Covariant Affine Integral Quantization and applications

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- The basic procedure starts from a phase space or symplectic manifold, e.g. $\mathbb{R}^2$,

$$\mathbb{R}^2 \ni (q,p)\,, \quad \{q,p\} = 1 \mapsto \textbf{self-adjoint}\,(Q,P)\,, \quad [Q,P] = \mathrm{i}\hbar I\,,$$
$$f(q,p) \mapsto f(Q,P) \mapsto (\mathrm{Sym}f)(Q,P)\,.$$

- Remind that $[Q,P] = \mathrm{i}\hbar I$ holds true with self-adjoint Q, P, **only if** both have continuous spectrum $(-\infty,+\infty)$
- But then what about singular f, e.g. the angle $\arctan(p/q)$? What about other phase space geometries? barriers or other impassable boundaries? The motion on a circle? In a bounded interval? **On the positive half-line**? ....

Despite their elementary aspects, examples like the motion on the circle, on the positive half-line,...., leave open many questions both on mathematical and physical levels, irrespective of the manifold quantization procedures, like Path Integral Quantization (Feynman, thesis, 1942), or, after approaches by Weyl (1927), Groenewold (1946), Moyal (1947), Geometric Quantization, Kirillov (1961), Souriau (1966), Kostant (1970), Deformation Quantization, Bayen, Flato, Fronsdal, Lichnerowicz, Sternheimer (1978), Fedosov (1985), Kontsevich (2003), Coherent state or anti-Wick or Toeplitz quantization with Klauder (1961), Berezin (1974) ....

- The canonical procedure is quasi-universally accepted in view of its numerous experimental validations, one of the most famous and simplest one going back to the early period of Q.M. with the quantitative prediction (1925) of the isotopic effect in vibrational spectra of diatomic molecules.
- These data validated the canonical quantization, contrary to the Bohr-Sommerfeld ansatz (which predicts no isotopic effect).
- Nevertheless this does not prove that another method of quantization fails to yield the same prediction.
- Moreover, as already mentioned above, the canonical quantization is too rigid or even untractable in some circumstances. As a matter of fact, the canonical or the Weyl-Wigner integral quantization maps $f(q)$ to $f(Q)$ (resp. $f(p)$ to $f(P)$), and so might be unable to cure a given classical singularity.
- Nevertheless, physics works mostly with effective models, and an effective quantum model is expected to be more regular than a classical one.

- Quantization is

  (i) a <u>linear</u> map

$$\mathfrak{Q} : \mathscr{C}(X) \mapsto \mathscr{A}(\mathscr{H})$$

  $\mathscr{C}(X)$: vector space of complex-valued functions $f(x)$ on a set X
  $\mathscr{A}(\mathscr{H})$: vector space of linear operators

$$\mathfrak{Q}(f) \equiv A_f$$

  in some complex Hilbert space $\mathscr{H}$ such that

  (ii) $f = 1 \mapsto$ identity operator I on $\mathscr{H}$,

  (iii) real $f \mapsto$ (essentially) self-adjoint operator A_f in $\mathscr{H}$ (if not, should be at least symmetric)

- Add preservation of symmetry ("covariance")
- Add further requirements on X and $\mathscr{C}(X)$ (e.g., measure, topology, manifold, closure under algebraic operations, time evolution or dynamics...)
- Add physical interpretation about measurement of spectra of classical $f \in \mathscr{C}(X)$ or quantum $A \in \mathscr{A}(\mathscr{H})$ to which are given the status of *observables*.
- Add requirement of unambiguous classical limit of the quantum physical quantities, **the limit operation being associated to a change of scale**

- (X,ν): measure space.
- $X \ni x \mapsto \mathrm{M}(x) \in \mathscr{L}(\mathscr{H})$: X-labelled family of bounded operators on Hilbert space $\mathscr{H}$ resolving the identity I:

$$\int_X \mathrm{M}(x)\,\mathrm{d}\nu(x) = I, \quad \text{in a weak sense}$$

- If the $\mathrm{M}(x)$'s are positive semi-definite and unit trace,

$$\mathrm{M}(x) \equiv \rho(x) \quad (\textit{density operator})$$

and if X is space with suitable topology, the map

$$\mathscr{B}(X) \ni \Delta \mapsto \int_\Delta \rho(x)\,\mathrm{d}\nu(x)$$

may define a normalized positive operator-valued measure (POVM) on (Borel) subsets of X, **with probabilistic content.**

- Quantization of complex-valued functions $f(x)$ (on more singular objects!) on X is the linear map:

$$f \mapsto A_f = \int_X \mathsf{M}(x)\, f(x)\, \mathrm{d}\nu(x)\,,$$

- understood as the sesquilinear form,

$$B_f(\psi_1, \psi_2) = \int_X \langle \psi_1 | \mathsf{M}(x) | \psi_2 \rangle\, f(x)\, \mathrm{d}\nu(x)\,,$$

defined on a dense subspace of $\mathscr{H}$.

- If f is real and at least semi-bounded, and if the $M(x)$'s are positive operators, then the Friedrich's extension of B_f univocally defines a self-adjoint operator.
- If f is not semi-bounded, no natural choice of a self-adjoint operator associated with B_f, a subtle question. More information on $\mathscr{H}$ is needed.

- Quantization issues, e.g. spectral properties of A_f, quantum dynamics, may be understood from functional properties of lower (Lieb) or covariant (Berezin) symbols (generalize Husimi function or Wigner function) $\rightarrow$ **semi-classical portraits**

$$f \mapsto A_f \mapsto \check{f}(x) := \mathrm{tr}(\mathsf{M}(x)\, A_f)\,,$$

- If $\mathsf{M} = \rho$, then $\check{f}(x)$ is the local averaging of the original f with respect to the probability distribution $x' \mapsto \mathrm{tr}(\rho(x)\rho(x'))$

$$f(x) \mapsto \check{f}(x) = \int_X f(x')\, \mathrm{tr}(\rho(x)\rho(x'))\, \mathrm{d}\nu(x')\,.$$

- The Bargmann-Segal-like map $f \mapsto \check{f}$ is in general a regularization of the original, possibly extremely singular, f.
- The classical limit itself means: given one or more scale parameter(s) $\varepsilon_{(i)}$ and a distance $d(f, \check{f})$:

$$d(f,\check{f}) \rightarrow 0 \quad \text{as} \quad \varepsilon_{(i)} \rightarrow 0\,.$$

- Let G be a Lie group with left Haar measure $\mathrm{d}\mu(g)$, and let $g \mapsto U(g)$ be a ***unitary irreducible representation*** (UIR) of G in a Hilbert space $\mathscr{H}$.
- Let M a bounded operator on $\mathscr{H}$. Suppose that the operator

$$R := \int_G \mathsf{M}(g)\,\mathrm{d}\mu(g)\,, \quad \mathsf{M}(g) := U(g)\mathsf{M}\,U^{\dagger}(g)\,,$$

  is defined in a weak sense. From the left invariance of $\mathrm{d}\mu(g)$ the operator R commutes with all operators $U(g)$, $g \in G$, and so from Schur's Lemma, $R = c_{\mathsf{M}} I$ with

$$c_{\mathsf{M}} = \int_G \mathrm{tr}\left(\rho_0 \mathsf{M}(g)\right)\,\mathrm{d}\mu(g)\,,$$

  where the unit trace positive operator ρ_0 is chosen in order to make the integral convergent.
- **Resolution of the identity** follows:

$$\boxed{\int_G \mathsf{M}(g)\,\mathrm{d}\nu(g) = I\,, \quad \mathrm{d}\nu(g) := \mathrm{d}\mu(g)/c_{\mathsf{M}}\,.}$$

- For square-integrable UIR U for which ρ is an "admissible" density operator,

$$c_\rho = \int_G \mathrm{d}\mu(g)\,\mathrm{tr}\left(\rho U(g)\rho U^\dagger(g)\right) < \infty$$

- Resolution of the identity then is obeyed by the family:

$$\rho(g) = U(g)\rho U^\dagger(g)$$

- This allows *covariant* integral quantization of complex-valued functions on the group

$$\boxed{f \mapsto A_f = \frac{1}{c_\rho}\int_G \rho(g)\, f(g)\, \mathrm{d}\mu(g)\,,}$$

$$U(g) A_f U^\dagger(g) = A_{\mathfrak{U}(g)f}\,,$$

where

$$(\mathfrak{U}(g)f)(g') := f(g^{-1}g')$$

(regular representation if $f \in L^2(G, \mathrm{d}\mu(g))$).

- Generalization of the Berezin or heat kernel transform on G:

$$\check{f}(g) := \int_G \mathrm{tr}(\rho(g)\,\rho(g'))\, f(g')\, \mathrm{d}\nu(g')$$

Positive operator-valued measure (POVM) based on $x \mapsto \rho(x)$,
particularly "coherent states" (CS) $\rho(x) = |x\rangle\langle x|$:
a bridge classical $\leftrightarrow$ quantum models

- Integral quantizations, particularly CS quantizations, are suitable when we have to deal with some singularities
- POVM afford a semi-classical phase space portrait of quantum states and quantum dynamics together with a probabilistic interpretation

- As the complex plane is viewed as the phase space for the motion of a particle on the line, the half-plane is viewed as the phase space for the motion of a particle on the half-line.
- One equips the upper half-plane $\Pi_+ := \{(q,p)\,|\, p\in\mathbb{R}\,,\, q>0\}$ with the measure $\mathrm{d}q\mathrm{d}p$.
- Together with
  (i) the multiplication law
  $$(q,p)(q_0,p_0) = \left(qq_0, \frac{p_0}{q} + p\right), \, q\in\mathbb{R}_+^*, \, p\in\mathbb{R},$$
  (ii) the unity $(1,0)$
  (iii) and the inverse
  $$(q,p)^{-1} = \left(\frac{1}{q}, -qp\right)\,,$$
  Π_+ is viewed as the affine group $\mathrm{Aff}_+(\mathbb{R})$ of the real line
- The measure $\mathrm{d}q\,\mathrm{d}p$ is left-invariant with respect to this action.

