# Twisted K-Classes and 1-Cocycles

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**Pedram Hekmati**
School of Mathematical Sciences
THE UNIVERSITY of ADELAIDE

# Outline

- Preliminaries on twisted K-theory and loop groups $LG$.
- Dirac family construction of classes in $K^*_G(G, \Omega)$.
- Generalisation to gauge groups $\mathcal{G}$ and obstacles.
- Fractional loop group $L_q G$ and 1-cocycles.

(Joint work with Jouko Mickelsson)

# Twisted K-theory

- Let $X$ be a paracompact, Hausdorff topological space and $\mathcal{H}$ an infinite-dimensional complex separable Hilbert space.

- Let $Fred(\mathcal{H})$ denote the space of bounded Fredholm operators on $\mathcal{H}$ (with norm topology), and $Fred_*(\mathcal{H})$ the subspace of bounded self-adjoint Fredholm operators with both positive and negative essential spectrum.

- (Atiyah-Jänich)

$$K^0(X) = [X, Fred(\mathcal{H})], \quad K^1(X) = [X, Fred_*(\mathcal{H})]$$

- A gerbe introduces a twist in K-theory on $X$ and is characterised by an element $\Omega \in H^3(X, \mathbb{Z})$.

# Twisted K-theory

- Recall that $PU(\mathcal{H}) = U(\mathcal{H})/S^1 \simeq K(\mathbb{Z}, 2)$. Let $P_\Omega$ denote a principal $PU(\mathcal{H})$-bundle determined by its Dixmier-Douady class $\Omega \in H^3(X, \mathbb{Z})$.

- The projective unitary group $PU(\mathcal{H})$ acts continuously on $Fred(\mathcal{H})$ by the conjugation action of $U(\mathcal{H})$. Let

$$P_\Omega(Fred) = P_\Omega \times_{PU(\mathcal{H})} Fred(\mathcal{H})$$

denote the associated bundle of Fredholm operators on $X$.

- Twisted K-theory groups are defined by

$$K^0(X, \Omega) = \pi_0(\Gamma_c(X, P_\Omega(Fred)))$$

$$K^1(X, \Omega) = \pi_0(\Gamma_c(X, P_\Omega(Fred_*)))$$

i.e. homotopy classes of compactly supported continuous sections of the associated bundles.

# Twisted K-theory

- A twisted $K$-class is a family of locally defined Fredholm operators $T_i : U_i \to Fred(\mathcal{H})$ satisfying

$$T_j(x) = Ad_{\hat{g}_{ij}}(T_i)(x)$$

on contractible intersections, where $\hat{g}_{ij}$ are lifts of the transition functions $g_{ij} : U_i \cap U_j \to PU(\mathcal{H})$ to the unitary group $U(\mathcal{H})$.

- Equivalently, a twisted $K$-class is a $PU(\mathcal{H})$-equivariant map $T : P_\Omega \to Fred(\mathcal{H})$, i.e.

$$T(pg) = g^{-1} T(p) g$$

for all $g \in PU(\mathcal{H})$.

# Loop group

- Let $G$ be a compact Lie group. The loop group $LG = C^\infty(S^1, G)$ has many central extensions

$$1 \to S^1 \to \widehat{LG} \to LG \to 1 \ .$$

For $G$ simple and simply connected, there is a *universal* central extension.

- Construction: $\widehat{LG} = (DG \times_\gamma S^1)/C^\infty(S^2, G)_0$

- The principal $S^1$-bundle $\widehat{LG}$ is determined by its first Chern class up to isomorphism. When $G$ is connected and simply connected, the transgression map

$$H^2(LG, \mathbb{Z}) \to H^3(G, \mathbb{Z})$$

is an isomorphism.

# Central extension

- The corresponding Lie algebra extension

$$0 \to i\mathbb{R} \to \widehat{L\mathfrak{g}} \to L\mathfrak{g} \to 0$$

is determined by the 2-cocycle

$$c_0(X, Y) = \frac{k}{2\pi} \int_{S^1} \langle X, dY \rangle_{\mathfrak{g}} \ ,$$

where $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ is a symmetric invariant bilinear form on $\mathfrak{g}$.

