Noncommutative Nonunital Index Theory

This talk is about work in progress with Gayral, Rennie and Sukochev motivated by index theory on non-compact manifolds, in particular, theorems of Gromov-Lawson type, also applications to pseudo-Riemannian manifolds, generalisations of Atiyah's L^2 index theorem and foliations of non-compact manifolds.

Specifically: is there a good notion of spectral triple for nonunital algebras that will lead to new index theorems on noncompact manifolds via a nonunital version of the local index formula in noncommutative geometry? Our aim is to avoid the use of local units (i.e. compact support assumptions).

Tasks: (i) to explain how to define index pairings in terms of spectral triples in the nonunital situation.
(ii) to extend the local index formula in noncommutative geometry to the nonunital case.
(iii) to allow for the case when the spectral triple is semifinite.

Part I: Kasparov picture.

Let $(\mathcal{N}, \tau)$ be a semifinite von Neumann algebra, τ, a semifinite normal faithful trace. We denote by $\mathcal{K}(\mathcal{N}, \tau)$ the ideal of τ-compact operators in $\mathcal{N}$. A nonunital Fréchet sub-$*$-algebra $\mathcal{A}$ of $\mathcal{N}$ is called a *pre-C^*-algebra* if it is stable under the holomorphic functional calculus. This means that its minimal unitalization $\mathcal{A}^\sim := \mathcal{A} \oplus \mathbf{C}$ is stable under the (ordinary) holomorphic functional calculus in the minimal unitalization of its C^*-completion.

Definition 1 *A nonunital semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, relative to $(\mathcal{N}, \tau)$, is given by a Hilbert space $\mathcal{H}$, a pre-C^*-algebra $\mathcal{A} \subset \mathcal{N}$ acting on $\mathcal{H}$, and a densely defined unbounded self-adjoint operator $\mathcal{D}$ affiliated to $\mathcal{N}$ such that we firstly have a semifinite unbounded Kasparov module (recall that this means:*
1) $da := [\mathcal{D}, a]$ is densely defined and extends to a bounded operator in $\mathcal{N}$ for all $a \in \mathcal{A}$,
2) $a(1 + \mathcal{D}^2)^{-1/2} \in \mathcal{K}(\mathcal{N}, \tau)$ for all $a \in \mathcal{A}$)
$(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is even if there is a $\mathbf{Z_2}$-grading such that $\mathcal{A}$ is even and $\mathcal{D}$ is odd. Otherwise we say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is odd.

Remarks. 'Nonunital' is really referring to the case where $(1+\mathcal{D}^2)^{-1}$ is not compact so that even if $\mathcal{A}$ has a unit it cannot be the unit of $\mathcal{N}$.

δ is the unbounded derivation given by $\delta(T) := [|\mathcal{D}|, T]$, $T \in \mathcal{N}$.

Definition 2 *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a nonunital semifinite spectral triple, relative to $(\mathcal{N}, \tau)$. We say $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is QC^k if for all $b \in \mathcal{A} \cup [\mathcal{D}, \mathcal{A}]$ we have for all $0 \leq j \leq k$, that the operator $\delta^j(b) \in \mathcal{N}$. We say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is QC^∞ if it is QC^k for all $k \in \mathbf{N}$.*

The Kasparov class of a spectral triple

We follow ideas of Kaad, Nest and Rennie (2008) that are related to older results of Connes-Cuntz (1987). Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}$ be a nonunital semifinite spectral triple relative to $(\mathcal{N}, \tau)$. Set $F_\mathcal{D} = \mathcal{D}(1+\mathcal{D}^2)^{-1/2}$. Then for all $a \in \mathcal{A}$ and $\varphi \in C_0(\mathbf{R})$ the following operators are τ-compact

$$[F_\mathcal{D}, a], \quad a\varphi(\mathcal{D}).$$

Regarding $\mathcal{K}_\mathcal{N}$ as a right $\mathcal{K}_\mathcal{N}$ C^*-module via $(b_1|b_2) := b_1^* b_2$, we see immediately that left multiplication by $F_\mathcal{D}$ on $\mathcal{K}_\mathcal{N}$ gives $F_\mathcal{D} \in End_{\mathcal{K}_\mathcal{N}}(\mathcal{K}_\mathcal{N})$.

Left multiplication by $a \in A$, the C^*-completion of $\mathcal{A}$, gives a representation of A as adjointable endomorphisms of $\mathcal{K}_{\mathcal{N}}$ and $[F_{\mathcal{D}}, a] \in \mathcal{K}_{\mathcal{N}} = End^0_{\mathcal{K}_{\mathcal{N}}}(\mathcal{K}_{\mathcal{N}})$, the compact endomorphisms, for all $a \in A$.