- The affine group $\mathrm{Aff}_+(\mathbb{R})$ has two non-equivalent UIR $U_\pm$ ($\sim$ carried on by Hardy spaces)
- Both are square integrable: this is the rationale behind *continuous wavelet analysis* resulting from a resolution of the identity.
- The UIR $U_+ \equiv U$ is realized in the Hilbert space $\mathscr{H} = L^2(\mathbb{R}_+^*, \mathrm{d}x)$:

$$U(q,p)\psi(x) = (e^{\mathrm{i}px}/\sqrt{q})\psi(x/q)\,.$$

- By adopting the integral quantization scheme described above, we restrict to the specific case of rank-one density operator or projector

$$\rho = |\psi\rangle\,\langle\psi|$$

where ψ is a unit-norm state in $L^2(\mathbb{R}_+^\dagger, \mathrm{d}x) \cap L^2(\mathbb{R}_+^\dagger, \mathrm{d}x/x)$ (also called "fiducial vector" or "wavelet").

- The action of UIR U produces all affine coherent states, i.e. wavelets, defined as $|q,p\rangle = U(q,p)|\psi\rangle$.

- Due to the irreducibility and square-integrability of the UIR U, the following resolution of the identity holds

$$\int_{\Pi_+} |q,p\rangle\langle q,p| \, \frac{\mathrm{d}q\mathrm{d}p}{2\pi c_{-1}} = I,$$

where

$$c_\gamma := \int_0^\infty |\psi(x)|^2 \, \frac{\mathrm{d}x}{x^{2+\gamma}} \, .$$

- Thus, a necessary condition for resolution of the identity holding true is that $c_{-1} < \infty$, which implies $\psi(0) = 0$, a well-known requirement in wavelet analysis.
- Corresponding quantization reads as

$$\boxed{f \;\mapsto\; A_f = \int_{\Pi_+} f(q,p)|q,p\rangle\langle q,p| \frac{\mathrm{d}q\,\mathrm{d}p}{2\pi c_{-1}} \, ,}$$

- As expected, the map $f \mapsto A_f$ is covariant with respect to the unitary affine action U:

$$U(q_0,p_0)A_f U^{\dagger}(q_0,p_0) = A_{\mathfrak{U}(q_0,p_0)f}\,,$$

with

$$(\mathfrak{U}(q_0,p_0)f)(q,p) = f\left((q_0,p_0)^{-1}(q,p)\right) = f\left(\frac{q}{q_0}, q_0(p-p_0)\right)\,,$$

- $\mathfrak{U}$ is the left regular representation of the affine group.

- Quantization of the momentum

$$A_p = P = -\mathrm{i}\frac{\partial}{\partial x}\,.$$

- Quantization of powers of the position

$$A_{q^\beta} = \frac{c_{\beta-1}}{c_{-1}}\,Q^\beta\,, \quad Qf(x) = xf(x)\,.$$

- Whereas Q is self-adjoint, operator P is symmetric but has no self-adjoint extension.

- This affine quantization is, up to a multiplicative constant, canonical,

$$[Q,P] = \mathrm{i}c_0/c_{-1}\,I$$

(The constant can be brought to 1 through a suitable rescaling.)

- The quantization of the product *qp* yields:

$$A_{qp} = \frac{c_0}{c_{-1}} \frac{QP + PQ}{2} \equiv \frac{c_0}{c_{-1}} D \, ,$$

where D is the dilation generator. As one of the two generators (with Q) of the UIR U of the affine group, it is essentially self-adjoint, with continuous spectrum $\lambda \in \mathbb{R}$ and corresponding eigendistributions $x^{\frac{1}{2}+\mathrm{i}\lambda}$.

- The quantization of kinetic energy gives

$$A_{p^2} = P^2 + \frac{K(\psi)}{Q^2}\,, \quad K(\psi) = \int_0^\infty (\psi'(u))^2\, u \frac{\mathrm{d}u}{c_{-1}} > 0.$$

- Therefore, this "wavelet" quantization prevents a quantum free particle moving on the positive line from reaching the origin.
- While the operator $P^2 = -d^2/dx^2$ in $L^2(R_+^*, \mathrm{d}x)$ is not essentially self-adjoint, the above regularized operator, defined on the domain of smooth function of compact support, is essentially self-adjoint[1] for $K \geqslant 3/4$. Then quantum dynamics of the free motion is unique.

[1] Reed M. and Simon B., Methods of Modern Mathematical Physics, II. Fourier Analysis, Self-Adjointness Volume 2 Academic Press, New York, 1975

- The quantum states and their dynamics have phase space representations through wavelet symbols. For the state $|\phi\rangle$ one has

$$\Phi(q,p) = \langle q,p|\phi\rangle/\sqrt{2\pi}\,,$$

- The associated probability distribution on phase space given by

$$\rho_\phi(q,p) = \frac{1}{2\pi c_{-1}}|\langle q,p|\phi\rangle|^2.$$

- Having the (energy) eigenstates of some quantum Hamiltonian H, e.g. the ACS quantized A_{h} of a classical $\mathrm{h}(q,p)$, at our disposal, we can compute the time evolution

$$\rho_\phi(q,p,t) := \frac{1}{2\pi c_{-1}}|\langle q,p|e^{-\mathrm{iH}t}|\phi\rangle|^2$$

for any state ϕ.

Affine CS quantization of the half-plane in a nutshell

e.g. Volume - Expansion pair : $(q,p) \in \mathbb{R}_+ \times \mathbb{R}$

Affine group : $(q,p)(q_0,p_0) = (qq_0, \frac{p_0}{q} + p)$, ***Left invariant measure:*** $\mathrm{d}q\,\mathrm{d}p$

UIR : $L^2(\mathbb{R}_+, \mathrm{d}x) \ni \psi(x) \mapsto (U(q,p)\psi)(x) = \frac{e^{\mathrm{i}px}}{\sqrt{q}}\,\psi\left(\frac{x}{q}\right)$

Affine CS : $L^2(\mathbb{R}_+, \mathrm{d}x) \cap L^2(\mathbb{R}_+, \mathrm{d}x/x) \ni \psi \mapsto |q,p\rangle = U(q,p)\psi$

ACS-integral quantization : $f(q,p) \mapsto A_f = \mathrm{Cst}_\psi \int_{\mathbb{R}_+\times\mathbb{R}} \mathrm{d}q\,\mathrm{d}p\, f(q,p)\, |q,p\rangle\langle q,p|$

Probe operator to be affine transported

- Given a weight function $\varpi(q,p)$ one defines the operator

$$\int_{\Pi_+} \mathsf{C}_{\mathrm{DM}}^{-1} U(q,p) \mathsf{C}_{\mathrm{DM}}^{-1} \varpi(q,p)\, \mathrm{d}q\, \mathrm{d}p := \mathsf{M}^{\varpi} .$$

- The appearance of the positive self-adjoint and invertible Duflo-Moore operator $\mathsf{C}_{\mathrm{DM}} := \sqrt{2\pi/Q}$ is due to the non-modularity of the affine group. This operator is needed to establish the square-integrability of the UIR U

$$\int_{\Pi_+} \mathrm{d}q\, \mathrm{d}p \langle U(q,p)\psi|\phi\rangle \overline{\langle U(q,p)\psi'|\phi'\rangle} = \langle \mathsf{C}_{\mathrm{DM}}\psi|\mathsf{C}_{\mathrm{DM}}\psi'\rangle \langle \phi'|\phi\rangle ,$$

for any pair (ψ,ψ') of admissible vectors, i.e. which obey $\|\mathsf{C}_{\mathrm{DM}}\psi\| < \infty$, $\|\mathsf{C}_{\mathrm{DM}}\psi'\| < \infty$, and any pair (ϕ,ϕ') of vectors in $L^2(\mathbb{R}_+^*, \mathrm{d}x)$.

- Operator M^{ϖ} is symmetric if $\varpi(q,p)$ obeys $\varpi(q,p) = \frac{1}{q}\overline{\varpi\left(\frac{1}{q}, -qp\right)}$

- Corresponding integral quantization

$$f \mapsto A_f^{\varpi} = \int_{\Pi_+} \frac{\mathrm{d}q\,\mathrm{d}p}{c_{\mathrm{M}^{\varpi}}}\, f(q,p)\mathrm{M}^{\varpi}(q,p)\,, \quad \mathrm{M}^{\varpi}(q,p) = U(q,p)\mathrm{M}^{\varpi}U^{\dagger}(q,p)$$

- The constant $c_{\mathrm{M}^{\varpi}}$ is given by

$$c_{\mathrm{M}^{\varpi}} = \sqrt{2\pi}\int_0^{+\infty}\frac{\mathrm{d}q}{q}\,\hat{\varpi}_p(1,-q)\,,$$

where $\hat{\varpi}_p$ is the partial Fourier transform of ϖ with respect to the variable p.

