T-duality and parametrized strict deformation quantization

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Plan of talk

1. Review of Rieffel's strict deformation quantization;
2. Parametrized strict deformation quantization, properties and examples;
3. Motivation of definition of principal NC principal torus bundles;
4. Review classification of Echterhoff, Nest, Oyono-Oyono;
5. Classification via parametrized strict deformation quantization;
6. Classification of twisted versions;
7. Link with T-duality in String theory.

Rieffel's strict deformation quantization

We begin by recalling the construction by Rieffel which realizes the smooth noncommutative torus as a strict deformation quantization of the smooth functions on a torus $T = \mathbf{R}^n/\mathbf{Z}^n$ of dimension equal to n.

Recall that the Poisson bracket for $a, b \in C^\infty(T)$ is just

$$\{a, b\} = \sum \theta_{ij} \frac{\partial a}{\partial x_i} \frac{\partial b}{\partial x_j},$$

where (θ_{ij}) is a skew symmetric matrix with constant entries.

The action of T on itself is given by translation. The Fourier transform is an isomorphism between $C^\infty(T)$ and $\mathcal{S}(\hat{T})$, taking the pointwise product on $C^\infty(T)$ to the convolution product on $\mathcal{S}(\hat{T})$ and taking differentiation with respect to a coordinate function to multiplication by the dual coordinate.

Noncommutative torus

In particular, the Fourier transform of the Poisson bracket gives rise to an operation on $\mathcal{S}(\hat{T})$ denoted the same. For $\phi, \psi \in \mathcal{S}(\hat{T})$, define

$$\{\psi, \phi\}(p) = -4\pi^2 \sum_{p_1+p_2=p} \psi(p_1)\phi(p_2)\gamma(p_1, p_2)$$

where γ is the skew symmetric form on $\hat{T}$ defined by

$$\gamma(p_1, p_2) = \sum \theta_{ij}\, p_{1,i}\, p_{2,j}.$$

For $\hbar \in \mathbf{R}$, define a skew bicharacter $\sigma_\hbar$ on $\hat{T}$ by

$$\sigma_\hbar(p_1, p_2) = \exp(-\pi\hbar\gamma(p_1, p_2)).$$

Using this, define a new associative product $\star_\hbar$ on $\mathcal{S}(\hat{T})$,

$$(\psi \star_\hbar \phi)(p) = \sum_{p_1+p_2=p} \psi(p_1)\phi(p_2)\sigma_\hbar(p_1, p_2).$$

This is precisely the smooth noncommutative torus $A^\infty_{\sigma_\hbar}$.

Noncommutative torus

The norm $||\cdot||_\hbar$ is defined to be the operator norm for the action of $\mathcal{S}(\hat{T})$ on $L^2(\hat{T})$ given by $\star_\hbar$. From this point of view, the norm completion of $A^\infty_{\sigma_\hbar}$ is just the twisted group C*-algebra, $C^*(\hat{T},\sigma_\hbar)$.

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Via the Fourier transform, carry this structure back to $C^\infty(T)$, to obtain the smooth noncommutative torus as a strict deformation quantization of $C^\infty(T)$, with respect to the translation action of T.

Strict deformation quantization of T-C^*-algebras

Let A be a C^*-algebra with a continuous action α of T. Then we define the strict deformation quantization of A, denoted A_θ as follows, where θ is a skew symmetric matrix with constant entries as before.

We have the direct sum decomposition,

$$\begin{aligned} A &\cong \widehat{\bigoplus}_{\chi \in \hat{T}} A_\chi \\ a &= \sum_{\chi \in \hat{T}} a_\chi, \end{aligned}$$

where for $\chi \in \widehat{T}$,

$$A_\chi := \{ a \in A \mid \alpha_t(a) = \chi(t) \cdot a \quad \forall\, t \in T \}.$$

Since T acts by $\star$-automorphisms, we have

$$A_\chi \cdot A_\eta \subseteq A_{\chi\eta} \quad \text{and} \quad A_\chi^* = A_{\chi^{-1}} \qquad \forall\, \chi, \eta \in \widehat{T}.$$

Strict deformation quantization of T-C^*-algebras

The completion of the direct sum is explained as follows. The representation theory of T shows that $\bigoplus_{\chi \in \widehat{T}} A_\chi$ is a T-equivariant dense subspace of A, where T acts on A_χ as follows: $\hat{\alpha}_t(a_\chi) = \chi(t) a_\chi$ for all $t \in T$. Then $\widehat{\bigoplus}_{\chi \in \hat{T}} A_\chi$ is the completion in the C^*-norm of A.

