}\, \Psi(t_0, \vec{r})\,, \qquad \text{(Norm Preserving)}$$

H is the Generator ! (modulo multiplicative constant)

OBS: By the SPECTRAL THM, **SPECTRUM of H Determines the Time Evolution** !

COMPLICATIONS: $\dim \mathcal{H}$ may be ∞; $V(\vec{r})$ may be SINGULAR!

Convenient for Analysis: IMAGINARY TIME (Euclidean) $it \rightarrow t$

$$\Psi(t,\vec{r}) = e^{-H(t-t_0)}\,\Psi(t_0,\vec{r})$$

$H = H_0 + V > 0$, for a **CONTRACTION**

$\|\Psi\|$ Does NOT Increase! (Guarantees PROBABILISTIC Interpretation)

For This: V Bounded from Below. Physics OK! In **TIME EVOLUTION SEMI-GROUP**, APPLY:

Lie-Trotter Product Formula: (May have $[A,B] \neq 0$)

$$e^{-(A+B)} = \lim_{n\to\infty} \left(e^{-A/n}\,e^{-B/n}\right)^n$$

HERE: $A = H_0$ e $B = V$, Both Bounded from Below in $\mathcal{H}$.

BESIDES: REQUIRE $H = H_0 + V$ to be **ESSENTIALLY SELF-ADJOINT** in H_0 & V DOMAIN INTERSECTION.

<u>ESSENTIALLY SELF-ADJOINT</u>: $\mathcal{O} = \mathcal{O}^{**}$

[$V \in \mathcal{L}^2(\mathbb{R}^d, d^dx) + \mathcal{L}^\infty(\mathbb{R}^d)$ such that H is self-adjoint in $D(H_0)$]

OBS: Under These CONDITIONS: Strongly Continuous SEMI-GROUP + CONVERGENCE of LIMIT $n \to \infty$ in Lie-Trotter (in Strong Topology).

FEYNMAN-KAC FORMULA

$\Phi \in \mathcal{L}^2(\mathbb{R}^d, d^d x)$, $2m = 1$

$$\left(e^{-H_0\, t}\,\Phi\right)(x) \;=\; \left(\frac{1}{4\pi t}\right)^{d/2} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}}\,\Phi(y)\, d^d y\,,$$

($\int_{\mathbb{R}^d} \equiv \lim_{R\to\infty} \int_{|x|\le R}$ with limit in $\mathcal{L}^2$ norm)

$$\left(e^{-(H_0+V)t}\,\Phi\right)(x) = \lim_{n\to\infty} \left(\frac{n}{4\pi t}\right)^{\frac{dn}{2}} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} e^{-t\, S_n(x, y_1, \ldots, y_n)}\,\Phi(y_n)\, d^d y_1 \ldots d^d y_n\,,$$

$$S_n(x \equiv y_0, y_1, \ldots, y_n) = \sum_{i=1}^{n} \frac{1}{n} \left[\frac{1}{4}\left(\frac{|y_i - y_{i-1}|}{t/n}\right)^2 - V(y_i)\right]$$

UNCERTAINTY Principle: The y´s are in the WHOLE SPACE. Given INITIAL/FINAL Point, **SUM Over ALL TRAJECTORIES With Corresponding Statistical Weight!**

FEYNMAN: Sum Over ALL HISTORIES!

For Ω_x being the SET of ALL PATHS ω with $\omega(0) = x$,

$$\left(e^{-(H_0+V)\,t}\,\Phi\right)(x) = \int_{\Omega_x} e^{-tS(\omega)}\,\Phi(\omega(t))\,d\omega\,.$$

OBS: the WIENER Measure of all these RANDOM PROCESSES can be CONSTRUCTED.

S **is the System ACTION !**

2) QFT: INCORPORATING **SPECIAL RELATIVITY** to QM

EUCLIDEAN QFT: EQUIVALENT to Classical Statistical Mechanics.

Similarly to QM: DEFINED by **FUNCTIONAL Integral**

HERE: INTEGRAL is OVER the Φ FIELD SPACE $\mathcal{S}'$

- $x \in \mathbb{R}^d$ denotes a d-dimensional SPACETIME Point.
- $\Phi(x)$ Field Variable defined at x
- $\ell \in \{0, 1, 2, 3, \ldots\}$

Normalized Correlations or Schwinger Functions with ℓ-points are *formally* given by:

$$S_\ell^{norm}(x_1, \ldots, x_\ell) = \frac{1}{Z} S_\ell(x_1, \ldots, x_\ell),$$

Partition Function: Normalization $Z \equiv S_0$

UNNORMALIZED CORRELATIONS: Are the Moments

$$S_\ell(x_1, \ldots, x_\ell) = \int \Phi(x_1) \cdots \Phi(x_\ell) \, d\nu(\Phi),$$

of the measure $d\nu(\Phi) = \exp\left\{\int [-\lambda V(\Phi(z))] \, d^d z\right\} d\mu_C(\Phi)$.

Gaussian Measure: $d\mu_C(\Phi)$ - Normalized, Zero Mean & Covariance C.

OBS: $S_\ell^{norm}(x_1, \ldots, x_\ell)$ is the $\ell \in \mathbb{N}$ order moment of the Gibbs Probability Measure $d\nu(\Phi)/Z$.