- Resolution of the identity holds for $c_{\mathrm{M}^{\varpi}} < \infty$.
- By construction, this quantization map is covariant

$$U(q_0,p_0)A_f^{\varpi}U^{\dagger}(q_0,p_0) = A\varpi_{\mathfrak{U}(q_0,p_0)f}\,.$$

Proposition

The action on ϕ in $\mathscr{H}$ of the operator A^{ϖ}_f defined by the integral quantization map is given by

$$(A^{\varpi}_f \phi)(x) = \int_0^{+\infty} \mathscr{A}^{\varpi}_f(x,x')\,\phi(x')\,\mathrm{d}x'\,,$$

where the kernel $\mathscr{A}^{\varpi}_f$ is defined as

$$\mathscr{A}^{\varpi}_f(x,x') = \frac{1}{c_{\mathrm{M}^{\varpi}}}\,\frac{x}{x'}\int_0^{+\infty}\frac{\mathrm{d}q}{q}\,\hat{\varpi}_p\left(\frac{x}{x'},-q\right)\,\hat{f}_p\left(\frac{x}{q},x'-x\right)\,.$$

Here $\hat{f}_p$ is the partial Fourier transform of f with respect to the variable p.

- Position dependent function $f(q,p) \equiv u(q)$. Its quantum version is the multiplication operator

$$A^{\varpi}_{u(q)} = \frac{2\pi}{c_{\mathrm{M}^{\varpi}}} \int_0^{+\infty} \frac{\mathrm{d}q}{q} \mathscr{M}^{\varpi}(q,q)\, u\left(\frac{Q}{q}\right) = \frac{\sqrt{2\pi}}{c_{\mathrm{M}^{\varpi}}} \int_0^{+\infty} \frac{\mathrm{d}q}{q} \hat{\varpi}_p(1,-q)\, u\left(\frac{Q}{q}\right)$$

i.e. the multiplication by the convolution on the multiplicative group $\mathbb{R}^*_+$ of $u(x)$ with $\frac{\sqrt{2\pi}}{c_{\mathrm{M}^{\varpi}}} \hat{\varpi}_p(1,-x)$.

- An interesting more particular case is when u is a simple power of q, say $u(q) = q^{\beta}$. Then we have

$$\boxed{A^{\varpi}_{q^{\beta}} = \frac{\sqrt{2\pi}}{c_{\mathrm{M}^{\varpi}}} \int_0^{+\infty} \frac{\mathrm{d}q}{q^{1+\beta}} \hat{\varpi}_p(1,-q)\, Q^{\beta} \equiv \frac{d_{\beta}}{d_0} Q^{\beta}\,,}$$

where $d_{\beta} = \int_0^{+\infty} \frac{\mathrm{d}q}{q^{1+\beta}} \hat{\varpi}_p(1,-q)$

- Momentum dependent functions $f(q,p) \equiv v(p)$

$$\mathscr{A}^{\varpi}_{v(p)}(x,x') = \frac{1}{c_{\mathrm{M}\varpi}}\,\hat{v}(x'-x)\,\frac{x}{x'}\int_0^{+\infty}\frac{\mathrm{d}q}{q}\,\hat{\varpi}_p\left(\frac{x}{x'},-q\right) \equiv \frac{1}{c_{\mathrm{M}\varpi}}\,\hat{v}(x'-x)\,\frac{x}{x'}\,\Omega\left(\frac{x}{x'}\right).$$

- As a simple but important example, let us examine the case $v(p)=p^n$, $n\in\mathbb{N}$. From distribution theory

$$\hat{v}(x'-x) = \sqrt{2\pi}\,\mathrm{i}^n\,\delta^{(n)}(x'-x)\,,$$

we derive the differential action of the operator $A^{\varpi}_{p^n}$ in $\mathscr{H}$ as the polynomial in $P=-\mathrm{i}d/dx$

$$A^{\varpi}_{p^n} = \frac{\sqrt{2\pi}}{c_{\mathrm{M}\varpi}}\sum_{k=0}^{n}\binom{n}{k}\left(-\mathrm{i}\frac{d}{dx'}\right)^{n-k}\frac{x}{x'}\,\Omega\left(\frac{x}{x'}\right)\Big|_{x'=x} P^k = P^n + \cdots.$$

- In particular

$$A^{\varpi}_{p} = P + \frac{\mathrm{i}}{x}\left[1+\frac{\Omega'(1)}{\Omega(1)}\right].$$

- This operator is symmetric but has no self-adjoint extension
- The commutation rule $[A_q, A_p] = \frac{d_1}{d_0}\,\mathrm{i}I$ holds canonical up to a factor which can be easily put equal to one through a rescaling of the weight function.

- For the kinetic energy we have

$$A^{\varpi}_{p^2} = P^2 + \frac{2\mathrm{i}}{Q}\left[1 + \frac{\Omega'(1)}{\Omega(1)}\right] P - \frac{1}{Q^2}\left[2 + 4\,\frac{\Omega'(1)}{\Omega(1)} + \frac{\Omega''(1)}{\Omega(1)}\right].$$

- This symmetric operator is essentially self-adjoint or not, depending on the strength of the (attractive or repulsive) potentiel $1/x^2$.
- With the choice of a weight function such that $-2 - 4\,\frac{\Omega'(1)}{\Omega(1)} - \frac{\Omega''(1)}{\Omega(1)} \geqslant 3/4$, it is essentially self-adjoint and so quantum dynamics of the free motion on the half-line is unique.

- Separable functions $f(q,p) \equiv u(q)\,v(p)$

$$\mathscr{A}^{\varpi}_{u(q)v(p)}(x,x') = \frac{1}{c_{\mathrm{M}^{\varpi}}}\,\hat{v}(x'-x)\,\frac{x}{x'}\int_0^{+\infty}\frac{\mathrm{d}q}{q}\,\hat{\varpi}_p\left(\frac{x}{x'},-q\right)\,u\left(\frac{x}{q}\right).$$

- The elementary example is the quantization of the function qp which produces the integral kernel and its corresponding operator

$$\mathscr{A}^{\varpi}_{qp}(x,x') = \frac{\sqrt{2\pi}}{c_{\mathrm{M}^{\varpi}}}\,\mathrm{i}\,\delta'(x'-x)\,\frac{x^2}{x'}\int_0^{+\infty}\frac{\mathrm{d}q}{q^2}\,\hat{\varpi}_p\left(\frac{x}{x'},-q\right),$$

$$A^{\varpi}_{qp} = \frac{\Omega_1(1)}{\Omega(1)}\,D + \mathrm{i}\left[\frac{3}{2}\,\frac{\Omega_1(1)}{\Omega(1)} + \frac{\Omega_1'(1)}{\Omega(1)}\right],$$

where $D = \frac{1}{2}(QP+PQ)$ is the dilation generator. Here

$$\Omega_\beta(u) = \int_0^{+\infty}\frac{\mathrm{d}q}{q^{1+\beta}}\,\hat{\varpi}_p(u,-q)\,, \quad \Omega_0(u) = \Omega(u)\,.$$

- Given a weight function $\varpi(q,p)$ yielding a symmetric unit trace operator M^{ϖ}, we define the semi-classical or lower symbol of an operator A in $\mathscr{H}$ as the function

$$\check{A}(q,p) := \mathrm{Tr}\left(A\,U(q,p)\mathsf{M}^{\varpi}\,U^{\dagger}(q,p)\right) = \mathrm{Tr}\left(A\mathsf{M}^{\varpi}(q,p)\right).$$

- When the operator A is the affine integral quantized version of a classical $f(q,p)$ with the same weight ϖ, we get the transform

$$f(q,p) \mapsto \check{f}(q,p) \equiv \check{A}^{\varpi}_{f}(q,p) = \int_{\Pi_+} \frac{\mathrm{d}q'\,\mathrm{d}p'}{c_{\mathsf{M}^{\varpi}}}\, f\left(qq', \frac{p'}{q} + p\right) \mathrm{Tr}\left(\mathsf{M}^{\varpi}(q',p')\mathsf{M}^{\varpi}\right).$$

- Of course, this expression has the meaning of an averaging of the classical f if the function

$$\begin{aligned}(q,p) \equiv g \mapsto \frac{1}{c_{\mathsf{M}^{\varpi}}}\mathrm{Tr}\left(\mathsf{M}^{\varpi}(g)\mathsf{M}^{\varpi}\right) =\\ = \frac{1}{c_{\mathsf{M}^{\varpi}}}\frac{1}{2\pi q}\int_0^{+\infty}\mathrm{d}x\int_0^{+\infty}\mathrm{d}y\, e^{-\mathrm{i}p(y-x)}\,\hat{\varpi}_p\left(\frac{x}{y}, -\frac{x}{q}\right)\hat{\varpi}_p\left(\frac{y}{x}, -y\right).\end{aligned}$$

is a true probability distribution on the half-plane.

- With the specific weight $\varpi_{a\mathscr{W}}(q,p) = \frac{e^{-\mathrm{i}\sqrt{q}p}}{\sqrt{q}}$ we obtain twice the affine inversion operator

$$\mathrm{M}^{a\mathscr{W}} \equiv 2\mathscr{I} = \int_{\Pi_+} U(q,p)\,\varpi_{a\mathscr{W}}(q,p)\,\mathrm{d}q\,\mathrm{d}p\,, \quad (\mathscr{I}\psi)(x) := \frac{1}{x}\,\psi\left(\frac{1}{x}\right)\,, \quad \mathscr{I}^2 = I\,.$$

- This operator is the affine counterpart of the operator yielding the Weyl-Wigner integral quantization when the phase space is $\mathbb{R}^2$, i.e. we deal with Weyl-Heisenberg symmetry.

