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On Moduli space of a quantum Heisenberg manifold

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I learned this subject and problem from Sooran Kang at GPOTS 08 who provided me a generous introduction. Her approach is somewhat different with mine even if we share the set-up for the Yang-Mills problem.

deformation quantization and non-commutative manifolds

Example:Non-commutative tori

Let $\mathbb{T}^n$ be an ordinary n-torus, and let θ be a real skew-symmetric $n \times n$ matrix. Let $A = C^\infty(\mathbb{T}^n)$. If we view $\mathbb{T}$ as $\mathbb{R}/\mathbb{Z}$, then we can define a Poisson bracket on A by

$$\{f, g\} = \sum \theta_{jk}(\partial f/\partial x_j)(\partial g/\partial x_k).$$

As the Lie-group G we will take $\mathbb{T}^n$, acting on itself by translation. The Fourier transform carries $C^\infty(\mathbb{T}^n)$ to $S(\mathbb{Z}^n)$ with convolution, and takes the Poisson bracket to the operation

$$\{\phi, \psi\} = -4\pi^2 \sum_q \phi(q)\psi(p-q)\gamma(q, p-q)$$

where γ is the skew-bilinear form on $\mathbb{Z}^n$ defined by

$$\gamma(p, q) = \sum \theta_{jk} p_j q_j.$$

deformation quantization and non-commutative manifolds

For any $\hbar \in \mathbb{R}$ we define a skew-bicharacter $\sigma_\hbar$ on $\mathbb{Z}^n$ by

$$\sigma_\hbar(p, q) = e^{-2\pi^2 \hbar \gamma(p,q) i},$$

and then define a product $*_\hbar$ on $S(\mathbb{Z}^n)$ by

$$(\phi *_\hbar \psi)(p) = \sum_q \phi(q)\psi(p - q)\sigma_\hbar(q, p - q).$$

The involution $*$ is the complex conjugation for all $\hbar$. We define the operator norm(C^*-norm) for $S(\mathbb{T}^n)$ acting on $L^2(\mathbb{Z}^n)$ by the same formula as defines $*_\hbar$.

Then we can verify two important properties:

1. For every $f \in A$ the function $\hbar \to \|f\|_\hbar$ is continuous (C^*-bundle condition),
2. For every $f, g \in A$, $\|f *_\hbar g - g *_\hbar f / i\hbar - \{f, g\}\|_\hbar$ converges to 0 as $\hbar$ goes to 0 (asymtotic Dirac's condition).

Following Rieffel, we call this construction a strict deformation quantization of $\mathbb{T}^n$. Invariance of the action $G = \mathbb{T}^n$ on "bundles of algebras" can be verified. This is a C^*-algebraic deformation quantization of an ordinary manifold. The algebra corresponding to $\hbar = 1$ is called the non-commutative n-torus which can be regarded as a non-commutative differential manifold.

Example:Non-commutative Heisenberg manifolds

Let G be the Heisenberg Lie-group. For any non-zero integer c, we can parametrize G as $\mathbb{R}^3$ with the product given by

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + cyx').$$

Let D be the discrete subgroup of G consisting of elements with integer entries. The $M_c = G/D$ is the Heisenberg manifold on which G acts on the left.

The Poisson bracket is given by skew 2-vector fields on G $(\mu\partial_1 + \nu\partial_2) \wedge \partial_3$ for some $\mu, \nu \in \mathbb{R}$. For each $\hbar \in \mathbb{R}$, $D_\hbar$ is the space of C^∞ functions ϕ on $\mathbb{R} \times \mathbb{T} \times \mathbb{Z}$ which satisfy

(a) $\phi(x + k, y, p) = e(ckpy)\phi(x, y, z)$ for all $k \in \mathbb{Z}$

(b) $\sup_K \|p^k \frac{\partial^{m+n}}{\partial x^m \partial y^n}\phi(x, y, p)\| < \infty$ for all $k, m, n \in \mathbb{N}$ and any compact set K of $\mathbb{R} \times \mathbb{T}$

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We can give product $*_\hbar$ and involution $*$ for each $\hbar \in \mathbb{R}$ as follows;

(c) $\phi * \psi(x, y, p) =$
$\sum_q \phi(x - \hbar(q-p)\mu, y - \hbar(q-p)\nu, q)\psi(x - \hbar q\mu, y - \hbar q\nu, p - q)$

(d) $\phi^*(x, y, p) = \overline{\phi}(x, y, -p)$

with the norm coming from the representation on $L^2(\mathbb{R} \times \mathbb{T} \times \mathbb{Z})$ defined by

$$\phi(f)(x, y, p) = \sum_q \phi(x - \hbar(q-2p)\mu, y - \hbar(q-2p)\nu, q)f(x, y, p-q)$$

where μ, ν are non-zero real numbers. $D_\hbar$ will denote the norm completion also. The algebras $D_\hbar$ then provide a strict deformation quantization of M_c.