- In Fourier basis, the generators $T^a_m = T^a z^m$ of the Lie algebra $\widehat{L\mathfrak{g}}$ satisfy

$$[T^a_m, T^b_n] = \sum_{c=1}^{\dim\mathfrak{g}} \lambda^{abc} T^c_{m+n} + km\delta_{m,-n} \langle T^a, T^b \rangle_{\mathfrak{g}}$$

where the central element $k$ is called the *level*.

# Positive energy representations of $LG$

- $LG$ has a distinguished class of unitary irreducible integrable projective highest weight representations $V_{(k,\lambda)}$, labelled by the level $k \in \mathbb{Z}_+$ and dominant integral weights $\lambda$ of $\mathfrak{g}$.

- $V_{(k,\lambda)}$ can be obtained from geometric quantization of affine coadjoint orbits at level $k$.

- For a given level $k$, there are only finitely many irreducible representations.

- The free abelian group generated by isomorphism classes of irreducible representations,

$$R(LG, k) = R(G)/\mathcal{I}_k$$

forms a ring under fusion product, the Verlinde algebra.

# Twisted K-theory class on $G$

- Let $G$ be a compact, connected, simply connected, simple Lie group. Then
$$H^3(G,\mathbb{Z}) = H^3_G(G,\mathbb{Z}) = \mathbb{Z}$$
(with generator $\Omega_0 = \frac{1}{24\pi^2}\mathrm{Tr}(g^{-1}dg)^3$, so that $\Omega = k\Omega_0$).

- Let $\mathcal{A}_{S^1}$ denote the affine space of connections on $Q = S^1 \times G$.

- We have the universal $\Omega G$-bundle
$$\Omega G \to \mathcal{A}_{S^1} \to G\ ,$$
where $\Omega G$ is the group of based loops, acting on $\mathcal{A}_{S^1}$ by gauge transformations
$$\Omega G \times \mathcal{A}_{S^1} \to \mathcal{A}_{S^1}, \quad (g, A) \mapsto A^g = g^{-1}Ag + g^{-1}dg\ ,$$
and the projection $\mathcal{A}_{S^1} \to G$ is given by the holonomy around the circle. Note that $LG = \Omega G \rtimes G$.

# Twisted K-theory class on $G$

- The gerbe associated to $\Omega = k\Omega_0$ is given by

$$P_\Omega = \mathcal{A}_{S^1} \times_\Phi PU(\mathcal{H})$$

where $\Phi : LG \to PU(\mathcal{H})$ is a level $k$ projective representation.

- Next we want to construct a $PU(\mathcal{H})$-equivariant family of Fredholm operators $T : P_\Omega \to Fred(\mathcal{H})$.

- Let $V_{(k,\lambda)}$ be a level $k$ representation and $S_{(h^\vee,\rho)}$ denote the spin representation of $\widehat{L\mathfrak{g}}$.

- $S_{(h^\vee,\rho)}$ is constructed by fixing a representation of the Clifford algebra $Cliff(L\mathfrak{g})$,

$$\{\psi_m^a, \psi_p^b\} = 2\delta^{ab}\delta_{m,-p} \ .$$

# Twisted K-theory class on $G$

- The operators are realised explicitly as bilinears in the Clifford generators

$$K_m^a = -\frac{1}{4}\sum_{b,c,n} \lambda^{abc} : \psi^b_{m-n}\psi^c_n : \ .$$

- The full Hilbert space $\mathcal{H} = V_{(k,\lambda)} \otimes S_{(h^\vee,\rho)}$ carries a tensor product representation of $\widehat{L\mathfrak{g}}$ of level $k + h^\vee$. The DD-class of the gerbe is $\Omega = (k + h^\vee)\Omega_0$, where the degree shift $h^\vee$ is the dual Coxeter number of $G$.

- Consider the affine cubic Dirac operator

$$\begin{aligned} \partial\!\!\!/ &= i\sum_{a,m} : \left( T^a_m \otimes \psi^a_{-m} + \frac{1}{3}\mathbf{1} \otimes \psi^a_m K^a_{-m} \right) : \\ &= i : \left( \sum_{a,m} T^a_m \otimes \psi^a_{-m} - \frac{1}{12}\sum_{a,b,c,m,n} \mathbf{1} \otimes \lambda^{abc}\psi^a_m\psi^b_{-m-n}\psi^c_n \right) : \end{aligned}$$

acting on the Hilbert space $\mathcal{H} = V_{(k,\lambda)} \otimes S_{(h^\vee,\rho)}$.