The product then also decomposes as,

$$(ab)_\chi = \sum_{\chi_1 \chi_2 = \chi} a_{\chi_1} b_{\chi_2}$$

for $\chi_1, \chi_2, \chi \in \widehat{T}$. The product can be deformed by setting

$$(a \star_\sigma b)_\chi = \sum_{\chi_1 \chi_2 = \chi} a_{\chi_1} b_{\chi_2} \exp(2\pi i \theta_{\chi_1, \chi_2})$$

This is the strict deformation quantization A_θ of A, which is associative because of the skew-symmetry of θ.

Strict deformation quantization of T-C^*-algebras

An important special case when (M, ω) is a symplectic manifold, with a smooth action of T on M preserving the symplectic form ω. If $(C_0^\infty(M)\{,\})$ is the Poisson algebra, then we can strict deform quantize it with this as choice of A, generalizing the example of the torus.

C^*-bundles over X

We begin by recalling the notion of C^*-bundles over X and the special case of noncommutative principal bundles.

Let X be a locally compact Hausdorff space and let $C_0(X)$ denote the C^*-algebra of continuous functions on X that vanish at infinity. A C^*-**bundle** $A(X)$ over X in the sense of [ENOO] is exactly a $C_0(X)$-algebra in the sense of Kasparov. That is, $A(X)$ is a C^*-algebra together with a non-degenerate $*$-homomorphism

$$\Phi : C_0(X) \to ZM(A(X)),$$

called the **structure map**, where $ZM(A)$ denotes the center of the multiplier algebra $M(A)$ of A.

C^*-bundles over X

The **fibre** over $x \in X$ is then $A(X)_x = A(X)/I_x$, where

$$I_x = \{\Phi(f) \cdot a;\ a \in A(X) \text{ and } f \in C_0(X) \text{ such that } f(x) = 0\},$$

and the canonical quotient map $q_x : A(X) \to A(X)_x$ is called the **evaluation map** at x.

NB. This definition does ***not*** require local triviality of the bundle, or for the fibres of the bundle to be isomorphic.

Let G be a locally compact group. One says that there is a **fibrewise action** of G on a C^*-bundle $A(X)$ if there is a homomorphism $\alpha : G \longrightarrow \mathrm{Aut}(A(X))$ which is $C_0(X)$-linear in the sense that

$$\alpha_g(\Phi(f)a) = \Phi(f)(\alpha_g(a)), \qquad \forall g \in G,\ a \in A(X),\ f \in C_0(X).$$

That is α induces an action α^x on the fibre $A(X)_x$ for all $x \in X$.

Parametrised strict deformation quantization

Let $A(X)$ be a C^*-algebra bundle over X with a fibrewise action α of a torus T. Let $\sigma : X \to Z^2(\widehat{T}, \mathbf{T})$ be a continuous map, called a *deformation parameter*.

Then define the *parametrised strict deformation quantization* of $A(X)$, denoted $A(X)_\sigma$ as follows. We have the direct sum decomposition,

$$\begin{aligned} A(X) &\cong \widehat{\bigoplus}_{\chi \in \hat{T}} A(X)_\chi \\ \phi(x) &= \sum_{\chi \in \hat{T}} \phi_\chi(x) \end{aligned}$$

for $x \in X$, where for $\chi \in \widehat{T}$,

$$A(X)_\chi := \{a \in A(X) \mid \alpha_t(a) = \chi(t) \cdot a \quad \forall\, t \in T\}.$$

Parametrised strict deformation quantization

Since T acts by $\star$-automorphisms, we have

$$A(X)_\chi \cdot A(X)_\eta \subseteq A(X)_{\chi\eta} \quad \text{and} \quad A(X)_\chi^* = A(X)_{\chi^{-1}} \qquad \forall\, \chi, \eta \in \widehat{T}.$$

Therefore the spaces $A(X)_\chi$ for $\chi \in \widehat{T}$ form a Fell bundle $A(X)$ over $\widehat{T}$; there is no continuity condition because $\widehat{T}$ is discrete.