Proposition

The integral kernel of the quantization of a function $f(q,p)$ through the weight function has the following expression,

$$\mathscr{A}_f^{a\mathscr{W}}(x,x') = \frac{1}{\sqrt{2\pi}}\,\hat{f}_p\left(\sqrt{\frac{x'}{x}}, x'-x\right)\,.$$

Proposition

(i) *The quantization of a function of q, $f(q,p) = u(q)$ provided by the weight $\varpi_{a\mathscr{W}}$ is $u(Q)$.*

(ii) *Similarly, the quantization of a function of p, $f(q,p) = v(p)$ provided by the weight $\varpi_{a\mathscr{W}}$ is $v(P)$ (in the pseudo-differential sense).*

(iii) *More generally, the quantization of a separable function $f(q,p) = u(q)\, v(p)$ provided by the weight $\varpi_{a\mathscr{W}}$ is the integral operator*

$$\left(A^{a\mathscr{W}}_{u(q)v(p)}\,\psi\right)(x) = \frac{1}{\sqrt{2\pi}}\int_0^{+\infty} \mathrm{d}x'\, \hat{v}(x'-x)\, u\left(\sqrt{x\,x'}\right)\, \psi(x')\,.$$

(iv) *In particular, the quantization of $u(q)\,p^n$, $n\in\mathbb{N}$, yields the symmetric operator,*

$$A^{a\mathscr{W}}_{u(q)p^n} = \sum_{k=0}^{n}\binom{n}{k}(-\mathrm{i})^{n-k}u^{(n-k)}(Q)\,P^k\,,$$

and for the dilation, $A^{a\mathscr{W}}_{qp} = D$

Therefore, this affine integral quantization is the exact counterpart of the Weyl-Wigner integral quantization

In quantum cosmology

- Quantum dynamics of isotropic, anisotropic non-oscillatory and anisotropic models
- Singularity resolution
- Unitary dynamics without boundary conditions
- (Consistent) semi-classical description of involved quantum dynamics

Link with Klauder's approach : proceeding in quantum theory with an "affine" quantization instead of the Weyl-Heisenberg quantization was already present in Klauder's work devoted the question of dealing with singularities in quantum gravity (see e.g. An Affinity for Affine Quantum Gravity, *Proc. Steklov Inst. of Math.* **272**, 169-176 (2011); gr-qc/1003.261 for recent references). The procedure rests on the representation of the affine Lie algebra. In this sense, it remains closer to the canonical one and it is not of the integral type.

Hamiltonian formulation from the solving of the constraint in modelling a closed Friedman universe

- FLRW models filled with barotropic fluid with equation of state $\mathrm{p} = w\rho$ and resolving Hamiltonian constraint leads to a model of singular universe $\sim$ particle moving on the half-line $(0,\infty)$.
- In appropriate affine canonical coordinates (q,p), Hamiltonian reads as

$$\{q,p\} = 1, \;\; h(q,p) = \alpha(w)p^2 + 6\tilde{k}q^{\beta(w)}, \;\; q > 0\,.$$

  with $\tilde{k} = (\int \mathrm{d}\omega)^{2/3}k$, $\alpha(w) = 3(1-w)^2/32$ and $\beta(w) = 2(3w+1)/(3(1-w))$. $k = 0, -1$ or 1 (in suitable unit of inverse area) depending on whether the universe is flat, open or closed.
- Assume a closed universe with radiation content : $w = 1/3$ and $k = +1$.

Isotropy singularity cured by affine CS quantization[a]

[a] H. Bergeron, A. Dapor, J.P. G. and P. Małkiewicz, *Phys. Rev. D* **89**, 083522 (2014); arXiv:1305.0653 [gr-qc]

- **Closed Friedman universe**:

$$\mathrm{h} = \frac{1}{24}p^2 + 6q^2 \mapsto A_{\mathrm{h}} = \frac{1}{24}P^2 + \frac{K(\psi)}{24}\frac{1}{Q^2} + 6M(\psi)Q^2$$
$$P \equiv -\mathrm{i}\frac{d}{dx}\,, \quad Q\phi(x) \equiv x\phi(x)\,, \quad K\,,\,M > 0\ \forall\,\psi$$

- **Quantum consistency**:

for $K \geqslant \frac{3}{4}$ quantum Hamiltonian A_{h} is self-adjoint, giving a unique unitary evolution

Semi-classical description

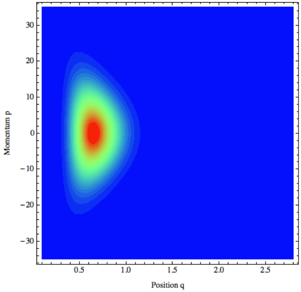
- **Semi-classical dynamics is ruled by** :

$$\langle q,p|A_{\mathrm{h}}|q,p\rangle = \mathrm{Cst}_{\psi}\int_{\mathbb{R}_+\times\mathbb{R}} \mathrm{d}q'\,\mathrm{d}p'\,|\langle q',p'|q,p\rangle|^2 \mathrm{h}(q',p')\,,$$

- with a displacement of the equilibrium point of the potential at

$$Q_{\mathrm{eq}}^4 = \frac{1}{144}\frac{K}{M}$$

Ground state ϕ_0

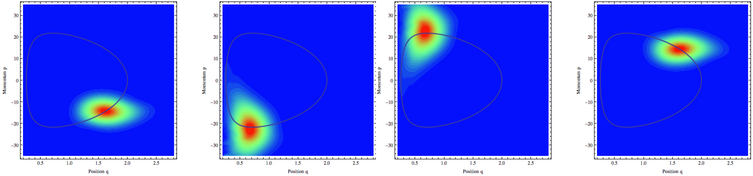


Phase space probability distribution of the ground state with a certain choice of ψ[2]

$$|\phi_0\rangle \mapsto \rho_{|\phi_0\rangle}(q,p) = \mathrm{Cst}_\psi |\langle q,p|\phi_0\rangle|^2$$

[2] This stationary quantum state of the universe is distributed around the equilibrium point q_e (minimum of the potential curve involved in the Hamiltonian). The existence of the semi-classical equilibrium point $q_e \neq 0$ is a consequence of the repulsive part of the potential.

Dynamics

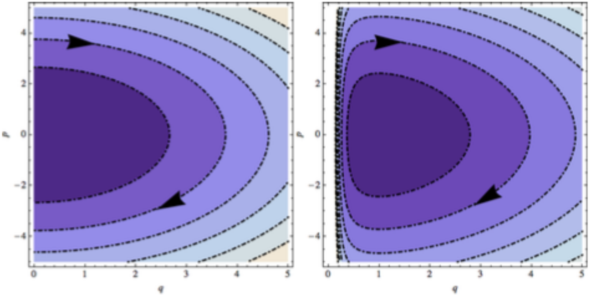


Phase space distribution

$$\rho_{|q_0,p_0\rangle}(q,p,t) = \mathrm{Cst}_{\psi} |\langle q,p|e^{-\mathrm{i}A_{\mathrm{h}}t}|q_0,p_0\rangle|^2$$

for some selected values of time t. (Fluid configuration variable is chosen as a clock of universe). Black curves are phase trajectories obtained from semi-classical ($\sim$ effective) dynamics.

Phase space trajectories



Compared contour plot of phase space trajectories for classical Hamiltonian $h(q,p)$ (left) and semi-classical Hamiltonian (right)

$$\langle q,p|A_{\mathrm{h}}|q,p\rangle = \mathrm{Cst}_{\psi}\int_{\mathbb{R}_+\times\mathbb{R}} \mathrm{d}q'\,\mathrm{d}p'\,|\langle q',p'|q,p\rangle|^2 \mathrm{h}(q',p')$$

A “semiclassical” Friedmann equation

- As the result of affine quantization we obtain two corrections to the Friedmann equation which reads in its semi-classical version as

$$\left(\frac{\dot{a}}{a}\right)^2 + c^2 a_P^2 (1-w)^2 A \frac{1}{V^2} + B\frac{kc^2}{a^2} = \frac{8\pi G}{3c^2}\rho \,,$$

where A and B are positive factor dependent on the fiducial ψ and can be adjusted at will in consistence with (so far very hypothetical!) observations.

- The first correction is the repulsive potential, which depends on the volume. As the singularity is approached $a \to 0$, this potential grows faster ($\sim a^{-6}$) than the density of fluid ($\sim a^{-3(1+w)}$) and therefore at some point the contraction must come to a halt.
- Second, the curvature becomes dressed by the factor B. This effect could in principle be observed far away from the quantum phase. However, we do not observe the intrinsic curvature neither in the geometry nor in the dynamics of space. Nevertheless, for a convenient choice of ψ, this factor ≈ 1
- **The form of the repulsive potential does not depend on the state of fluid filling the universe: the origin of singularity avoidance is quantum geometrical.**

Anisotropy singularity[a]

[a] H Bergeron, A Dapor, J.P.G., and P Małkiewicz, *Phys. Rev D* **91** 124002 (2015); arXiv:1501.07718 [gr-qc]

- **Bianchi type I model**:

$$\mathrm{h} = \frac{1}{24}p^2 - \frac{\mathrm{Cst}}{24}\left(p_+^2 + p_-^2\right) q^{-2}$$