# Twisted K-theory class on $G$

- Next perturbe $\not\partial$ by coupling to the Clifford action of $A \in \mathcal{A}_{S^1}$,

$$\not\partial_A = \not\partial + i\bar{k}\langle\psi, A\rangle$$

where $\langle\psi, A\rangle = \sum_{a,m} \psi^a_m A^a_{-m}$ and $\bar{k} = \left(\frac{k+h^\vee}{4}\right)$.

- This produces a continuous family of self-adjoint unbounded Fredholm operators $\not\partial_A$ that is $\widehat{LG}$-equivariant,

$$\Phi(g)^{-1}\not\partial_A\Phi(g) = \not\partial_{A^g}$$

where $\Phi : LG \to PU(\mathcal{H})$ is the level $k + h^\vee$ embedding of the loop group.

# Twisted K-theory class on $G$

- Replacing $\partial\!\!\!/_A$ by the approximate sign operator

$$F_A = \frac{\partial\!\!\!/_A}{(1 + \partial\!\!\!/_A^2)^{\frac{1}{2}}} \ ,$$

  we obtain a bounded family of Fredholm operators.

- $T : P_\Omega \to Fred(\mathcal{H})$, given by $T = g^{-1} F_A g$ with $g \in PU(\mathcal{H})$, determines a class in $G$-equivariant twisted K-theory on $G$. Parity of dim$G$ determines whether the class is in even or odd K-theory.

## Theorem. (Freed-Hopkins-Teleman)

*The bounded Dirac family provides an isomorphism of graded free abelian groups*

$$R(LG, k) \cong K_G^{dim\ G}(G, k + h^\vee) \ .$$

# Gauge groups

- $LG$ is the gauge group of the trivial bundle $Q = S^1 \times G$.
- Natural generalisation is to replace $S^1$ by a higher dimensional compact manifold $X$.
- We have a principal $G$-bundle $Q \to X$ with gauge group $\mathcal{G} = \Gamma(X, \text{Ad } Q)$.
- Objective: construct classes in $K^*(\mathcal{A}/\mathcal{G}, \Omega)$.
- Obstacles to merely reproducing the standard theory for the circle:

  1. No natural triangular decomposition giving meaning to the highest weight condition.
  2. Divergencies and absence of a canonical central extension.

# Example: Fock representation

- Let $\mathcal{B} = \{\psi(u), \psi^\dagger(v) \mid u, v \in \mathcal{H}\}$ denote the CAR $C^*$-algebra.
- Fix a polarisation $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ with projections $P_\pm : \mathcal{H} \to \mathcal{H}_\pm$ and $\epsilon = P_+ - P_-$ $\left(= \frac{\not{D}}{|\not{D}|}\right)$.
- Free Fock space

$$\mathcal{F}_0 = \mathcal{B}/\langle \psi^\dagger(P_- u), \psi(P_+ v)\rangle = \bigoplus_{p,q} \bigwedge^p \mathcal{H}_+ \otimes \bigwedge^q \mathcal{H}_-^* \ .$$

- (Shale-Stinespring) $g \in \mathcal{G}$ is implementable on $\mathcal{F}_0$ if and only if $[\epsilon, g]$ is Hilbert-Schmidt (i.e. $|[\epsilon, g]|^2$ is trace class).
- Asymptotic analysis: $[\epsilon, g]$ is Hilbert-Schmidt if and only if $\mathrm{ord}([\epsilon, g]) < -\mathrm{dim}(X)/2$.

# Example: Fock representation

- However $\mathrm{ord}([\epsilon, g]) = -1$, so there is a UV-divergency when $\dim(X)>1$.

- Regularization: Pick an appropriate family of unitaries $R : \mathcal{A} \to \mathcal{U}(\mathcal{H})$ and introduce

$$\omega(g; A) = R^{\dagger}_{A^g} g R_A$$

such that $\mathrm{Tr}|[\epsilon, \omega(g; A)]|^2 < \infty$.