The completion of the direct sum is explained as follows. The representation theory of T shows that $\bigoplus_{\chi \in \hat{T}} A(X)_\chi$ is a T-equivariant dense subspace of $A(X)$, where T acts on $A(X)_\chi$ as follows: $\hat{\alpha}_t(\phi_\chi(x)) = \chi(t)\phi_\chi(x)$ for all $t \in T$, $x \in X$. Then $\widehat{\bigoplus}_{\chi \in \hat{T}} A(X)_\chi$ is the completion in the C^*-norm of $A(X)$.

Parametrised strict deformation quantization

The product then also decomposes as,

$$(\phi\psi)_\chi(x) = \sum_{\chi_1\chi_2=\chi} \phi_{\chi_1}(x)\psi_{\chi_2}(x)$$

for $\chi_1, \chi_2, \chi \in \widehat{T}$. The product can be deformed by setting

$$(\phi \star_\sigma \psi)_\chi(x) = \sum_{\chi_1\chi_2=\chi} \phi_{\chi_1}(x)\psi_{\chi_2}(x)\sigma(x; \chi_1, \chi_2)$$

In [HM10], this is the parametrised strict deformation quantization of the $T - C^*$-bundle $A(X)$, is the C^*-algebra denoted by $A(X)_\sigma$, which is associative because of the cocycle property of σ.

Parametrised strict deformation quantization

Theorem (HM09)

Let $A(X)$ be a C^*-bundle with a fibrewise action of T. Let $\sigma, \sigma' : X \longrightarrow Z^2(\widehat{T}, \mathbf{Z})$ be two deformation parameters.

Then $A(X)_\sigma$ is a C^*-bundle over X with a fibrewise T-action, and there is a natural isomorphism,

$$(A(X)_\sigma)_{\sigma'} \cong A(X)_{\sigma\sigma'}$$

Parametrised strict deformation quantization

We next consider a special case of this construction. Consider a smooth fiber bundle of smooth manifolds,

$$\begin{tikzcd} Z \arrow[r] & Y \arrow[d, "\pi"] \\ & X. \end{tikzcd} \tag{1}$$

Suppose there is a fibrewise action of a torus T on Y. That is, assume that there is an action of T on Y satisfying,

$$\pi(t.y) = \pi(y), \qquad \forall\, t \in T,\, y \in Y.$$

Let $\sigma \in C_b(X, Z^2(\widehat{T}, \mathbf{T}))$ be a deformation parameter. $C_0(Y)$ is a C^*-bundle over X, and as above, form the parametrised strict deformation quantization $C_0(Y)_\sigma$.

Parametrised strict deformation quantization

In particular, let Y be a principal G-bundle over X, where G is a compact Lie group such that $\mathrm{rank}(G) \geq 2$. (e.g. $G = \mathrm{SU}(n),\ n \geq 3$ or $G = \mathrm{U}(n),\ n \geq 2$). Let T be a maximal torus in G and $\sigma \in C_b(X, Z^2(\widehat{T}, \mathbf{T}))$ be a deformation parameter. Then $C_0(Y)$ is a C^*-bundle over X, and as above, form the parametrised strict deformation quantization $C_0(Y)_\sigma$.

Noncommutative principal torus bundles

Let T denote the torus of dimension n. The authors of [ENOO] define a **noncommutative principal T-bundle** (or **NCP T-bundle**) over X to be a separable C^*-bundle $A(X)$ together with a fibrewise action $\alpha : T \to \mathrm{Aut}(A(X))$ such that there is a Morita equivalence,

$$A(X) \rtimes_\alpha T \cong C_0(X, \mathcal{K}),$$

as C^*-bundles over X, where $\mathcal{K}$ denotes the C^*-algebra of compact operators.