EXPLANATIONS

- $d\nu(\Phi)$ the formal product ' $e^{-A(\Phi)}d\Phi$ ', where the Action $A(\Phi)$ in the Exponential has a FREE Part (quadratic form)

$$\exp -\frac{1}{2}\left[\int \Phi(z_1)C^{-1}(z_1,z_2)\Phi(z_2)d^dz_1d^dz_2\right],$$

defined with the Covariance C, Times the Exponential of the Interaction Among the Fields Φ

$$\exp\left[\int [-\lambda V(\Phi(x))]\, d^dx\right],$$

where $\underline{\lambda \in \mathbb{R}}$ is the <u>COUPLING Constant</u> measuring the Intensity of the <u>Interaction Potential</u> $V(\Phi)$.

- Usually, $V(\Phi)$ is Local, combination of monomials in $\Phi(x)$, and eventually some derivatives of Φ.

- For unbounded Φ: $V(\Phi)$ must be physically stable, Bound From Below, such that $\exp\left[\int -\lambda V(\Phi(x))d^dx\right]$ DECAYS at INFINITY, for a fixed SIGN of λ.

Quantization is IMPLICIT in Wiener INTEGRATION. FIELDS Scan ALL the SPACE $\mathcal{S}'$.

Φ are random processes in $\mathcal{S}'$ and are Defined Over SPACETIME $\mathbb{R}^d$.

AGAIN, the Math Complexity: INFINITE, UNCOUNTABLE Number of Degrees of Freedom $\Phi(x)$, $x \in \mathbb{R}^d$.

Convenient: UNDERSTAND the INFINITE NUMBER of Degrees of Freedom LIMIT in TWO STEPS.

Example: $\mathbb{R}^d$ is APPROXIMATED by a Compact Domain Λ in a **LATTICE** $(a\mathbb{Z})^d$, of FINITE VOLUME $|\Lambda|$ and SPACING $a > 0$.

AFTERWARDS: REMOVE the CUTOFFS! Take, in Some Order, the **INFINITE-VOLUME Limit** (Thermodynamic Limit) $|\Lambda| \nearrow \infty$ and the **CONTINUUM Limit** $a \searrow 0$.

By FOURIER

CONTINUUM Limit is the Ultraviolet Limit (UV).

INFINITE VOLUME Limit is the INFRARED Limit (IR).

CUTOFF Parameters: IR (Λ) and UV (a), May be Implemented Both in CONFIGURATION SPACE Space x or in the Fourier Dual Space of **MOMENTA**.

Minlos-Bochner Thm: Gaussian Measure $d\mu_C(\Phi)$ can be Realized in Schwartz Space $\mathcal{S}'$ of TEMPERED DISTRIBUTIONS, ensuring the EXISTENCE of Fourier Transforms, for $C > 0$, almost everywhere.

OBSERVATION: Interest for Distributions is Already Manifested in the **FREE CASE** $\lambda = 0$.

See e.g CLASSICAL PDE in Relativistic Minkowski space for Klein-Gordon fields.

Notational ABUSE! *Misleading* **but SIMPLIFIED** $\Phi(x)$

UNDERSTOOD: Canonical Pairing, (*test function f smearing*)

$$\Phi(f) = \int \Phi(x)\, f(x)\, d^d x \quad ; \quad x \equiv (t, \vec{x}) \equiv (x^0, \vec{x}) = (x^0, x^1, \ldots, x^{d-1}),$$

$f \in \mathcal{S}$ in DUAL SPACE of SMOOTH Fcts with FAST DECAY at ∞.

In FINITE LATTICE $\Lambda \subset a\mathbb{Z}^d$: Fields are Sufficiently Smoothed. $\Phi(x)$ is a Realistic Notation. Gaussian Measure $d\mu_C(\Phi)$ is Given by **EXP Quadratic Part** (with Covariance C) **times Product Measure** of Lebesgue Measures $d\Phi(x)$.

MAY USE a Larger or Matrix Valued TENSORED SPACE $\mathcal{S}' \otimes W$ to Include Interesting Features as MANY FIELD COMPONENTS, SPIN, ISOSPIN, COLOR, etc.

CONCRETE EXAMPLE: $d \geq 2$ & *Four-Body* Potentials s.t. **Scalar Bosonic Model** $\lambda\phi^4$ and the **Purely FERMIONIC** Gross-Neveu Model.

$Modelo\ \phi^4$	$Modelo\ GN$
$\Phi = \phi$	$\Phi = \bar{\psi}_\alpha, \psi_\alpha\ ;\ \alpha \in \{spin\}$
$C \equiv C_b = \left(-\zeta_b \Delta + m^2\right)^{-1}$	$C \equiv C_f = \left[\left(i\zeta_f \not{\partial} + m\mathcal{I}\right)\right]^{-1}$
$-\lambda V(\Phi(z)) = -\lambda\,[\phi(z)]^4,\ \lambda > 0$	$-\lambda V(\Phi(z)) = \lambda\left[\bar{\psi}(z)\psi(z)\right]^2$

,

ϕ describes a **Bosonic** neutral particle of Spin 0. C_b (inverse of *Klein-Gordon*) Operator defined in Sobolev Space $\mathcal{H}_1 \sim \mathcal{L}_2\left((p^2 + m^2)\, d^d p, \tilde{\Lambda}\right)$,where $\tilde{\Lambda}$ is the Fourier dual space for Λ.