- For instance when $\dim X = 3$, set $R_A = \exp(\frac{i}{4}|\not{D}|^{-1}[\not{D}, \not{A}]|\not{D}|^{-1})$.

- $\omega$ is an operator-valued 1-cocycle:

$$\omega(gg'; A) = \omega(g; A^{g'})\omega(g'; A)$$

- Associated Hilbert bundle $\mathcal{F} = \mathcal{A} \times_{\omega} \mathcal{F}_0$, where

$$(A, v) \sim (A^g, \widehat{\omega}(g; A)^{-1} v) .$$

# Fractional loop group: motivation

- There are many different kinds of loop groups,

$$L_{pol}G \subset L_{rat}G \subset L_{anal}G \subset LG \subset L^cG$$

  where $L^cG = C^0(S^1, G)$ is the Banach Lie group of continuous loops.

- We wish to study the "thicker" loop group $L_qG$, by relaxing the smoothness property of maps $g : S^1 \to G$.

- There is still a good notion of triangular decomposition, but the central extension breaks down.

# The fractional loop group $L_q G$

Definition.
*Let $G$ denote a compact Lie group and fix an embedding $G \subset U_N(\mathbb{C})$. The fractional loop group $L_q G$ for Sobolev exponent $q > \frac{1}{2}$ is the Hilbert Lie group defined by*

$$L_q G = \{g \in Map(S^1, G) \mid \|g\|^2_{2,q} = \sum_{m \in \mathbb{Z}} (1 + m^2)^q |\hat{g}_m|^2 < \infty\}$$

*where $|\hat{g}_m|$ is the standard matrix norm of the $m^{th}$ Fourier component of $g : S^1 \to G$.*

Remark: Clearly $LG \subset L_q G$, and by Rellich-Kondrachov theorem, also $L_q G \subset L^c G$.

# Spectral triple

- There is a natural spectral triple arising here,

$$(A, D^q, \mathcal{H})$$

  where $\mathcal{H} = L^2(S^1)$, an associative, commutative $*$-algebra $A = L_q\mathbb{C}$ and a *fractional Dirac operator* on the circle defined by

$$D^q f(x) = \sum_{m \in \mathbb{Z}} \text{sign}(m)|m|^q \hat{f}_m e^{imx} \ .$$

- The spectral dimension is given by $\frac{1}{q}$.
- $L_q G$ is the gauge group in "non-commutative" Yang-Mills theory.

# Critical value $q = \frac{1}{2}$

- Recall that the central extension $\widehat{L\mathfrak{g}}$ is fixed by the 2-cocycle

  $$c_0(T^a_m, T^b_n) = km\delta_{m,-n}\langle T^a, T^b\rangle_{\mathfrak{g}} \ .$$

  By

  $$\left|\left|\frac{df}{dx}\right|\right|^2_{L^2} = \sum_{m\in\mathbb{Z}} m^2|\hat{f}_m|^2 \ ,$$

  it follows that $c_0(X, Y)$ is well-defined for $X, Y \in L_q\mathfrak{g}$ if and only if $q \geq \frac{1}{2}$.

- $LG$ is not dense in $L_{\frac{1}{2}}G$.

- For $q \leq \frac{1}{2}$, the Sobolev spaces $H^q(S^1)$ are not algebras!

# The fractional loop group $L_qG$

- $L_qG$ acts as operators on the Hilbert space $\mathcal{H} = L^2(S^1, \mathbb{C}^N)$, $M : L_qG \to GL(\mathcal{H}),\ g \mapsto M_g$ by pointwise multiplication

$$(M_g\psi)(x) = g(x)\psi(x)\ .$$

- The sign operator $\epsilon = \frac{D^q}{|D^q|}$ defines an orthogonal decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ into positive and negative Fourier modes.

- Consider the p-th Schatten class

$$\mathcal{L}_{2p} = \{A \in \mathcal{B}(\mathcal{H}) \mid \ \|A\|_{2p} = \left[\mathrm{Tr}(A^\dagger A)^p\right]^{\frac{1}{2p}} < \infty\}$$

which is a two-sided ideal in the algebra of bounded operators $\mathcal{B}(\mathcal{H})$.