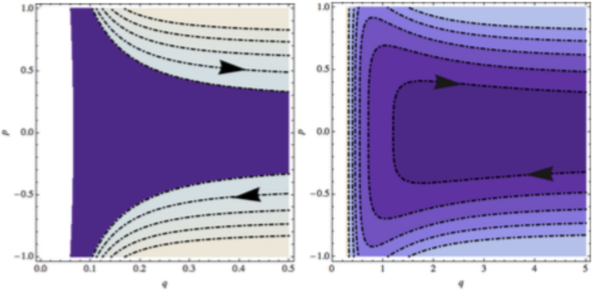
- **Positivity constraint**:

$$\mathrm{h} > 0$$

- **Affine CS quantization of the singular $\theta(\mathrm{h})\,\mathrm{h}$, θ is Heaviside**:

$$\theta(\mathrm{h})\,\mathrm{h} \mapsto A_{\theta(\mathrm{h})\,\mathrm{h}} = \frac{1}{24}P^2 + \frac{K(\psi)}{24}\frac{1}{Q^2} - 6N(\psi)\frac{\mathrm{Cst}}{Q^2} + \cdots$$

Phase space trajectories



Compared contour plot of phase space trajectories for classical Hamiltonian $\theta(h)h(q,p)$ (left) and semi-classical Hamiltonian (right)

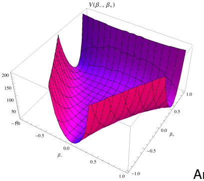
$$\langle q,p|A_{\mathrm{h}}|q,p\rangle = \mathrm{Cst}_\psi \int_{\mathbb{R}_+\times\mathbb{R}} \mathrm{d}q'\,\mathrm{d}p'\,|\langle q',p'|q,p\rangle|^2\,\theta(\mathrm{h})\,\mathrm{h}(q',p')$$

$$\sim p^2 + \widetilde{K}(\psi)\,\frac{1}{q^2} - \widetilde{L}(\psi)\,\frac{k^2}{q^2} + F(k^2, p^2 q^2)$$

Oscillatory singularity[a]

[a] H. Bergeron, E. Czuchry, J.P. G., P. Małkiewicz, and W. Piechocki, *Phys. Rev. D*; arXiv:1501.02174 [gr-qc]; *Phys. Rev. D*, **92**, 124018; arXiv:1501.07871 [gr-qc]

- **Vacuum Bianchi type IX (Mixmaster)**:



Anisotropy potential $V(\beta_\pm)$

$$\mathscr{C} = \frac{3}{16}p^2 + \frac{3}{4}q^{2/3} - \mathrm{h}_q^{\mathrm{anis}} \equiv \mathrm{h}_q^{\mathrm{is}} - \mathrm{h}_q^{\mathrm{anis}}$$

$$\mathrm{h}_q^{\mathrm{anis}} = \frac{1}{12q^2}\left(p_+^2 + p_-^2\right) + \frac{3}{4}q^{2/3}V(\beta_\pm)$$

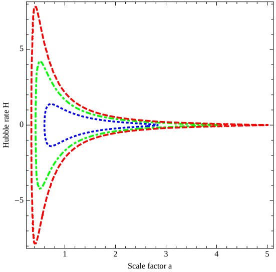
- **Affine CS + canonical quantization**:

$$\mathscr{C} \mapsto A_{\mathscr{C}} = \frac{3}{16}\left(P^2 + \hbar^2\frac{\mathscr{K}_1(\psi)}{Q^2}\right) + \frac{3}{4}\mathscr{K}_3(\psi)\,Q^{2/3} - A_{\mathrm{h}_q^{\mathrm{anis}}} \equiv A_{\mathrm{h}_q^{\mathrm{is}}} - A_{\mathrm{h}_q^{\mathrm{anis}}}$$

$$A_{\mathrm{h}_q^{\mathrm{anis}}} = \sum_N E_N(Q)|e_N(Q)\rangle\langle e_N(Q)|$$

Adiabatic ($\sim$ Born-Oppenheimer) approximation

$$|\Psi\rangle = |q,p\rangle \otimes |e_N\rangle$$



Three periodic semiclassical trajectories in the half-plane (a,H)

$$\langle q,p|A_{\mathscr{C}}|q,p\rangle = \frac{3}{16}\left(p^2 + \hbar^2\frac{\mathscr{K}_4(\psi)}{q^2}\right) + \frac{3}{4}\mathscr{K}_5(\psi)\, q^{2/3} - E_N(q)$$

Vibronic approach[a]

Mix semi-classical dynamics for (q,p) with quantum dynamics for anisotropy

[a] H. Bergeron, E. Czuchry, J.P. G., and P. Małkiewicz, submitted; arXiv: 1511.05790[gr-qc]; arXiv:1512.00304v1 [gr-qc]
D. R. Yarkony, "Nonadiabatic Quantum ChemistryPast, Present, and Future", Chem. Rev. **112**, 481 (2012).

- **General state**:

$$|\Psi\rangle = |q,p\rangle \otimes |e\rangle \qquad |e\rangle = \sum_N \lambda_N |e_N\rangle$$

- **Semi-classical Hamiltonian dynamics (Klauder)**:

$$\dot{q} = \mathscr{N}\frac{\partial\langle\Psi|A_{\mathscr{C}}|\Psi\rangle}{\partial p}\,,\quad \dot{p} = -\mathscr{N}\frac{\partial\langle\Psi|A_{\mathscr{C}}|\Psi\rangle}{\partial q}\,,\quad \frac{\hbar}{\mathrm{i}}\frac{\partial|e\rangle}{\partial t} = \mathscr{N}\langle q,p|A_{\mathrm{h}_q^{\mathrm{anis}}}|q,p\rangle\,|e\rangle$$

- **Quantized constraint is semi-classically consistent**:

$$\mathsf{L}\left(\Psi,\dot{\Psi},\mathscr{N}\right) = \langle\Psi(t)\left|\left(i\hbar\frac{\partial}{\partial t} - \mathscr{N}\mathscr{C}\right)\right|\Psi(t)\rangle \Rightarrow \frac{\partial \mathsf{L}}{\partial\mathscr{N}} = \langle\Psi|A_{\mathscr{C}}|\Psi\rangle = 0$$

Beyond BO approximation: vibronic approach continued 1

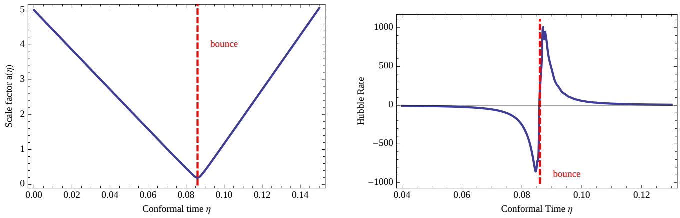
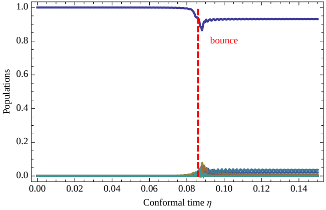


Figure: The evolution of the scale factor $a(\eta)$ (left panel) and the Hubble rate (right panel) as a function of the conformal time η. The initial value of a is $a_0 = 5$ and the initial state is $|\phi_0^{(\mathrm{int})}\rangle = |0\rangle$.

Vibronic approach continued 2



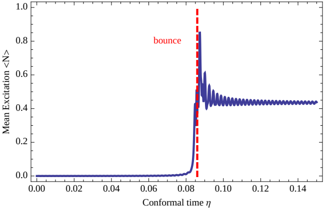


Figure: Evolution of the quantum state with conformal time when the initial value of a is $a_0 = 5$ and the initial state is $|\phi_0^{(\mathrm{int})}\rangle = |0\rangle$. On the left panel the evolution of the populations $|c_n(\eta)|^2$ for $n = 0, 1, \ldots, 12$. $|c_0(\eta)|^2$ corresponds to the curve on the top. On the right panel, the mean excitation $< \hat{\mathrm{N}} > (\eta)$.

Beyond BO approximation: vibronic approach 3

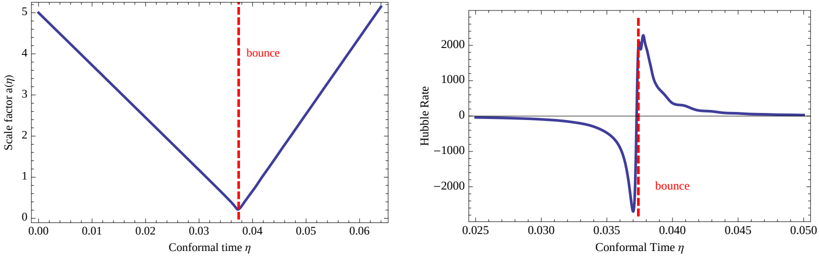


Figure: The evolution of the scale factor $a(\eta)$ (left panel) and the Hubble rate (right panel) as a function of the conformal time η. The initial value of a is $a_0 = 5$ and the initial state is $|\phi_0^{(\mathrm{int})}\rangle = |n = 2\rangle$.

Vibronic approach continued 4

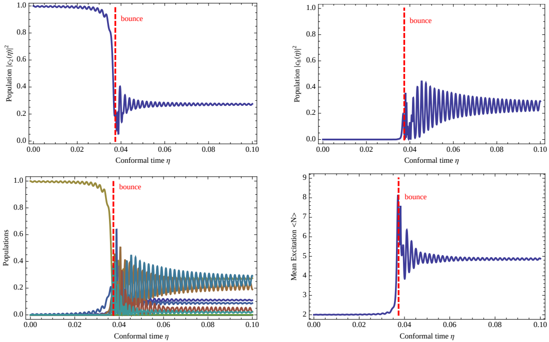


Figure: Evolution of the quantum state with conformal time when the initial value of a is $a_0 = 5$ and the initial state is $|\phi_0^{(\mathrm{int})}\rangle = |n=2\rangle$. On the top left panel the decay of the initial level $n=2$. On the top right panel the excitation of the level $n=8$. On the bottom left panel the evolution of the populations $|c_n(\eta)|^2$ for $n = 0, 1, \ldots, 12$. On the bottom right panel, the mean excitation $<\hat{\mathrm{N}}>(\eta)$.