ψ_α, $\bar{\psi}_\alpha$ associated with **Fermionic** charged particles Spin $\frac{1}{2}$ (electron, positron, etc). C_f : inverse of *Dirac* Operator in $\mathcal{H}_{\frac{1}{2}} \sim \mathcal{L}_2\left((p^2 + m^2)^{\frac{1}{2}}\, d^d p, \tilde{\Lambda}\right)$.

$\bar{\psi}_\alpha(x)$, $\psi_\alpha(x)$, $\forall x \in \Lambda$, obey Pauli Exclusion, and are generators of a (anticommutative) Grassmann Algebra $\mathcal{A}_\Lambda$, $\alpha = 1, ..., s$ (s= Dimension of Spinorial Degrees of Freedom).

Defining: on the Lattice, $\forall x \in \Lambda$, associate a Complex Vector Space V_x,

$$V_x = V_x^1 \oplus V_x^2 \quad , \quad \dim(V_x^1, V_x^2) = s$$

For V_x^1 take the basis $\bar{\psi}_\alpha(x)$, $\alpha = 1, ..., s$, and for V_x^2 a basis $\psi_\alpha(x)$, $\alpha = 1, ..., s$.

In general, $V_x^i = (V_x^i)_{\text{spin}} \otimes (V_x^i)_{\text{int}}$, where $(V_x^i)_{\text{int}}$ is an INTERNAL Index Space e $N = \dim(V_x^i)_{\text{int}}$.

$\mathcal{A}_\Lambda$ is the Grassmann Algebra over $V_\Lambda = \oplus_{x\in\Lambda} V_x$.

Berezin (Fermionic) INTEGRALS: are used for the integral in $\mathcal{A}_\Lambda$. Explicitly, ($\bigwedge$ denotes the Exterior Product.)

$$\int \bigwedge_{\alpha,a} (\bar{\psi}_{\alpha,a}(x) \wedge \psi_{\alpha,a}(x)) \bigwedge_{\alpha,a} d\bar{\psi}_{\alpha,a}(x) d\psi_{\alpha a}(x) = 1$$

The integral

$$\int (\text{monomials in } \bar{\psi}_{\alpha,a}(x) \text{ and } \psi_{\alpha,a}(x) \text{ with degree less than the maximum}) \\ \times \quad \bigwedge_{\alpha,a} d\bar{\psi}_{\alpha,a}(x) d\psi_{\alpha,a}(x) = 0\,,$$

Integral is Extended for ALL $\mathcal{A}_\Lambda$ using Linearity.

$\bigwedge$ will be OMITTED in Notation.

SPECIAL ROLE Played by Gaussian Integrals in $d\nu(\Phi)$ since **we can compute them!**

$$\int \phi(x) d\mu_C(\phi) \;=\; 0\,,$$
$$\int \phi(x_1)\phi(x_2) d\mu_C(\phi) \;=\; \int \phi(x_2)\phi(x_1) d\mu_C(\phi) \;=\; C_b(x_1, x_2)\,,$$

$$\int \bar{\psi}_\alpha(x) d\mu_C(\bar{\psi},\psi) = \int \psi_\alpha(y) d\mu_C(\bar{\psi},\psi) = 0\,,$$
$$\int \bar{\psi}_\alpha(x_1)\bar{\psi}_\beta(x_2) d\mu_C(\bar{\psi},\psi) = -\int \bar{\psi}_\beta(x_2)\bar{\psi}_\alpha(x_1) d\mu_C(\bar{\psi},\psi) = 0\,,$$
$$\int \psi_\alpha(y_1)\psi_\beta(y_2) d\mu_C(\bar{\psi},\psi) = -\int \psi_\beta(y_2)\psi_\alpha(y_1) d\mu_C(\bar{\psi},\psi) = 0\,,$$
$$\int \bar{\psi}_\alpha(x)\psi_\beta(y) d\mu_C(\bar{\psi},\psi) = -\int \psi_\beta(y)\bar{\psi}_\alpha(x) d\mu_C(\bar{\psi},\psi) = \left[C_f\right]_{\alpha\beta}(x,y)\,.$$

for the one-component case.

For Higher Degree Monomials, we have Wick's Thm for even ℓ,

$$\int \phi(x_1)\ldots\phi(x_\ell) d\mu_C(\phi) = \sum_{\text{pairings } \{(x_i,x_j),\ldots\}} \prod_{\text{pairs}} C_b(\text{pair})\,,$$

where the (*pairing*) set exhaust the set of Points $\{x_1,\ldots,x_\ell\}$, each x_i is taken only ONCE and $C_b(\text{pair}) = C_b(x_i,x_j)$ if the pair (x_i,x_j) is taken.

There are $(\ell-1)!! = (\ell-1)\ (\ell-3)\ (\ell-5)\ldots 1$ possible pairings , since the first field has $(\ell-1)$ possible *contractions* (pairing), There are $(\ell-3)$ for the second field, etc.

The integral VANISHES if ℓ is ODD.

FERMIONIC CASE:

$$\int \bar\psi_{\alpha_1}(x_1)\ldots\bar\psi_{\alpha_\ell}(x_\ell)\ \psi_{\beta_1}(y_1)\ldots\psi_{\beta_\ell}(y_\ell)\ d\mu_C(\bar\psi,\psi) = \sum_{\text{pairings } \{(x_i,\alpha_i\ ;\ y_j,\beta_j),\ldots\}} \ \prod_{\text{pairs}} \text{sign}\ C_f(\text{pair}),$$

$sign$: determined by the Number of Anticommutations to Perform a Permutation such that $\psi_{\beta_2}(x_2)$ is Placed on the Right of the Contracting $\bar\psi_{\alpha_1}(x_1)$ Leading to a PROPAGATOR $[C_f]_{\alpha_1\beta_2}(x_1,x_2)$.