Since $a(F_{\mathcal{D}}^2 - 1) \in \mathcal{K}_{\mathcal{N}}$ and $F_{\mathcal{D}} = F_{\mathcal{D}}^*$ by construction, we obtain a Kasparov module $({}_A(\mathcal{K}_{\mathcal{N}})_{\mathcal{K}_{\mathcal{N}}}, F_{\mathcal{D}})$ with class $[(\mathcal{K}_{\mathcal{N}}, F_{\mathcal{D}})] \in KK^j(A, \mathcal{K}_{\mathcal{N}})$, where j is 0 iff our spectral triple was $\mathbf{Z_2}$-graded.

Using the Kasparov product we now have a well-defined map

$$\cdot \otimes_A [(\mathcal{K}_{\mathcal{N}}, F_{\mathcal{D}})] \,:\, K_j(A) \to K_0(\mathcal{K}_{\mathcal{N}})$$

To make the Kasparov product explicit we need to choose a representative with $F^2 = 1 (=: Id_X)$.

I will explain how to do this later.

Suppose that e and f are projections in a (matrix algebra over the minimal unitization $A^{\sim}$) and suppose also that we have a class $[e]-[f] \in K_0(A)$.

For brevity write B for $\mathcal{K}_{\mathcal{N}}$ acting on our Kasparov module on the right and, ${}_AX_B$ for the Kasparov module $({}_A(\mathcal{K}_{\mathcal{N}})_{\mathcal{K}_{\mathcal{N}}}, F_{\mathcal{D}})$. We represent elements $a+\lambda Id_{A^{\sim}}$ on X as $a+\lambda Id_X$, $\lambda \in \mathbf{C}$.

If $F^2 = 1$ we claim:
$eF_+e : e\frac{1+\gamma}{2}X \to e\frac{1-\gamma}{2}X$ is Fredholm, with eF_-e being an inverse up to compacts.

In this case $([e]-[f]) \otimes_A [({}_AX_B, F)]$ is given by

$$[\text{Index}(eF_+e)] - [\text{Index}(fF_+f)]$$

$$= [\ker eF_+e] - [\text{coker}\, eF_+e] - [\ker fF_+f] + [\text{coker}\, fF_+f],$$

and the individual terms are the classes of finite projective modules.

Similarly, in the odd case

$$[u] \otimes_A [({}_AX_B, F)] = [\mathrm{Index}(\frac{1+F}{2}u\frac{1+F}{2} - \frac{1-F}{2})]$$
$$\in K_0(B),$$

where $[u] \in K_1(A)$.

Writing $(1+F)/2 = P$ for the positive spectral projection of F, we have

$$[u] \otimes_A [({}_AX_B, F)] = [\mathrm{Index}(PuP)] \in K_0(B),$$

and both $\ker PuP$ and $\mathrm{coker} PuP$ are finite projective B-modules. A parametrix of PuP is given by Pu^*P.

Fredholm modules: numerical index pairing

A semifinite **pre-Fredholm module** for a $*$-algebra $\mathcal{A}$ relative to $(\mathcal{N}, \tau)$ is a pair $(\mathcal{H}, F)$ where $\mathcal{A}$ is (continuously) represented in $\mathcal{N}$ and F is a self-adjoint operator in $\mathcal{N}$ satisfying:

1. $a(1-F^2) \in \mathcal{K}_\mathcal{N}$, *and*
2. $[F, a] \in \mathcal{K}_\mathcal{N}$ *for* $a \in \mathcal{A}$.

Summability:
if $[F,a] \in \mathcal{L}^{p+1}(\mathcal{N},\tau)$ for $a \in \mathcal{A}$, we say that $(\mathcal{H},F)$ is $p+1$-summable. The spectral dimension of such a module is the infimum of those n such that $[F,a] \in \mathcal{L}^{n}(\mathcal{N},\tau)$ for all $a \in \mathcal{A}$.

In the spectral triple version this is the requirement $a(1+\mathcal{D}^2)^{-n/2} \in \mathcal{L}^1(\mathcal{N},\tau)$ for all $n > p$.

Lemma Given a semifinite finitely summable spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$, with spectral dimension p, then setting $F_{\mathcal{D}} := \mathcal{D}(1+\mathcal{D}^2)^{-1/2}$ yields, a semifinite $\lfloor p \rfloor + 1$-summable pre-Fredholm module for $\mathcal{A}$.