Yang-Mills for non-commutative manifolds

Classical Yang-Mills theory is concerned with the set of connections (i.e. gauge potentials) on a vector bundle of a smooth manifold. The Yang-Mills functional YM measures the "strength" of the curvature of a connection. The Yang-Mills problem is determining the nature of the set of connections where YM attains its minimum, or more generally the nature of the set of critical points for YM.

Definition

A triple " (A, G, α) ", where G is a locally compact group, A is a C^*-algebra, is called a C^*-dynamical system if $\alpha : G \to \mathrm{Aut}(A)$ is continuous which means the function from G to A: $g \to \|\alpha_g(x)\|$ is continuous for each $x \in A$.

Yang-Mills for non-commutative manifolds

Definition (Connes)

It is said that x in A is C^∞-vector if and only if $g \to \alpha_g(x)$ from G to the normed space is of C^∞. Then $A^\infty = \{x \in A| \quad x \text{ is of } C^\infty \ \}$ is norm dense in A. In this case we call A^∞ the smooth dense subalgebra of A.

Since a C^*-algebra with a smooth dense subalgebra is an analogue of a smooth manifold(recall a strict deformation quantization), finitely generated projective A^∞-modules are the appropriate generalizations of vector bundles over the manifold.

Let $\mathfrak{g}$ be the Lie-algebra of (unbounded) derivations of A^∞ given by the representation δ from the Lie-algebra $\mathfrak{g}$ of G where

$\delta_X(a) = \lim_{t\to 0} \dfrac{1}{t}(\alpha_{exp(tX)}(a) - a)$ for $X \in \mathfrak{g}$ and $a \in A^\infty$.

Definition (Cones, Rieffel)

Given a finitely generated projective module Ξ^∞, a connection on Ξ^∞ is a linear map $\nabla : \Xi^\infty \to \Xi^\infty \otimes \mathfrak{g}^*$ such that, for all $X \in \mathfrak{g}$, $\xi \in \Xi^\infty$ and $x \in A^\infty$ one has $\nabla_X(\xi \cdot x) = \nabla_X(\xi) \cdot x + \xi \cdot \delta_X(x)$.

We shall say that ∇ is compatible with the A^∞-valued hermitian metric if and only if $< \nabla_X(\xi), \eta >_A + < \xi, \nabla_X(\eta) >_A = \delta_X(< \xi, \eta >_A)$ for all $\xi, \eta \in \Xi^\infty$, $X \in \mathfrak{g}$.

Definition

Let ∇ be a connection on Ξ^∞, the curvature of ∇ is the element Θ of $\mathrm{End}_{A^\infty}(\Xi^\infty) \otimes \Lambda^2(\mathfrak{g})^*$ given by

$$\Theta_\nabla(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

for all $X, Y \in \mathfrak{g}$.

Yang-Mills for non-commutative manifolds

The classical Yang-Mills functional on a principal bundle with structure group G, base M, connection ∇, and curvature (Yang-Mills field tensor) K is

$$\mathrm{YM}(\nabla) = \int |K|^2 d\mathrm{vol}_M,$$

To get an analogue of this functional, we need a bilinear form and an integration. Given alternating bilinear forms Ψ and Φ we let

$$\{\Phi, \Psi\}_E = \sum_{i<j} \Phi(Z_i, Z_j)\Psi(Z_i, Z_j)$$

where $E = End_A(\Xi)$.

Yang-Mills for non-commutative manifolds

We assume given a faithful trace τ on A, which is invariant under the action of $\mathfrak{g}$. Then τ determines a canonical faithful trace τ_E on E. Thus for compatible connections ∇ we define the Yang-Mills functional YM by

$$YM(\nabla) = -\tau_E(\{\Theta_\nabla, \Theta_\nabla\}).$$

following Connes and Rieffel.
This (non-commutative) Yang-Mills functional is invariant under the unitary group of E which will be denoted by $U(E)$ as the classical one does under the gauge groups. Thus we can define the moduli space by $\{$compatible connections YM attains its minimum$\}/U(E)$.

Yang-Mills for non-commutative manifolds

This (topological) space was motivated to give an answer about how to associate to a non-commutative algebra an ordinary manifold which would be its "manifold shadow"

Non-commutative torus case:

Since the Lie-algebra of $\mathbb{T}^n$ is Abelian, the analysis of Yang-Mills problem in this case is not much complicated. In fact, Connes and Rieffel studied Yang-Mills for the non-commutative 2-torus and classified the moduli space for Ξ^d as $(\mathbb{T}^2)^d/\Sigma_d$ where Σ_d is the group of permutations of d objects and Ξ is a module which is not a multiple of any other module.

Non-commutative Heisenberg manifold case:
We define non-commutative or quantum Heisenberg manifolds as $D^c_{\mu\nu}$. Now we fix one set $\{\mu, \nu\}$ and call it a quantum Heisenberg manifold. As far as I know, no one has been able to classify the moduli space for this case. However, the ground-breaking work about some critical points of Yang-Mills functional on a module $\Xi^\infty = S(\mathbb{R} \times \mathbb{T})$ was done by Sooran Kang in 2009. This depends on Abadie's description on a quantum Heisenberg manifold which gives (strong) Morita equivalence of $D^c_{\mu\nu}$ via Ξ^∞ and an idea of Kang introducing "multiplication type" elements in $\mathrm{End}(\Xi^\infty)$.