The Integral VANISHES if the NUMBER of $\bar\psi$ and ψ are DIFFERENT.

In **Cayley's Notation** (up to a global multiplicative factor (-1))

$$\int \bar{\psi}_{\alpha_1}(x_1)\ldots\bar{\psi}_{\alpha_\ell}(x_\ell)\psi_{\beta_1}(y_1)\ldots\psi_{\beta_\ell}(y_\ell)d\mu_C(\bar{\psi},\psi)$$

$$= \begin{pmatrix} x_1,\alpha_1 & x_2,\alpha_2 & \ldots & x_\ell,\alpha_\ell \\ y_1,\beta_1 & y_2,\beta_2 & \ldots & y_\ell,\beta_\ell \end{pmatrix}$$

$$\equiv det \begin{bmatrix} C_{\alpha_1,\beta_1}(x_1,y_1) & \ldots & C_{\alpha_1\beta_\ell}(x_1,y_\ell) \\ \vdots & \ddots & \vdots \\ C_{\alpha_\ell,\beta_1}(x_\ell,y_1) & \ldots & C_{\alpha_\ell\beta_\ell}(x_\ell,y_\ell) \end{bmatrix}$$

and we let $C \equiv C_f$.

IMPORTANT OBSERVATION: The ABOVE **det** is a GRAM Determinant

$\det(f_i, g_j)$, of a **Matrix with Elements Given by a SCALAR PRODUCT** in $\mathcal{L}_2(d^dx, \mathbb{R}^d)$.

MATH DIFFICULTY: Control the PERTUBATIONS ABOUT GAUSSIAN MEASURES

3) PROBLEMS: PRESENCE of SINGULARITIES

For Example, in the **Continuum Infinite Volume** BOSONIC Case:

$$(-\Delta + m^2) C_b(x,y) = \delta(x-y)$$

$$|\mathbf{x}-\mathbf{y}| \ll \mathbf{1}: \quad C_b(x,y) \approx \begin{cases} -\frac{1}{2\pi} \ln(m\,|x-y|) \quad , \quad d=2 \\ \frac{\Gamma(\frac{d-2}{2})}{4\pi^{d/2}} \, |x-y|^{-d+2} \quad , \quad d \geq 3 \end{cases}$$

$$|\mathbf{x}-\mathbf{y}| \gg \mathbf{1}: \quad C_b(x,y) \approx \sqrt{\pi/2}\,(2\pi)^{-d/2}\, m^{(d-3)/2}\, |x-y|^{-(d-1)/2}\, e^{-m|x-y|}\,.$$

IR (Large-Distance) SINGULARITIES: $m = 0$ (Critical Phenomena, Phase transitions). NOT INTEGRABLE when $|x-y| \nearrow \infty$ OR $|\Lambda| \nearrow \infty$.

UV (Short-Distance) SINGULARITIES: LOOSE INTEGRABILITY When $|x-y| \searrow 0$ OR $a \searrow 0$.

NATURAL to TRY: PERTURBATION Series for the CUTOFF Model

Taylor Expansion in λ for $e^{-\lambda \int V(\Phi)}$,

$$S_\ell(x_1, \ldots, x_\ell) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \int \Phi(x_1) \ldots \Phi(x_\ell)$$
$$\times \left[\int V(\Phi(z_1)) d^d z_1 \ldots \int V(\Phi(z_n)) d^d z_n \right] d\mu_C(\Phi) \, .$$

HERE THEY COME: Feynman Diagrams, Amplitudes & Integrals!

SIMILAR PROBLEM: We CAN COMPUTE the GAUSSIAN Integral

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{2\pi}{\alpha}} \qquad , \qquad \alpha > 0 \, ,$$

BUT NOT ($0 < \epsilon \ll 1$)

$$I(\alpha, \epsilon) \equiv \int_{-\infty}^{\infty} e^{-\alpha x^2} e^{-\epsilon x^4} dx \; ; \; \alpha \, , \epsilon > 0 \, .$$

Even Though we KNOW IT EXISTS and IS FINITE!

OBS: Note the ϵ Expansion DIVERGES, Otherwise There Would Be a Convergence Domain Around $\epsilon = 0$. But we know the Integral DIVERGES for $\epsilon < 0$!

HERE: The Series is BOREL SUMMABLE!

Meaning: 1) Perturbation Series With an EXTRA $(1/n!)$ FACTOR Converges, which comes from

$$\left| \frac{1}{n!} \int_{-\infty}^{\infty} e^{-\alpha x^2} \left[-\epsilon x^4 \right]^n dx \right| \leq \text{const}^n \, n!$$

where we used $ue^{-u} \leq e^{-1}$, $u > 0$, and Stirling's Formula in

$$x^{4n} e^{-\alpha x^2/2} = \left[\frac{\alpha \, x^2}{4n} e^{-\alpha \frac{x^2}{4n}} \right]^{2n} \left(\frac{4n}{\alpha} \right)^{2n} \leq \left(\frac{4n}{\alpha e} \right)^{2n} = [\text{const}]^n \, (n!)^2$$

and the fact the integral of the remaining $e^{-\alpha x^2/2}$ factor is finite.