# The fractional loop group $L_q G$

- The subgroup $GL_p \subset GL(\mathcal{H})$ is defined by

$$GL_p = \{A \in GL(\mathcal{H}) \mid [\epsilon, A] \in \mathcal{L}_{2p}\} \, .$$

Writing elements in $GL(\mathcal{H})$ in block form with respect to the Hilbert space polarisation,

$$A = \begin{pmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{pmatrix}$$

the condition

$$[\epsilon, A] = 2 \begin{pmatrix} 0 & A_{+-} \\ -A_{-+} & 0 \end{pmatrix} \in \mathcal{L}_{2p}$$

means that the off-diagonal blocks are not “too large".

# The fractional loop group $L_q G$

- Given the topology defined by the norm

$$\|A_{++}\| + \|A_{+-}\|_{2p} + \|A_{-+}\|_{2p} + \|A_{--}\|$$

where

$$\|a\| = \sup_{\|\psi\|=1} \|a\psi\| \,,$$

$GL_p$ is a Banach Lie group with the Lie algebra

$$\mathfrak{gl}_p = \{X \in \mathcal{B}(\mathcal{H}) \mid [\epsilon, X] \in \mathcal{L}_{2p}\} \ .$$

# Fractional loop group $L_q G$

**Proposition.**

*$L_q G$ is contained in $GL_p$, if $p \geq \frac{1}{2q}$.*

**Definition.**

*The fractional loop group $L_q G$ for real index $0 < q < \frac{1}{2}$ is defined to be $L^c G \cap GL_{\frac{1}{2q}}$, with the induced Banach-Lie structure coming from the embedding.*

# Regularisation

- The cocycle defining the central extension $\widehat{L_q\mathfrak{g}}$ can be written

$$c_0(X, Y) = \frac{1}{8}\mathrm{Tr}\Big(\epsilon[[\epsilon, X], [\epsilon, Y]]\Big) = \mathrm{Tr}\Big(X_{+-}Y_{-+} - Y_{+-}X_{-+}\Big)$$

for $X, Y \in L_q\mathfrak{g}$. It diverges unless $[\epsilon, X]$ and $[\epsilon, Y]$ belong to $\mathcal{L}_2$.

- We regularise by shifting by a 1-cochain $\eta_p(X; F)$,

$$T(X) \mapsto T(X) + \eta_p(X; F),$$

where $X \in L_q\mathfrak{g}$ and in component notation $T(X) = \sum_{a,m} T^a_m X^a_{-m}$.

- The new commutation relations will be

$$[T(X), T(Y)] = T([X, Y]) + c_p(X, Y; F)$$

where

$$c_p(X, Y; F) = c_0(X, Y) + (\delta\eta_p)(X, Y; F)\ .$$

# Construction of 1-cochain

- Here $\eta_p(X;F)$ is a 1-cochain parametrised by points on the Grassmannian,

$$Gr_p = GL_p/B = \{F \in GL_p | F = F^*, F^2 = 1, F - \epsilon \in L_{2p}\} \ .$$

- Group action:

$$L_qG \times Gr_p \to Gr_p \ , \quad (g, F) \mapsto g^{-1}Fg$$

- Infinitesimal action:

$$L_q\mathfrak{g} \times Gr_p \to Gr_p \ , \quad (X, F) \mapsto [F, X]$$

- Note: $F - \epsilon = g^{-1}[\epsilon, g]$ is a “flat connection".

# Construction of 1-cochain

- The abelian group $\mathrm{Map}(Gr_p, \mathbb{C})$ is naturally a $L_q\mathfrak{g}$ module:

$$(X, f) \mapsto \mathcal{L}_X f(F) = \frac{d}{dt} f\Big(e^{-tX}(F - \epsilon)e^{tX} + t[\epsilon, X]\Big)\Big|_{t=0}$$

- Define a 1-cochain by

$$\eta_p(X; F) = \sum_{k=0}^{p-1} Tr(\epsilon(F - \epsilon)^{2k+1}[\epsilon, X]) \ .$$

- This leads to an abelian extension $\widehat{L_q\mathfrak{g}} = L_q\mathfrak{g} \oplus \mathrm{Map}(Gr_p, \mathbb{C})$.