Summary[a]

[a] for other approaches based on affine symmetry see
M. Fanuel and S. Zonetti, Affine Quantization and the Initial Cosmological Singularity, *Eur. Phys. Lett.* **101**, 10001 (2013);
J. R. Klauder, An Affinity for Affine Quantum Gravity, *Proc. Steklov Institute of Mathematics* **272**, 169-176 (2011); gr-qc/1003.261

- **ACS resolve the hardest singularities**
- **ACS provide a manageable semiclassical description**
- **ACS combined with molecular physics like Born-Oppenheimer-Huang approximations provide a description of oscillatory singularities**
- **Other developments: "multiple choice problems" in QG, quantum theory of cosmological perturbations on quantum backgrounds**

The knowledge of anything, since all things have causes, is not acquired or complete unless it is known by its causes.

Ibn Sînâ 980-1037

- In the absence of square-integrability over G, (e.g. Weyl Heisenberg group, Euclidean group, Galileo group, Poincaré group ....), there exists a definition of square-integrable representation with respect to a left coset manifold $X = G/H$, with H a closed subgroup of G, equipped with a quasi-invariant measure ν.[3]
- For a global Borel section $\sigma : X \to G$ of the group, let ν_σ be the unique quasi-invariant measure defined by

$$\mathrm{d}\nu_\sigma(x) = \lambda(\sigma(x), x)\,\mathrm{d}\nu(x)\,,$$

where $\lambda(g,x)\,\mathrm{d}\nu(x) = \mathrm{d}\nu(g^{-1}x),\ (\forall g \in G)$

- A UIR U is said square integrable $\mathrm{mod}(H,\sigma)$ with respect to the density operator ρ if

$$c_\rho := \int_X \mathrm{tr}\left(\rho\,\rho_\sigma(x)\right)\,\mathrm{d}\nu_\sigma(x) < \infty$$

with $\rho_\sigma(x) = U(\sigma(x))\rho U(\sigma(x))^\dagger$.

- Then we have the resolution of the identity and the resulting quantization

$$f \mapsto A_f = \frac{1}{c_\rho}\int_X f(x)\,\rho_\sigma(x)\,\mathrm{d}\nu_\sigma(x)$$

[3] S. T. Ali, J.-P. Antoine, and J.-P. G., *Coherent States, Wavelets and their Generalizations* (Graduate Texts in Mathematics, Springer, New York, 2000). New edition in 2014

- Covariance holds here too in the following sense.

$$U(g)A_f U(g)^\dagger = A^{\sigma_g}_{\mathfrak{U}_r(g)f}\,, \text{ with } A^{\sigma_g}_f = \frac{1}{c_\rho}\int_X f(x)\,\rho_{\sigma_g}(x)\,\mathrm{d}\nu_{\sigma_g}(x)\,.$$

- Here, the sections $\sigma_g : X \mapsto G$, $g \in G$ are covariant translates of σ under g,

$$\sigma_g(x) = g\sigma(g^{-1}\cdot x) = \sigma(x)\,h\left(g, g^{-1}x\right)$$

where h is the cocycle defined by the factorisation

$$g\sigma(x) = \sigma(g\cdot x)\,h(g,x)\,, \quad h(g_1g_2,x) = h(g_1, g_2\cdot x)h(g_2,x)\,,$$

and the measure $\mathrm{d}\nu_{\sigma_g}$ is defined consistently to (**??**) by

$$\mathrm{d}\nu_{\sigma_g}(x) = \lambda\left(\sigma_g(x), x\right)\,\mathrm{d}\nu(x)\,.$$

Besides the Weyl-Heisenberg group, another example concerns the motion on the circle for which G is the group of Euclidean displacements in the plane, i.e. the semi-direct product $\mathbb{R}^2 \rtimes \mathrm{SO}(2)$, and the subgroup H is isomorphic to $\mathbb{R}$. Other examples involve the relativity groups, Galileo, Poincaré, 1+ 1 Anti de Sitter (unit disk and SU(1,1)).

- Let U be a UIR of G and $\varpi(x)$ be a function (the "weight") on the coset $X = G/H$. We will explain later the meaning of this function from a physical point of view.
- Suppose that it allows to define a bounded operator $\mathsf{M}^{\varpi}_{\sigma}$ on $\mathscr{H}$ through the operator-valued integral

$$\mathsf{M}^{\varpi}_{\sigma} = \int_X \varpi(x)\,\mathsf{C}^{1/2}\,U(\sigma(x))\,\mathsf{C}^{1/2}\,\mathrm{d}\nu_{\sigma}(x)\,.$$

  where the positive invertible operator C is (optionnally) included in order to make the above operator-valued integral converge in a weak sense.
- Then, under appropriate conditions on C and on the weight function $\varpi(\sigma(x))$ such that U be a UIR which is square integrable $\mathrm{mod}(H)$ and M is admissible in the above sense, the family of transported operators $\mathsf{M}^{\varpi}_{\sigma}(x) := U(\sigma(x))\mathsf{M}^{\varpi}_{\sigma}U(\sigma(x))^{\dagger}$ resolves the identity.

Weyl-Heisenberg group and algebra, Fock or number representation

- Weyl-Heisenberg group $G_{\mathrm{WH}} = \{(s,z)\,,\, s \in \mathbb{R}\,,\, z \in \mathbb{C}\}$ with multiplication law
$$(s,z)(s',z') = (s+s'+\mathrm{Im}(z\bar{z}'), z+z')$$
- Let $\mathscr{H}$ be a separable (complex) Hilbert space with orthonormal basis $e_0, e_1, \dots, e_n \equiv |e_n\rangle, \dots$, (e.g. the Fock space with $|e_n\rangle \equiv |n\rangle$).
- Lowering and raising operators a and $a^\dagger$:
$$a|e_n\rangle = \sqrt{n}|e_{n-1}\rangle\,, \quad a|e_0\rangle = 0\,,$$
$$a^\dagger|e_n\rangle = \sqrt{n+1}|e_{n+1}\rangle\,.$$
- Operator algebra $\{a, a^\dagger, 1\}$ obeys the ccr
$$[a,a^\dagger] = 1\,,$$
and represents the Lie Weyl-Heisenberg algebra
- Number operator: $N = a^\dagger a$, spectrum $\mathbb{N}$, $N|e_n\rangle = n|e_n\rangle$.

Unitary Weyl-Heisenberg group representation and standard CS

- Consider the center $C = \{(s, 0), s \in \mathbb{R}\}$ of G_{WH}. Then, set X is the coset $X = G_{\mathrm{WH}}/C \sim \mathbb{C}$ with measure $\mathrm{d}^2 z/\pi$.
- To each $z \in \mathbb{C}$ corresponds the (unitary) displacement ($\sim$ Weyl) operator $D(z)$:

$$\mathbb{C} \ni z \mapsto D(z) = e^{za^\dagger - \bar{z}a}.$$

- Space inversion $\rightarrow$ Unitarity:

$$D(-z) = (D(z))^{-1} = D(z)^\dagger .$$

- Addition formula (Quantum Mechanics in a nutshell!):

$$D(z)D(z') = e^{\frac{1}{2}(z\bar{z}' - \bar{z}z')} D(z + z') = e^{(z\bar{z}' - \bar{z}z')} D(z')D(z),$$

  i.e. $z \mapsto D(z)$ is a projective representation of the abelian group $\mathbb{C}$.
- Standard (i.e., Schrödinger-Klauder-Glauber-Sudarshan) CS

$$|z\rangle = D(z)|e_0\rangle,$$

Quantization(s) with weight function(s) I

- Let $\varpi(z)$ be a function on the complex plane obeying $\varpi(0) = 1$. Suppose that it allows to define a bounded operator M on $\mathscr{H}$ through the operator-valued integral

$$\mathrm{M}^{\varpi} = \int_{\mathbb{C}} \varpi(z) D(z) \, \frac{\mathrm{d}^2 z}{\pi} \, .$$

- Then, the family of displaced $\mathrm{M}^{\varpi}(z) := D(z)\mathrm{M}^{\varpi} D(z)^{\dagger}$ under the unitary action $D(z)$ resolves the identity

$$\int_{\mathbb{C}} \mathrm{M}^{\varpi}(z) \, \frac{\mathrm{d}^2 z}{\pi} = I \, .$$

- It is a direct consequence of $D(z)D(z')D(z)^{\dagger} = e^{z\bar{z}' - \bar{z}z'} D(z')$, of $\int_{\mathbb{C}} e^{z\bar{\xi} - \bar{z}\xi} \, \frac{\mathrm{d}^2 \xi}{\pi} = \pi \delta^2(z)$, and of $\varpi(0) = 1$ with $D(0) = I$.

Quantization(s) with weight function(s); in variable z

- The resulting quantization map is given by

$$f \mapsto A_f^{\varpi} = \int_{\mathbb{C}} \mathrm{M}^{\varpi}(z)\, f(z)\, \frac{\mathrm{d}^2 z}{\pi} = \int_{\mathbb{C}} \varpi(z)\, D(z)\, \overline{\mathfrak{f}_{\mathfrak{s}}}[f](z)\, \frac{\mathrm{d}^2 z}{\pi}\,,$$

- where are involved the symplectic Fourier transforms $\mathfrak{f}_{\mathfrak{s}}$ and its space reverse $\overline{\mathfrak{f}_{\mathfrak{s}}}$

$$\mathfrak{f}_{\mathfrak{s}}[f](z) = \int_{\mathbb{C}} e^{z\bar{\xi} - \bar{z}\xi}\, f(\xi)\, \frac{\mathrm{d}^2 \xi}{\pi}\,, \quad \overline{\mathfrak{f}_{\mathfrak{s}}}[f](z) = \mathfrak{f}_{\mathfrak{s}}[f](-z)$$

Both are unipotent $\mathfrak{f}_{\mathfrak{s}}[\mathfrak{f}_{\mathfrak{s}}[f]] = f$ and $\overline{\mathfrak{f}_{\mathfrak{s}}}[\overline{\mathfrak{f}_{\mathfrak{s}}}[f]] = f$.