2) Use the Borel Series SUM to Define the 'ORIGINAL FUNCTION' $I(\alpha, \epsilon)$ VIA a BOREL TRANSFORM (Type of Laplace Transform with $e^{-t/s}$ Weight)

SCALAR MODEL: $C_b = \left(-\zeta_b \Delta + m^2\right)^{-1}$ and $-\lambda V(\Phi(z)) = -\lambda\,[\phi(z)]^4$

nth Order Perturbation: THERE ARE: $(4n-1)!! \simeq const^n\,(n!)^2$ Contributions (Feynman Diagrams): Instanton Problem!

NOTE: EACH TERM is MULTIPLIED by λ^n.
λ MAY BE AS SMALL AS YOU WANT!
TOO MANY CONTRIBUTIONS TIMES λ^n **MAY STILL BE VERY BIG!** The SERIES MAY DIVERGE! (Proved for ϕ_2^4 and ϕ_3^4)

THIS MAY BE THE CASE OF QED !!!!
Perturbation may be ONLY an ASYMPTOTIC SERIES!

Pure FERMIONS $\bar{\psi}$, ψ: BETTER Situation. By the Pauli Principle, FERMIONS Are BOUNDED Operators.

Gram Determinant admits the Estimate

$$\left|\det\left(f_i, g_j\right)\right| \leq \left[\prod_i \|f_i\|_2\right] \left[\prod_j \|g_j\|_2\right],$$

which SAVES an $n!$ FACTOR.

After the UV Treatment that follows, BOREL SUMMABILITY Works Sometimes. OK e.g. for $\lambda\phi^4_{d=1,2,3}$.

More General Solution: NON-PERTURBATIVE ANALYTICAL METHODS

POLYMER or CLUSTER + MAYER EXPANSIONS: Truncated, Finite Order Expansions in λ, KEEPING the $e^{-\lambda V}$ Factor. Need Control of Remainders.

Use Extensively the FAST DECAY of COVARIANCE (Free Propagator)

CURE: Infinite Volume Limit $|\Lambda| \nearrow \infty$ **EXISTS WHENEVER** $m \neq 0$.

Gives, for the NORMALIZED CORRELATIONS

$$S_\ell^{norm}(x_1, \dots, x_\ell) = \frac{1}{Z} S_\ell(x_1, \dots, x_\ell)$$

a Convergent Expansion UNIFORMLY in VOLUME $|\Lambda|$, if $0 < \lambda \ll 1$.

OBS: Both Numerator & Denominator Diverge for $|\Lambda| \nearrow \infty$ by Translation Invariance in x.

$C(x,y) \equiv C(x-y)$ DECAY: (Spectral Mass Gap) Follows from Payley-Wiener Thm

Scalar Case: Fourier Transform of $C_b(x-y) = \left(-\zeta_b\Delta + m^2\right)^{-1}(x-y)$ is $\tilde{C}_b(p) = \left(-\zeta_b p^2 + m^2\right)^{-1}$, $p \in \mathbb{C}$, which is ANALYTIC in a Poly-STRIP $|\mathcal{I}\mathrm{m}\ \ p^\mu| \leq m$, $\mu = 0, 1, \ldots, d-1$, so that $C_b(x,y) \sim \exp(-m|x-y|)$, for $|x-y| >> 1$.

SOME WORDS in the END Regarding the MASSLESS CASE $m = 0$.

TASTE OF CLUSTER EXPANSION: In VOLUME Λ, we can take (assume there is no UV problem):

$$S^{norm}_{\ell,\Lambda}(x_1,\dots,x_\ell) = \frac{1}{Z_\Lambda}\, S_{\ell,\Lambda}(x_1,\dots,x_\ell) \quad , \quad Z_\Lambda \equiv S_{0,\Lambda}\,,$$

Unnormalized Correlations:

$$S_{\ell,\Lambda}(x_1,\dots,x_\ell) = \int \Phi(x_1)\cdots\Phi(x_\ell)\, e^{\int_\Lambda [-\lambda\, V(\Phi(z))]\, d^d z}\, d\mu_{C_\Lambda}(\Phi)$$

$C_\Lambda = \Lambda\, C\, \Lambda$ is interpolated as follows: Consider a covering of lambda with disjoint cubic cells Δ_i of size $\approx 1/m$ (m is the mass appearing in C): $\Lambda = \cup_i\, \Delta_i$.

Associate a real variable $s_{ij} \in [0,1]$ to each pair of distinct cells $\Delta_i \neq \Delta_j$ and write

$$C_\Lambda(\{s_{ij}\}) = \sum_{\{ij\}-\text{pairs}\,\in\mathcal{P}} s_{ij}\left[\Delta_i\, C\, \Delta_j + i \leftrightarrow j\right] + \sum_i \Delta_i\, C\, \Delta_i\,,$$

such that $C_\Lambda = C_\Lambda(\{s_{ij}\} = 1)$.

By $\frac{d}{ds_{ij}} C_\Lambda(\{s_{ij}\})$ we generate a **link between** Δ_i **and** Δ_j given by the explicit propagator $\left[\Delta_i\, C\, \Delta_j + i \leftrightarrow j\right]$.