# Abelian extension

- Explicit formula for the 2-cocycle:

$$c_p(X, Y; F) = \sum_{m=0}^{p} \mathrm{Tr}\left( (F-\epsilon)^{2m}[\epsilon, X](F-\epsilon)^{2p-2m} Y - (X \leftrightarrow Y) \right)$$

- Notice that $c_p(X, Y; F)$ respects the triangular decomposition

$$\widehat{L_q\mathfrak{g}} = \left(L_q\mathfrak{g}_+ \oplus \mathfrak{g}_+\right) \oplus \left(\mathfrak{h} \oplus Map(Gr_p, \mathbb{C})\right) \oplus \left(\mathfrak{g}_- \oplus L_q\mathfrak{g}_-\right) .$$

- The corresponding abelian group extension $\widehat{L_qG}$ by $Map(Gr_p, \mathbb{C}^*)$ can be constructed using the method of path fibration.

- Equivalently, this can be viewed as a $S^1$-central extension of the Banach Lie groupoid $Gr_p \rtimes L_qG \rightrightarrows Gr_p$.

# Generalised Verma modules

- There is an algebraic formulation of a generalised vacuum representation $V_{\lambda,k} = \mathcal{U}(\widehat{L_q\mathfrak{g}})/I_\lambda$, where $\mathcal{U}(\widehat{L_q\mathfrak{g}})$ is the universal enveloping algebra generated by the $T^a_n$'s and $\psi^a_n$'s at level $k + h^\vee$, and $I_\lambda$ is the left ideal generated by the annihilators.

- This means that the cocycle $c_p(X, Y; F)$, when restricted to the smooth subalgebra $LG$, is cohomologous to $k + h^\vee$ times the basic cocycle.

- However for $q < \frac{1}{2}$, due to the large abelian ideal in $\widehat{L_q\mathfrak{g}}$, we cannot construct any invariant hermitian semidefinite form on the Verma module.

# Homotopy 1-Cocycle

- Let $F : G \to H$ be a homotopy equivalence between topological groups $G$ and $H$, and fix a representation $\rho$ of $H$. We can then produce an operator-valued 1-cocycle by setting

$$\omega(f;g) = \rho\big(F(g)^{-1}F(fg)\big) \ .$$

- This corresponds to a representation of $G$ in the group of matrices with entries in the algebra of complex functions on $G$, but with a $G$-action on functions through right translation.

- Applying this to $LG \simeq L_qG \simeq L^cG$, we have $\widehat{\omega} : L_qG \times L_qG \to \widehat{LG}$.

# $\widehat{L_q G}$-equivariant Dirac family

- For any $g \in L_q G$, one has

$$\hat{\omega}(f;g)^{-1} \partial\!\!\!/ \hat{\omega}(f;g) = \partial\!\!\!/ + i\bar{k} < \psi, \omega(f;g)^{-1} \partial_\theta \omega(f;g) >$$

  where $\partial_\theta$ is the differentiation with respect to the loop parameter.

- In the case of central extension the connections on $LG$ can be taken to be left invariant and they are written as a fixed connection plus a left invariant 1-form $A$ on $LG$.

- The form $A$ at the identity element is identified as a vector in the dual $L\mathfrak{g}^*$ which again is identified, through an invariant inner product, as a vector in $L\mathfrak{g}$ defining a $\mathfrak{g}$-valued 1-form on the circle.

- The right translations on $LG$ induce the gauge action on the potentials $A$.

# $\widehat{L_qG}$-equivariant Dirac family

- Consider next a perturbation of $\not{\partial}$ by a function $A : L_qG \to L_q\mathfrak{g}$,

$$\not{\partial}_A = \not{\partial} + i\bar{k}\langle\psi, A\rangle \ .$$

- The group $L_qG$ acts on $A$ by right translation $(g \cdot A)(f) = A(fg)$.

- Let $\Phi(g)$ denote the operator consisting of right translation on functions and by $\widehat{\omega}(\cdot; g)$ on values of functions via the $LG$ representation in the Hilbert space $\mathcal{H} = V_{(k,\lambda)} \otimes S_{(h^\vee,\rho)}$.