Quantization(s) with weight function(s), in variables (q,p)

- The resulting quantization map is given by in terms of variables $q=(z+\bar{z})/\sqrt{2}$, $p=-i(z-\bar{z})/\sqrt{2}$ and Fourier transform, with $f(z)\equiv F(q,p)$ and $\varpi(z)\equiv \Pi(q,p)$,

$$
\begin{aligned}
A_f^{\varpi} &= \int_{\mathbb{R}^2} \mathscr{D}(q,p)\,\mathfrak{F}_{\mathfrak{s}}[F](-q,-p)\,\Pi(q,p)\,\frac{\mathrm{d}q\,\mathrm{d}p}{2\pi} \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\frac{iqp}{2}}\, e^{ipQ}\, e^{-iqP}\, e^{i(qy-px)}\, F(x,y)\,\Pi(q,p)\,\frac{\mathrm{d}q\,\mathrm{d}p}{2\pi}\,\frac{\mathrm{d}x\,\mathrm{d}y}{2\pi} \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{\frac{iqp}{2}}\, e^{-iqP}\, e^{ipQ}\, e^{i(qy-px)}\, F(x,y)\,\Pi(q,p)\,\frac{\mathrm{d}q\,\mathrm{d}p}{2\pi}\,\frac{\mathrm{d}x\,\mathrm{d}y}{2\pi}
\end{aligned}
$$

where
$\mathfrak{f}_{\mathfrak{s}}[f](z)\equiv\mathfrak{F}_{\mathfrak{s}}[F](q,p)=\int_{\mathbb{R}^2} e^{-i(qy-px)}\,F(x,y)\,\frac{\mathrm{d}x\,\mathrm{d}y}{2\pi}=\mathfrak{F}[F](-p,q)$
and $\mathfrak{F}$ denotes the standard two-dimensional Fourier transform,
$\mathfrak{F}[F](k_x,k_y)=\int_{\mathbb{R}^2} e^{-i(k_x x+k_y y)}\,F(x,y)\,\frac{\mathrm{d}x\,\mathrm{d}y}{2\pi}$

Quantization(s) with weight function(s) : Covariance

- Translation covariance:

$$A^{\varpi}_{f(z-z_0)} = D(z_0) A^{\varpi}_{f(z)} D(z_0)^{\dagger}\,.$$

- Parity covariance

$$A^{\varpi}_{f(-z)} = \mathsf{P} A^{\varpi}_{f(z)} \mathsf{P}, \forall f \quad \Longleftrightarrow \quad \varpi(z) = \varpi(-z), \forall z\,,$$

where $\mathsf{P} = \sum_{n=0}^{\infty} (-1)^n |e_n\rangle\langle e_n|$ is the parity operator.

- Complex conjugation covariance

$$A^{\varpi}_{\overline{f(z)}} = \left(A^{\varpi}_{f(z)}\right)^{\dagger}, \forall f \quad \Longleftrightarrow \quad \overline{\varpi(-z)} = \varpi(z), \forall z\,,$$

Quantization(s) with weight function(s): Rotational covariance

- Define the unitary representation $\theta \mapsto U_{\mathbb{T}}(\theta)$ of the torus $\mathbb{S}^1$ on the Hilbert space $\mathscr{H}$ as the diagonal operator

$$U_{\mathbb{T}}(\theta)|e_n\rangle = e^{i(n+\nu)\theta}|e_n\rangle\,,$$

where ν is arbitrary real.

- From the matrix elements of $D(z)$ one proves easily the rotational covariance property

$$U_{\mathbb{T}}(\theta)D(z)U_{\mathbb{T}}(\theta)^{\dagger} = D\left(e^{i\theta}z\right)\,,$$

- and its immediate consequence on the nature of M and the covariance of A^{ϖ}_f,

$$\begin{aligned} U_{\mathbb{T}}(\theta)A^{\varpi}_f U_{\mathbb{T}}(-\theta) = A^{\varpi}_{T(\theta)f} \quad &\Longleftrightarrow \quad \varpi\left(e^{i\theta}z\right) = \varpi(z)\,, \forall\, z\,,\theta \\ &\Longleftrightarrow \quad \text{M diagonal}\,, \end{aligned}$$

where $T(\theta)f(z) := f\left(e^{-i\theta}z\right)$.

CCR is always the rule!

- Canonical Commutation Rule is a permanent outcome of the above quantization, whatever the chosen complex function $\varpi(z)$, provided integrability and derivability at the origin is insured.

$$A_z^{\varpi} = a\varpi(0) - \partial_{\bar z}\varpi|_{z=0} = a - \partial_{\bar z}\varpi|_{z=0}\,, \quad A_{\bar z}^{\varpi} = a^{+}\varpi(0) + \partial_z\varpi|_{z=0} = a^{+} + \partial_z\varpi|_{z=0}\,,$$

- Equivalently, with $z = (q+ip)/\sqrt{2}$, As a result, we have

$$A_q^{\varpi} = \frac{1}{\sqrt{2}}\left[(a+a^{+}) - \partial_{\bar z}\varpi|_{z=0} + \partial_z\varpi|_{z=0}\right]\,,$$
$$A_p^{\varpi} = \frac{1}{\sqrt{2}i}\left[(a-a^{+}) - \partial_{\bar z}\varpi|_{z=0} - \partial_z\varpi|_{z=0}\right]\,,$$

- From this the commutation relation becomes ccr,

$$A_q^{\varpi}A_p^{\varpi} - A_p^{\varpi}A_q^{\varpi} = i\left[a, a^{+}\right] = iI\,,$$

- Moreover, if $|\varpi(z)| = 1$

$$\mathrm{tr}\left((A_f^{\varpi})^{\dagger}A_f^{\varpi}\right) = \int_{\mathbb{C}} |f(z)|^2 \frac{\mathrm{d}^2 z}{\pi}\,,$$

which means that the map $f \mapsto A_f^{\varpi}$ is invertible through a trace formula.

Wigner-Weyl, CS, normal, and other, quantizations

- The normal, Wigner-Weyl and anti-normal (i.e., anti-Wick or Berezin or CS) quantizations correspond to $s \to 1_-$, $s = 0$, $s = -1$ resp. in the specific choice [4]

$$\varpi_s(z) = e^{s|z|^2/2}\,, \quad \mathrm{Re}\, s < 1.$$

- This yields a diagonal $\mathrm{M}^{\varpi} \equiv \mathrm{M}_s$ with

$$\langle e_n|\mathrm{M}_s|e_n\rangle = \frac{2}{1-s}\left(\frac{s+1}{s-1}\right)^n\,,$$

and so

$$\mathrm{M}_s = \int_{\mathbb{C}} \varpi_s(z) D(z)\, \frac{\mathrm{d}^2 z}{\pi} = \frac{2}{1-s} \exp\left[\ln\left(\frac{s+1}{s-1}\right) a^{\dagger} a\right].$$

[4] K.E. Cahill and R. Glauber, Ordered expansion in Boson Amplitude Operators, *Phys. Rev.* **117** 1857-1881 (1969)

Wigner-Weyl, CS, normal, and other, quantizations II

- The case $s=-1$ corresponds to the CS (anti-normal) quantization, since

$$\mathsf{M} = \lim_{s\to -1} \frac{2}{1-s} \exp\left(\ln \frac{s+1}{s-1} a^\dagger a\right) = |e_0\rangle\langle e_0|\,,$$

and so

$$A_f^{\varpi} = \int_{\mathbb{C}} D(z)\mathsf{M}D(z)^\dagger f(z)\,\frac{\mathrm{d}^2 z}{\pi} = \int_{\mathbb{C}} |z\rangle\langle z|\, f(z)\,\frac{\mathrm{d}^2 z}{\pi}\,.$$

- The choice $s=0$ implies $\mathsf{M}=2\mathsf{P}$ and corresponds to the Wigner-Weyl quantization. Then

$$A_f^{\varpi} = \int_{\mathbb{C}} D(z)\,2\mathsf{P}\,D(z)^\dagger f(z)\,\frac{\mathrm{d}^2 z}{\pi}\,.$$

- The case $s=1$ is the normal quantization in an asymptotic sense.
- The parameter s was originally introduced by Cahill and Glauber in view of discussing the problem of expanding an arbitrary operator as an ordered power series in a and $a^\dagger$, a typical question encountered in quantum field theory, specially in quantum optics.

Canonical quantization with POVM or not

- Operator M_s is positive unit trace class for $s \leqslant -1$ (and only trace class if $\mathrm{Re}\, s < 0$; for $s = 0$, see e.g. [5]), i.e., is density operator: quantization has a consistent probabilistic content, the operator-valued measure

$$\mathbb{C} \supset \Delta \mapsto \int_{\Delta \in \mathscr{B}(\mathbb{C})} D(z)\mathrm{M}_s D(z)^\dagger \, \frac{\mathrm{d}^2 z}{\pi}\,,$$

  is a normalized positive operator-valued measure.