For $f \in \mathcal{C}^1(R)$, we define $\mathcal{I} f(s) = f(s=0)$ and $\mathcal{J} f(s) = \int_0^1 \frac{df(s)}{ds}\, ds$ such that by the Fundamental Theorem of Calculus, we have:

$$f(1) = [\mathcal{I} + \mathcal{J}] f(s)\,.$$

CLUSTER EXPANSION:

$$\prod_{\{ij\}-\text{pairs} \in \mathcal{P}} [\mathcal{I}_{ij} + \mathcal{J}_{ij}] = \sum_{\mathcal{Q} \subset \mathcal{P}} \left[\prod_{q \in \mathcal{Q}} \mathcal{J}_q\right] \left[\prod_{p \notin \mathcal{Q}} \mathcal{I}_p\right]$$

For the ACTION of $\mathcal{J}$: INTEGRATION by PARTS FORMULA:

$$\frac{d}{ds}\int f(\phi)\, d\mu_{C(s)}(\phi) = -\frac{1}{2}\int dxdy \frac{dC(s,x,y)}{ds} \int \left[\frac{\delta}{\delta\phi(x)}\,\frac{\delta}{\delta\phi(y)}\, f(\phi)\right] d\mu_{C(s)}(\phi)$$

Each $\dfrac{\delta}{\delta\phi}$ may generate new ϕ fields in the integrand.

By Wick's Thm, for boson fields, this leads to bad factorials!

However, the NUMBER of GENERATED Fields per PAIR of CELLS is FINITE and the propagators produced by $\dfrac{d}{ds}$, with ROOT in Cell Δ, MUST GO FURTHER AND FURTHER.

This Gives SMALL WEIGHTS Accompanying the bad factorials due to a Large Number of Produced Fields

NET RESULT: EXP DECAY FACTORS **CAN BEAT ANY FACTORIAL** Coming from Wick!.

THE EXPANSION CONVERGES FOR SMALL $|\lambda|$ **both in Numerator and Denominator** of Normalized Correlations.

USING A **MAYER** EXPANSION: The BAD Dependence on $|\Lambda|$ cancels out leading to the EXISTENCE of THERMODYNAMIC LIMIT.

OBS: NEED TO USE A BETTER (INDUCTIVE, TREE) Expansion Instead of the above PAIRWISE, to Preserve POSITIVITY of Covariance!

PHYSICS: GOING BACK TO PERTURBATION in λ...

Now Consider the CONTINUUM LIMIT $a \searrow 0$:

Depending on d: INDIVIDUAL CONTRIBUTIONS MAY DIVERGE Due to UV Singularities of $C(x, y)$

To Have the UV Limit: RENORMALIZATION

SUBTRACTION of UV SINGULARITIES By REDEFINING the PHYSICAL PARAMETERS ζ, m and λ.

HEPP's Thm (Perturbative Renormalization, 1965): For a Class of MODELS called PERTURBATIVELY RENORMALIZABLE, Very Dependent on d, we can **RENDER FINITE CORRELATIONS IN EACH FIXED PERTURBATION ORDER**.

How: Consider a CUTOFF MODEL

For Example: USE a SPACETIME Lattice of Spacing a or Replace e.g. in Bosonic Case the Fourier Transform $\tilde{C}(p) = \frac{1}{\zeta p^2 + m^2}$ by

$$\tilde{C}_a(p) = \frac{e^{-p^2 a}}{\zeta_a p^2 + m_a^2} \quad , \quad a > 0 \, ,$$

so that C is obtained from C_a in the $a \searrow 0$ Limit.

Replace the Covariance $C(x, y)$ of Gaussian Measure by $C_a(x, y)$, and Define UV Cutoff Correlations $S^{norm}_{\ell,a}(x_1, \dots, x_\ell) = \frac{1}{Z_a} S_{\ell,a}(x_1, \dots, x_\ell)$.

Fix the PHYSICAL (Renormalized) Model Parameters $\lambda_{ren} > 0$, $m_{ren} > 0$ and real ζ_{ren}.

Perturbative Renormalization: It is POSSIBLE to Define VIA PERTURBATION SERIES in λ_{ren} the BARE (cutoff dependent) PARAMETERS:

$$\lambda_a \equiv \lambda\,(a, \lambda_{ren}, m_{ren}, \zeta_{ren}) = \sum_{n\geq 0} \lambda_n(a, m_{ren}, \zeta_{ren})\, \lambda_{ren}^n\,,$$

$$m_a \equiv m\,(a, \lambda_{ren}, m_{ren}, \zeta_{ren}) = \sum_{n\geq 0} m_n(a, m_{ren}, \zeta_{ren})\, \lambda_{ren}^n\,,$$

$$\zeta_a \equiv \zeta\,(a, \lambda_{ren}, m_{ren}, \zeta_{ren}) = \sum_{n\geq 0} \zeta_n(a, m_{ren}, \zeta_{ren})\, \lambda_{ren}^n\,;$$

such that the limit $a \searrow 0$ exists for $S^{norm}_{\ell,a}(x_1, \ldots, x_\ell) = \frac{1}{Z_a}\, S_{\ell,a}(x_1, \ldots, x_\ell)$ **Order by Order in Perturbation**.

ABOVE SERIES: Obtained Using the Zimmermann's FOREST FORMULA (see e.g. Collin's Book).