- Then

$$\Phi(g)^{-1}\not{\partial}_A\Phi(g) = \not{\partial}_{A^g}$$

  where

$$(A^g)(f) = \omega(f;g)^{-1}A(fg)\omega(f;g) + \omega(f;g)^{-1}\partial\omega(f;g)$$

# $\widehat{L_qG}$-equivariant Dirac family

- The associated abelian extension by $Map(L_qG, i\mathbb{R})$ defined by the 2-cocycle

$$\tilde{c}_p(f; X, Y) = [\widehat{d\omega}(f; X), \widehat{d\omega}(f; Y)] -$$

$$\widehat{d\omega}(f; [X, Y]) - \mathcal{L}_X \widehat{d\omega}(f; Y) + \mathcal{L}_Y \widehat{d\omega}(f; X)$$

  is cohomologous to that previously defined by $c_p$.

- In the case of $L_qG$ and the abelian extension, the connections on $S^1 \times G$ are no longer preserved under the action of $L_qG$ because of the modified gauge transformation.

- Geometrically, the $L_qG$-action on functions $A$ has the following interpretation: the abelian extension $\widehat{L_qG}$ carries a natural connection form given by

$$\Psi = Ad_{\hat{g}}^{-1} \mathrm{pr}_c(d\hat{g}\hat{g}^{-1})$$

  where $\mathrm{pr}_c$ is the projection onto the abelian ideal $Map(L_qG, i\mathbb{R})$.

## $\widehat{L_qG}$-equivariant Dirac family

- Restriction to constant maps in $Map(L_qG, S^1)$ defines a circle bundle $L$ on $L_qG$, and it carries a connection $\nabla$ induced by the identification of $L$ as subbundle of $\widehat{L_qG} \to L_qG$.

- An arbitrary connection in the bundle $L$ is then written as a sum $\nabla + A$ with $A \in \mathcal{A}$, and right translation by $L_qG$ produces the above gauge transformation on $A$.

- Thus, it means that we have to consider the larger family of Dirac operators parametrized by the space $\mathcal{A}$ of all connections of a circle bundle over $L_qG$.

- This is still an affine space, the extension of $L_qG$ acts on it. The family of Dirac operators transforms equivariantly under the extension and it follows that it can be viewed as an element in twisted K-theory of the moduli stack $\mathcal{A}//L_qG$.

# Upshot

- The study of gauge groups and $L_q G$ suggests the following generalised notion of twisted K-classes.

- Let $P \to X$ be a principal $\mathcal{G}$-bundle and fix a 1-cocycle:

$$\omega : P \times \mathcal{G} \to PU(\mathcal{H}) \ , \quad \omega(gg;p) = \omega(g;pg')\omega(g';p) \ .$$

- A continuous map $T : P \to Fred(\mathcal{H})$ with

$$T(pg) = \omega(g;p)^{-1} T(p) \omega(g;p)$$

defines a class in the twisted K-theory group $K^0(X, \omega)$.

- Moreover, the abelian extension determined by $\omega$,

$$1 \to Map(P, S^1) \to \widehat{\mathcal{G}}_\omega \xrightarrow{\pi} \mathcal{G} \to 1$$

is related to the twisting as follows.

# Upshot

- Let $\tau : P^{[2]} \to \mathcal{G}$ denote the difference map defined by $p_2 = p_1\tau(p_1, p_2)$.

- Introduce the bundle of abelian groups $\mathbb{P} = P \times_{\mathcal{G}} Map(P, S^1)$, where
$$(p, a) \sim (pg, \hat{g}^{-1} a \hat{g})$$
and $\pi(\hat{g}) = \tau(p, pg)$.

- The Čech representative of the Dixmier-Douday class is then given by
$$\epsilon_{\alpha\beta\gamma} = [s_\beta, \hat{g}_{\beta\gamma}\hat{g}_{\gamma\alpha}\hat{g}_{\alpha\beta}] \in \check{H}^2(X, \underline{\mathbb{P}})$$
where $s_\alpha : U_\alpha \to P$, $s_\alpha = s_\beta g_{\beta\alpha}$ are local sections and $\hat{g}_{\alpha\beta}$ denote lifts of the transition functions to the group $\widehat{\mathcal{G}}_\omega$.

Thanks for your attention!