The motivation for calling such C^*-bundles $A(X)$ NCP T-bundles arises from a special case of a theorem of Rieffel, which states that if $q : Y \longrightarrow X$ is a principal T-bundle, then

$$C_0(Y) \rtimes T \quad \text{is Morita equivalent to} \quad C_0(X, \mathcal{K}).$$

Classification by Echterhoff, Nest, and Oyono-Oyono

Noncommutative principal torus bundles $A(X)$ were classified in [ENOO] and will be outlined in this section.

By Takai duality $A(X)$ is Morita equivalent to $C_0(X, \mathcal{K}) \rtimes \widehat{T}$, so they note that the NCPT-bundles can be classified by up to Morita equivalence by the outer equivalence classes $\mathcal{E}_{\widehat{T}}(X)$ of $\widehat{T}$-actions, and one has the sequence (Echterhoff-Williams)

$$0 \longrightarrow H^1(X, \underline{T}) \longrightarrow \mathcal{E}_{\widehat{T}}(X) \longrightarrow C(X, H^2(\widehat{T}, \mathbf{T})) \longrightarrow 0.$$

This leads to a classification in terms of a principal torus bundle $q : Y \to X$, from $H^1(X, \underline{T})$, and a map $\sigma \in C(X, H^2(\widehat{T}, \mathbf{T}))$, the equivalence classes of multipliers on the dual group $\widehat{T}$.

Classification by Echterhoff, Nest, and Oyono-Oyono

These data define a noncommutative torus bundle by forming the fixed point algebra

$$[C_0(Y) \otimes_{C_0(\widehat{Z})} C^*(H_\sigma))]^T$$

with $C^*(H_\sigma)$ being the bundle of group C^*-algebras of the central extensions of $\widehat{T}$ by $\widehat{Z} := H^2(\widehat{T}, \mathbf{T})$ defined by $\sigma(x)$ at x, the action of $C_0(\widehat{Z})$ on $C_0(Y)$ coming from the composition $\sigma \circ q : Y \to X \to \widehat{Z}$ and that on $C^*(H_\sigma)$ from the natural action of a subgroup algebra.

In [HM09], we are able to give a complete classification of NCP T-bundles over X via parametrised strict deformation quantization of principal torus bundles.

Classification of noncommutative principal torus bundles

Theorem (HM09)

Given a NCPT-bundle $A(X)$, there is a defining deformation $\sigma \in C(X, Z^2(\widehat{T}, \mathbf{T}))$ and a principal torus bundle $q : Y \to X$ such that $A(X)$ is the parametrised strict deformation quantization of $C(Y)$ with respect to σ, that is, there is a T-isomorphism

$$A(X) \cong C(Y)_\sigma \otimes \mathcal{K}.$$

where $\mathcal{K}$ denotes the algebra of compact operators.

Classification of fibrewise smooth noncommutative principal torus bundles

Proof By the construction given earlier, $C(Y)_\sigma$ is a noncommutative principal torus bundle.

Conversely, if $A(X)$ is a fibrewise smooth noncommutative principal torus bundle, then it defines a $\sigma \in C(X, Z^2(\widehat{T}, \mathbf{T}))$. Consider now the deformed algebra $A(X)_{\bar{\sigma}}$. It is equivariantly Morita isomorphic to $C(Y)$ for some principal torus bundle Y over X, since it is classified by an element in $H^2(X, \widehat{T})$. Then by the property of iterated parametrised deformation quantization, $(A(X)_{\bar{\sigma}})_\sigma \cong A(X)$ is equivariantly Morita isomorphic to $C(Y)_\sigma$.

Classification of twisted noncommutative principal torus bundles

Suppose that $A(X)$ is a C^*-bundle over a locally compact space X with a fibrewise action of a torus T, and that $A(X) \rtimes T \cong \mathrm{CT}(X, H_3)$, where $\mathrm{CT}(X, H_3)$ is a continuous trace algebra with spectrum X and Dixmier-Douady class $H_3 \in H^3(X; \mathbf{Z})$. We call such C^*-bundles, H_3-twisted NCPT bundles over X.