Renormalization (Subtraction Procedure) GOES FROM SMALLER DIVERGENT SUBGRAPHS of a Feynman Graph TO THE BIGGER SUBGRAPHS (Including Eventually the Graph Itself)

FOREST: Sets of EITHER DISJOINT (no common line) or CHAINED GRAPHS (with Strict Inclusion, $G_1 \subset G_2 \subset \ldots \subset G$)

The OVERLAPPING DIVERGENCE PROBLEM Has Been Cured LONG AGO! But Even Books Published After 1965 May Still Talk About This While Forgetting Forests...

BUT The Perturbation Convergence PROBLEM CONTINUES: BAD DEPENDENCE in Pertubation order n.

INDIVIDUAL Renormalized Contributions at Order n **MAY HAVE A** $(n!)$ **BEHAVIOR**.

This is the **RENORMALON Problem**!

Example: THE BUBBLE CHAIN CASE in the $\lambda\phi^4_4$ Model

BUBBLEs FIGURE

$\tilde{B}(p) \sim \int \tilde{C}(k)C(k+p)\, d^4k$ DIVERGES LOGARITHMICALLY

$\tilde{B}_a(p) \sim \int \tilde{C}_a(k)C_a(k+p)\, d^4k \ \sim -\log a$

$$\tilde{B}_{ren}(p) \equiv \lim_{a \searrow 0} \left[\tilde{B}_a(p) - \tilde{B}_a(p=0)\right] \sim \log|p|\,.$$

Graph with RENORMALIZED BUBBLE CHAIN With $n-1$ Bubbles as a Subgraph:

$$\int [\tilde{C}(p)]^3 \left[\tilde{B}_{ren}(p)\right]^{n-1} d^4p \simeq \int \frac{1}{(p^2+m^2)^3} (\log|p|)^n \, d^4p \sim n!$$

CURE: Consider the Gross-Neveu$_2$ Model (With More than One-Component Fermions)

The Scenario is the SAME. Re-Introduce the (RUNNING) Coupling Constant in the Story

$$\int \lambda_a^n(p)\,[\tilde{C}(p)]^3\,\left[\tilde{B}_a(p)-\tilde{B}_a(p=0)\right]^{n-1}\,d^4p$$

$\lambda_a^n(p)$ is Dimensionless and Behaves as $-\left(\log|p| \simeq \log a^{-1}\right)^{-1}|$, for large $|p|$ $(a \searrow 0)$.

This KILLS the $n!$ Behavior of the Bubble Chain

GN$_2$ **MODEL IS UV ASYMPTOTICALLY FREE!**
NOT the Case for ϕ_4^4 !

HOW CAN THIS BE RIGOROUSLY IMPLEMENTED?

MULTISCALE NON-PERTURBATIVE Polymer or Cluster + Mayer: Expansions.

CURE: Simultaneously Both Limits $a \searrow 0$ and $|\Lambda| \nearrow \infty$, resulting in Convergent Expansions, UNIFORMLY Both in a, $|\Lambda|$, for COMPLETELY CUTOFF Correlations

$$S^{norm}_{\ell,a,\Lambda}(x_1, \ldots, x_\ell) = \frac{1}{Z_{a,\Lambda}} S_{\ell,a,\Lambda}(x_1, \ldots, x_\ell)\,,$$

if $0 < \lambda_{ren} \ll 1$.

IMPLEMENTATION: Use ρ to play the role of UV Cutoff Parameter $a^{-1} > 0$

REPLACE C_b e C_f by $C_{b,\rho}$ e $C_{f,\rho}$ such that in Fourier Space

$$\tilde{C}_b(p) = \frac{1}{p^2 + m^2} \qquad , \qquad \tilde{C}_f(p) = \frac{-i\not{p} + m}{p^2 + m^2},$$

Become

$$\tilde{C}_{b,\rho}(p) = \frac{1}{\zeta_{b,\rho} p^2 + m_\rho^2} \eta_\rho(p) \qquad , \qquad \tilde{C}_{f,\rho}(p) = \frac{-i\zeta_{f,\rho}\not{p} + m_\rho}{\zeta_{f,\rho}^2 p^2 + m_\rho^2} \eta_\rho(p),$$

where $\eta_\rho(p)$ is a UV Cutoff Function of ρ. For example, as before

$$\eta_\rho(p) = e^{-p^2 M^{-2\rho}} \qquad , \qquad M \in \mathbb{N},\ M > 2.$$

$\eta_\rho(p)$ has support *concentrated* on $P = \{p \in \mathbb{R}^d \,/\, p^2 \leq M^{2\rho}\}$.

UV Limit is Given by $\rho \nearrow \infty$, in which $\eta_\rho(p) \to 1$.

Generically: CUTOFF FIELDS Φ_ρ (ϕ_ρ, $\bar{\psi}_\rho$ and ψ_ρ)

(Forgetting Λ Dependence) the UV Cutoff Correlations

$$S_{\ell,\rho}(x_1,\ldots,x_\ell) \;=\; \int \Phi(x_1)\cdots\Phi(x_\ell)\, d\nu_\rho(\Phi_\rho)\,,$$

with $d\nu_\rho(\Phi_\rho) = \exp\left\{\int -\lambda_\rho\, V(\Phi_\rho(z))\, d^d z\right\} d\mu_{C_\rho}(\Phi_\rho)$, where $d\mu_{C_\rho}(\Phi_\rho)$ Has Covariance C_ρ and

$$|C_\rho(x-y)| \simeq \exp[-M^\rho|x-y|] \quad , \quad |x-y| \gg 1\,.$$

MULTISCALE ANALYSIS (Renormalization Group):

Decompose $\eta_\rho(p) = \sum_{i=0}^{\rho} \eta^i(p)$ with

$$\eta^i(p) \equiv e^{-p^2 M^{-2i}} - e^{-p^2 M^{-2(i-1)}}\;,\; i = 1,\ldots,\rho \quad ; \quad \eta^0(p) = 1\,,$$

$\eta^i(p)$ is Supported on $|p| \simeq M^i$!