Yang-Mills for non-commutative manifolds

After reparametrization of Heisenberg manifold by introducing a positive integer c, the Lie algebra of the (reparametrized) Heisenberg group has an orthonornal basis consisting of

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & 0 & 1/c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Define a linear map $\nabla^0 : \Xi \to \Xi \otimes \mathfrak{g}^*$ as follows: For computational purposes the easiest way to define it is in terms of an orthonormal basis for $\mathfrak{g}$.

$$(\nabla^0_X f)(x, y) = -\frac{\partial}{\partial x} f(x, y), \tag{1}$$

$$(\nabla^0_Y f)(x, y) = -\frac{\partial}{\partial y} f(x, y) + \frac{\pi c i}{2\mu} x^2 f(x, y), \tag{2}$$

$$(\nabla^0_Z f)(x, y) = \frac{\pi i x}{\mu} f(x, y) \tag{3}$$

for $f \in \Xi$.

Yang-Mills for non-commutative manifolds

The representation δ of L as a Lie algebra of derivations on $D^c_{\mu\nu}$ is given by

$$\delta_X(\phi)(x,y,p) = -\frac{\partial}{\partial x}\phi(x,y,p)$$
$$\delta_Y(\phi)(x,y,p) = -\frac{\partial}{\partial y}\phi(x,y,p) + 2\pi icp(x-p\mu)\phi(x,y,p)$$
$$\delta_Z(\phi)(x,y,p) = 2\pi ip\,\phi(x,y,p)$$

Theorem

The linear map $\nabla^0 : \Xi \to \Xi \otimes \mathfrak{g}^*$ satisfying (1),(2),(3) is a compatible connection.

Theorem

The connection $\nabla^0 : \Xi^\infty \to \Xi^\infty \otimes \mathfrak{g}^*$ satisfying (1), (2), and (3) is a minimum of the Yang-Mills functional for the quantum Heisenberg manifold $D^c_\hbar$.

From the Morita equivalence, we know that End(Ξ) is isomorphic to $E^c_{\mu\,\nu}$. Note that the multiplicative structure of $E^c_{\mu\,\nu}$ is given by

$$\psi_1 * \psi_2(x, y, k) = \sum_{l} \psi_1(x, y, l)\psi_2(x + l, y, k - l) \tag{4}$$

for $\psi_1, \psi_2 \in E^c_{\mu\,\nu}$.
The left action of $E^c_{\mu\,\nu}$ on Ξ is given by

$$(\psi \cdot f)(x, y) = \sum_{k \in \mathbb{Z}} \bar{\psi}(x, y, k) f(x + k, y) \tag{5}$$

Yang-Mills for non-commutative manifolds

For a function $G \in C^\infty$, define a **multiplication − type** element $\mathbf{G}$ in $E^c_{\mu\nu}$ by

$$\mathbf{G}(x, y, p) = G(x, y)\delta_0(p).$$

Theorem (Kang 2009)

$\mathbf{G}$ is skew-symmetric if and only if $\mathbf{G}$ acts on Ξ^∞ as a multiplication operator, i.e.

$$(\mathbf{G} \cdot f)(x, y) = -G(x, y)f(x, y)$$

for $f \in \Xi^\infty$.

Yang-Mills for non-commutative manifolds

Then we perturb ∇^0 by a skew-symmetric linear map whose range belongs to the set of **multiplication − type** elements, and generate a family of critical points of the Yang-Mills functional as follows.

Theorem (H. Lee)

Let $\nabla = \nabla^0 + \omega$ be a compatible connection where ω is a linear map from the Heisenberg Lie-algebra $\mathfrak{g}$ to $E^c_{\mu\,\nu}$, whose range is the set of "multiplication type" skew-symmetric elements. Set $\omega(Z_i) = \mathbf{G_i}$ for $i = 1, 2, 3$ where $\mathbf{G_i}(x, y, k) = G_i(x, y)\delta_0(k)$. Then ∇ is a critical point of the Yang-Mills functional if and only if G_1, G_2, and G_3 satisfy the following equations

$$\frac{\partial}{\partial y} G_1 - \frac{\partial}{\partial x} G_2 + cG_3 = c_1 i \quad \textit{for some } c_1 \in \mathbb{R}$$

$$\frac{\partial^2 G_3}{\partial x^2} + \frac{\partial^2 G_3}{\partial y^2} = cc_1 i$$

Yang-Mills for non-commutative manifolds

Furthermore, such a critical point including ∇^0 is a minimum of the Yang-Mills functional if $c_1 = 0$. Then we can show the following theorem.

Theorem

Any two different minimum points of the type $\nabla^0 + \omega$ where ω is the linear map from $\mathfrak{g}$ to $E^c_{\mu\nu}$, whose range is the set of "multiplication type" skew-symmetric elements, are not in the same orbit under the action of $U(E)$. In particular, the moduli space $MC(\Xi^\infty)/U(E)$ contains uncountably many points.