- Given an elementary quantum energy, say $\hbar\omega$ and with the temperature T-dependent $s = -\coth \dfrac{\hbar\omega}{2k_B T}$ the density operator quantization is Boltzmann-Planck (*thermal state* in Quantum Optics)

$$\rho_s = \left(1 - e^{-\frac{\hbar\omega}{k_B T}}\right) \sum_{n=0}^{\infty} e^{-\frac{n\hbar\omega}{k_B T}} |e_n\rangle\langle e_n|\,.$$

- Interestingly, the temperature-dependent operators $\rho_s(z) = D(z)\,\rho_s\,D(z)^\dagger$ defines a Weyl-Heisenberg covariant family of POVM's on the phase space $\mathbb{C}$, the null temperature limit case being the POVM built from standard CS.

[5] A. Grossmann, Parity operator and quantization of δ-functions, *Commun. Math. Phys.*, **48** (1976); I. Daubechies, On the distributions corresponding to bounded operators in the Weyl quantization, *Commun. Math. Phys.* **75** (1980)

Variations on the Wigner function

- The Wigner function is (up to a constant factor) the Weyl transform of the quantum-mechanical density operator. For a particle in one dimension it takes the form (in units $\hbar = 1$)

$$\mathfrak{W}(q,p) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left\langle q - \frac{y}{2}\right| \rho \left| q + \frac{y}{2}\right\rangle e^{ipy}\,\mathrm{d}y\,.$$

- Adapting this definition to the present context, and given an operator A, the corresponding Wigner function is defined as

$$\mathfrak{W}_A(z) = \mathrm{tr}\left(D(z)2\mathrm{P}D(z)^{\dagger}A\right)\,,$$

- This becomes in the case of Weyl-Wigner quantization

$$\mathfrak{W}_{A^{\varpi}_f} = f$$

(this one-to-one correspondence of the Weyl quantization is related to the isometry property).

Variations on the Wigner function (continued)

- In the case of the anti-normal quantization, the above convolution corresponds to the Husimi transform (when f is the Wigner transform of a quantum pure state).
- In the case of the quantization map $f \mapsto A_f^{\varpi}$ based on a general weight function ϖ, we get the "lower symbol" $\check{f}$ of A_f^{ϖ}

$$\mathfrak{W}_{A_f^{\varpi}}(z) \equiv \check{f}(z) = \int_{\mathbb{C}} \mathfrak{f}_{\mathfrak{s}}\left[\varpi\,\widetilde{\varpi}\right](\xi - z)\, f(\xi)\, \frac{\mathrm{d}^2\xi}{\pi} = \int_{\mathbb{C}} \varpi(\xi)\,\varpi(-\xi)\,\overline{\mathfrak{f}_{\mathfrak{s}}}[\mathrm{t}_{-z} f](\xi)\, \frac{\mathrm{d}^2\xi}{\pi}$$

where $(\mathrm{t}_{z_0} f)(z) := f(z - z_0)$.

- Hence the map $f \mapsto \check{f}$ is an (Berezin-like) integral transform with kernel $\mathfrak{f}_{\mathfrak{s}}\left[\varpi\,\widetilde{\varpi}\right](\xi - z)$.
- If this kernel is positive, it is a probability distribution and the map $f \mapsto \check{f}$ is interpreted as an averaging.
- In general this map $A \mapsto \mathfrak{W}_A$ is only the dual of the quantization map $f \mapsto A_f^{\varpi}$ in the sense that

$$\int_{\mathbb{C}} \mathfrak{W}_A(z) f(z) \frac{\mathrm{d}^2 z}{\pi} = \mathrm{tr}(A A_f^{\varpi}).$$

- This dual map becomes the inverse of the quantization map only in the case of a Hilbertian isometry.

Quantum harmonic oscillator according to ϖ

- For real even ϖ,

$$A^{\varpi}_{q^2} = Q^2 - \partial_z\partial_{\bar z}\varpi|_{z=0} + \frac{1}{2}\left(\partial_z^2\varpi\Big|_{z=0} + \partial_{\bar z}^2\varpi\Big|_{z=0}\right) ,$$
$$A^{\varpi}_{p^2} = P^2 - \partial_z\partial_{\bar z}\varpi|_{z=0} - \frac{1}{2}\left(\partial_z^2\varpi\Big|_{z=0} + \partial_{\bar z}^2\varpi\Big|_{z=0}\right)$$

and so

$$A^{\varpi}_{|z|^2} \equiv A^{\varpi}_J = a^\dagger a + \frac{1}{2} - \partial_z\partial_{\bar z}\varpi|_{z=0} .$$

where $|z|^2 (= J)$ is the energy (or action variable) for the H.O.

Quantum harmonic oscillator according to ϖ (continued)

- The difference between the ground state energy $E_0 = 1/2 - \partial_z \partial_{\bar{z}} \varpi|_{z=0}$, and the **minimum** of the **quantum** potential energy $E_m = [\min(A^{\varpi}_{q^2}) + \min(A^{\varpi}_{p^2})]/2 = -\,\partial_z \partial_{\bar{z}} \varpi|_{z=0}$ is independent of the particular (regular) quantization chosen, namely $E_0 - E_m = 1/2$ (experimentally verified in 1925).
- In the exponential Cahill-Glauber case $\varpi_s(z) = e^{s|z|^2/2}$ the above operators reduce to

$$A^{\varpi}_{|z|^2} = a^{\dagger} a + \frac{1-s}{2}\,, A^{\varpi}_{q^2} = Q^2 - \frac{s}{2}\,,\; A^{\varpi}_{p^2} = P^2 - \frac{s}{2}\,.$$

- It has been proven [6] that these constant shifts in energy are inaccessible to measurement.

[6] H. Bergeron, J.P. G., A. Youssef, Are the Weyl and coherent state descriptions physically equivalent?, Physics Letters A 377 (2013) 598605

What is the meaning of ϖ?

- To one choice of ϖ corresponds a certain ordering
- From $\mathrm{Tr}\, D(z) = \pi\delta^{(2)}(z)$, M^{ϖ}, if $\int$ and Tr commute, is unit trace.
- Necessary condition on $\varpi(z)$ for that $\mathsf{M}^{\varpi}(z)$ define a normalized Positive Operator Valued Measure (POVM)

$$\forall z\,,\ 0 < \langle z|\mathsf{M}^{\varpi}|z\rangle = \overline{\mathfrak{f}_{\mathfrak{s}}}\left[e^{-\frac{|\xi|^2}{2}}\,\varpi(\xi)\right](z) = \frac{2}{\pi}\overline{\mathfrak{f}_{\mathfrak{s}}}\left[e^{-\frac{|\xi|^2}{2}}\right] * \overline{\mathfrak{f}_{\mathfrak{s}}}[\varpi(\xi)](z)\,.$$

Quantizations according to ϖ

- If ϖ is even and real, then

$$A^{\varpi}_{z} = a\,, \quad A^{\varpi}_{\overline{f(z)}} = \left(A^{\varpi}_{f(z)}\right)^{\dagger}\,.$$

- ϖ is isotropic then the quantization is rotational covariant

- If ϖ is real valued and depends on $\mathrm{Im}\,(z^2) = qp$ like the Born-Jordan weight $\varpi(z) = \dfrac{\sin qp}{qp}$, then

$$A^{\varpi}_{f(q)} = f(Q)\,, \quad A^{\varpi}_{f(p)} = f(P)\,.$$

Only one physical constant ($\sim \hbar$), is needed to quantize, but classical singularities are preserved.

- if $|\varpi(z)| = 1$ for all z then

$$\mathrm{tr}\left(\left(A^{\varpi}_{f}\right)^{\dagger} A^{\varpi}_{f}\right) = \int_{\mathbb{C}} |f(z)|^2 \frac{\mathrm{d}^2 z}{\pi}\,.$$

$f \mapsto A^{\varpi}_{f}$ is then invertible (the inverse is given by a trace formula), and we have the trace formula

$$\mathrm{tr}\left(\left(A^{\varpi}_{f}\right)^{\dagger} A^{\varpi}_{f}\right) = \int_{\mathbb{C}} \frac{\mathrm{d}^2 z}{\pi} |\varpi(z)|^2 |\overline{\mathfrak{f}_{\mathfrak{s}}}(z)|^2\,.$$

From the invariance of the L^2-norm under symplectic transform, we find that $f \mapsto A^{\varpi}_{f}$ is isometric.

Final comments

Beyond the freedom (think to analogy with Signal Analysis where different techniques are complementary) allowed by integral quantization, the advantages of the method with regard to other quantization procedures in use are of four types.

(i) The minimal amount of constraints imposed to the classical objects to be quantized.

(ii) Once a choice of (positive) operator-valued measure has been made, which must be consistent with experiment, there is no ambiguity in the issue, to one classical object corresponds one and only one quantum object. Of course different choices are requested to be physically equivalent

(iii) The method produces in essence a regularizing effect, at the exception of certain choices, like the Weyl-Wigner (i.e. canonical) integral quantization.

(iv) The method, through POVM choices, offers the possibility to take benefit of probabilistic interpretation on a semi-classical level. As a matter of fact, the Weyl-Wigner integral quantization does not rest on a POVM.

A world of mathematical models for one "thing"

- The physical laws are expressed in terms of combinations of mathematical symbols
- This mathematical language is in constant development since the set of phenomenons which are accessible to our understanding is constantly broadening.
- These combinations take place within a mathematical model.
- A model is usually scale dependent. It depends on a ratio of physical (i.e. measurable) quantities, like lengths, time(s), sizes, impulsions, actions, energies, etc
- Changing scale for a model amounts to "quantize" or "de-quantize". One changes perspective.
- The understanding changes its glasses!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