This Decomposition Induces a Field, a Covariance and a Gaussian Measure Decomposition. Namely,

$$\tilde{C}_\rho(p) = \sum_{i=0}^{\rho} \tilde{C}^i(p)$$
$$\Phi_\rho = \oplus_{i=1}^{\rho} \Phi^i$$

such that

$$S_{\ell,\rho}(x_1, \ldots, x_\ell) = \int \Phi(x_1) \cdots \Phi(x_\ell)\, d\nu_\rho(\Phi_\rho = \oplus_{i=1}^{\rho} \Phi^i)$$

with $d\nu_\rho(\Phi_\rho) = \exp\left\{\int -\lambda_\rho V\left(\oplus_{i=1}^{\rho} \Phi^i(z)\right) d^d z\right\} \prod_{i=0}^{\rho} d\mu_{C^i}(\Phi^i)$,

$d\mu_{C^i}(\Phi^i)$ has Covariance C^i.

C^i Has the SCALED DECAY $C^i(x-y) \simeq e^{-M^i|x-y|}$, for $|x-y| \gg 1$.

Starting with ρ Dependent Parameters λ_ρ, m_ρ and ζ_ρ no cutoff UV ρ, NON-PERTURBATIVE Renormalization Aims at FINDING **FUNCTIONS**

$$\begin{cases} \lambda_\rho \equiv \lambda(\rho, \lambda_{ren}, m_{ren}, \zeta_{ren}), \\ m_\rho \equiv m(\rho, \lambda_{ren}, m_{ren}, \zeta_{ren}), \\ \zeta_\rho \equiv \zeta(\rho, \lambda_{ren}, m_{ren}, \zeta_{ren}), \ \ldots \end{cases}$$

of ρ, and the Renormalized Parameters $(\lambda_{ren}, m_{ren}, \zeta_{ren})$, with $\lambda_{ren} > 0$, such that the limit $\rho \to \infty$ of Correlations EXISTS.

RG MAP: Dynamical System Generated by the EACH INTEGRATION over the SCALED FIELDS Φ^ρ, $\Phi^{\rho-1}$,..., Φ^i,..., Φ^0, RESPECTING THIS ORDER.

Equivalently: SMALL DISTANCES $\longrightarrow$ LARGE DISTANCES.

In PARAMETER SPACE: Before Integrating Over Φ^i

$$\begin{cases} \lambda_\rho \equiv \lambda(\rho, \lambda_{ren}, m_{ren}, \zeta_{ren}), \\ m_\rho \equiv m(\rho, \lambda_{ren}, m_{ren}, \zeta_{ren}), \\ \zeta_\rho \equiv \zeta(\rho, \lambda_{ren}, m_{ren}, \zeta_{ren}), \end{cases} \longrightarrow \ldots \longrightarrow \begin{cases} \lambda_i \equiv \lambda(i, \lambda_{ren}, m_{ren}, \zeta_{ren}), \\ m_i \equiv m(i, \lambda_{ren}, m_{ren}, \zeta_{ren}), \\ \zeta_i \equiv \zeta(i, \lambda_{ren}, m_{ren}, \zeta_{ren}), \end{cases}$$

POSSIBLE to Formulate in HAMILTONIAN or ACTION SPACE!

TAUTOLOGY: *UV LIMIT EXISTS if THERE IS A STABLE FIXED POINT of this MAP!*

OBSERVATION: ZERO MASS CASE ($m = 0$) the IR Limit Can Be Similarly Treated.

IMPORTANT: STABLE FIXED POINT Defines the CONCEPT of ASYMPTOTIC SAFETY

SUCCESS: MANY MODELS (QFT, Statistical Mech., Condensed Matter)

QFT: GN_2, YM_4, Infrared ϕ^4_4 (with $\lambda < 0$), Fermi Liquids in $d = 3$,...

IN ALL THIS: Still Perturbation Series Gives the INSPIRATION!

MANY NAMES ARE INVOLVED IN DEVELOPING THIS MATTER!

PERSONAL CONTRIBUTION: UV Renormalizability of the PERTURBATIVELY NON-RENORMALIZABLE GN_3 with a Large Number N of Components. **BUBBLE CHAIN SUMMATION** with Cutoff Leads to a **Perturbatively Renomrmalizable Model in a NEW PARAMETER** and then to a **NONTRIVIAL UV FIXED POINT**.

BUT, for GN_3: NO O-S AXIOMS YET!

While an Undergraduate Student, I profited from long conversations with Prof. Patricio Letelier during my several visits to the UNICAMP. After this, unfortunately, I never met him...

BUT HERE IS MY CONTACT POINT, TODAY, With PROF. LETELIER!

WHAT ABOUT GRAVITY?

Quantum Gravity is NOT Perturbatively Renormalizable. Any Way to See It is UV Asymptotically SAFE?

THANKS FOR YOUR ATTENTION!