Our first main result is that any H_3-twisted NCPT bundle $A(X)$ is equivariantly Morita equivalent to the parametrised deformation quantization of the continuous trace algebra

$$\mathrm{CT}(Y, q^*(H_3))_\sigma,$$

where $q : Y \to X$ is a principal torus bundle with Chern class equal to $H_2 \in H^2(X; H^1(T; \mathbf{Z}))$, and $\sigma \in C_b(X, Z^2(\widehat{T}, \mathbf{T}))$ a defining deformation such that $[\sigma] = H_1 \in H^1(X; H^2(T; \mathbf{Z}))$.

Classification of twisted noncommutative principal torus bundles, K-theory and T-duality

Theorem

In the notation above, $(X \times T, H_1 + H_2 + H_3)$ and the parametrised strict deformation quantization of $(Y, q^*(H_3))$ with deformation parameter σ, $[\sigma] = H_1$, are T-dual pairs, where the 1st Chern class $c_1(Y) = H_2$. That is,

$$\mathrm{CT}(Y, q^*(H_3))_\sigma \rtimes V \cong \mathrm{CT}(X \times T, H_1 + H_2 + H_3).$$

Proof. As before, let V be the vector group that is the universal covering group of the torus group T, and the action of V on the spectrum factors through T. As seen yesterday, the continuous trace algebra $\mathrm{CT}(X \times T, H_1 + H_2)$ is isomorphic to $\mathrm{Ind}_\Gamma^V(C_0(X, \mathcal{K}))$, where Γ is a lattice in V such that $T = V/\Gamma$.

Therefore the crossed product $\mathrm{CT}(X \times T, H_1 + H_2) \rtimes_\beta V$ is Morita equivalent to $C_0(X, \mathcal{K}) \rtimes_\sigma \widehat{T}$, where as before, the Pontryagin dual group $\widehat{T}$ acts fibrewise on $C_0(X, \mathcal{K})$.
Setting $A(X) = C_0(X, \mathcal{K}) \rtimes_\sigma \widehat{T}$, then it is a C^*-bundle over X with a fibrewise action of T. By Takai duality,

$$A(X) \rtimes T \cong C_0(X, \mathcal{K}).$$

Therefore $A(X)$ is a NCPT-bundle and by the main Theorem in [HM09], there is a T-equivariant Morita equivalence,

$$A(X) \sim C_0(Y)_\sigma,$$

where the notation is as in the statement of this Theorem. By the classification of H_3-twisted NCPT-bundles,

$$\mathrm{CT}(Y, q^*(H_3))_\sigma \rtimes V \cong (C_0(Y)_\sigma \rtimes V) \otimes_{C_0(X)} \mathrm{CT}(X, H_3).$$

By Takai duality, $C_0(Y)_\sigma \rtimes V \cong \mathrm{CT}(X \times T, H_1 + H_2)$. Therefore

$$\mathrm{CT}(Y, q^*(H_3))_\sigma \rtimes V \cong \mathrm{CT}(X \times T, H_1 + H_2 + H_3),$$

proving the result.

Using Connes Thom isomorphism theorem and the result above, one has

Corollary

The K-theory of $\mathrm{CT}(Y, q^*(H_3))_\sigma$ *depends on the deformation parameter in general. More precisely, in the notation above* $[\sigma] = H_1$, $c_1(Y) = H_2$,

$$K_\bullet(\mathrm{CT}(Y, q^*(H_3))_\sigma) \cong K^{\bullet + \dim V}(X \times T, H_1 + H_2 + H_3),$$

where the right hand side denotes the twisted K-theory.

Conclusions and an open question

We have seen that parametrised strict deformations of continuous trace algebras, $\mathrm{CT}(Y, q^*(H_3))_\sigma$ are precisely the T-duals of $\mathrm{CT}(X \times T, H_1 + H_2 + H_3)$, ie trivial torus bundles with H-flux.

Question Can the T-dual of a general torus bundle with H-flux be also described in terms of some generalised strict deformation quantisation? via a spectral sequence?

In relation to this is what Peter mentioned in his talk, namely the putative invariants of such a strict deformation quantisation are

$$dH^1 = 0, \qquad (d + H^1 \wedge) H^2 = 0$$

where H^1 and H^2 are certain differential forms on the base X.