So given a pre-Fredholm module $(\mathcal{H},F)$ relative to $(\mathcal{N},\tau)$ we obtain a Kasparov module $(\mathcal{K}_{\mathcal{N}},F)$, just as we did for a spectral triple. Also given $(\mathcal{A},\mathcal{H},\mathcal{D})$ relative to $(\mathcal{N},\tau)$, the following diagram commutes

$$\begin{array}{ccc} (\mathcal{A},\mathcal{H},\mathcal{D}) & \rightarrow & (\mathcal{K}_{\mathcal{N}},F_{\mathcal{D}}) \\ \downarrow & \nearrow & \\ (F_{\mathcal{D}},\mathcal{H}) & & \end{array} .$$

Thus we have a single well-defined Kasparov class arising from either the spectral triple or the associated pre-Fredholm module. Now we show how to obtain a representative of this class with $F^2 = 1$, so simplifying the index pairing.

Definition. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a semifinite spectral triple relative to $(\mathcal{N}, \tau)$. For any $\mu > 0$, define the 'double' of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ to be the semifinite spectral 'triple'

$$(\mathcal{A}, \mathcal{H}^2, \mathcal{D}_\mu, M_2(\mathcal{N}), \tau \otimes \mathrm{Tr}_2),$$

with $\mathcal{H}^2 = \mathcal{H} \oplus \mathcal{H}$ and the action of $\mathcal{A}$ and $\mathcal{D}_\mu$ given by

$$\mathcal{D}_\mu := \begin{pmatrix} \mathcal{D} & \mu \\ \mu & -\mathcal{D} \end{pmatrix}, \qquad a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \quad \forall a \in \mathcal{A}.$$

If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is even and graded by γ then the double is even and graded by $\gamma \oplus -\gamma$.

$\mathcal{D}_\mu$ always is invertible, and $F_\mu = \mathcal{D}_\mu |\mathcal{D}_\mu|^{-1}$ has square 1.

Lemma. The KK-classes associated with $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ and $(\mathcal{A}, \mathcal{H}^2, \mathcal{D}_\mu)$ coincide. A representative of this class is $(\mathcal{K}^2_{\mathcal{N}}, F_\mu)$ with $F_\mu = \mathcal{D}_\mu |\mathcal{D}_\mu|^{-1}$.

Corollary. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a nonunital semifinite spectral triple which is finitely summable, with spectral dimension p. Then the same is true for the double $(\mathcal{A}, \mathcal{H}^2, \mathcal{D}_\mu, M_2(\mathcal{N}), \tau \otimes \mathrm{Tr}_2)$ and $(\mathcal{K}^2_{\mathcal{N}}, F_\mu)$ is a $\lfloor p \rfloor +$ 1-summable semifinite Fredholm module. Moreover the $K_0(\mathcal{K}_{\mathcal{N}})$-valued index pairings defined by the two spectral triples and the semifinite Fredholm module all agree: for $x \in K_*(\mathcal{A})$ of the appropriate parity and $\mu > 0$

$$x \otimes_A [(\mathcal{A}, \mathcal{H}, \mathcal{D})] = x \otimes_A \left[\Big(\mathcal{A}, \mathcal{H}^2, \mathcal{D}_\mu, M_2(\mathcal{N}), \tau \otimes \mathrm{Tr}_2\Big)\right]$$

$$= x \otimes_A [(\mathcal{K}^2_{\mathcal{N}}, F_\mu)] \,.$$

The τ-finite operators $\mathcal{F}_{\mathcal{N}} \subset \mathcal{K}_{\mathcal{N}}$ are stable under the holomorphic functional calculus, and so $K_0(\mathcal{K}_{\mathcal{N}}) = K_0(\mathcal{F}_{\mathcal{N}})$. Thus we can always represent elements of $K_0(\mathcal{K}_{\mathcal{N}})$ by classes $[e] - [f]$ with $e, f \in \mathcal{F}^{\sim}_{\mathcal{N}}$ where $\sim$ denotes the one-point unitization. Thus the trace τ defines a homomorphism $\tau_* : K_0(\mathcal{K}_{\mathcal{N}}) \to \mathbf{R}$ thus we can define a numerical index from the Fredholm module.

Compatibility of the Kasparov product, numerical index and Chern character

Definition Let $(\mathcal{H}, F)$ be a Fredholm module relative to $\mathcal{N}, \tau$. We define the 'conditional trace' τ' by

$$\tau'(T) = \frac{1}{2}\tau(F(FT + TF)),$$

provided $FT + TF \in \mathcal{L}^1(\mathcal{N})$.

We will use the (b, B) normalisation, and so make the following definition.

Definition Let $(\mathcal{H}, F)$ be a semifinite $n+1$-summable Fredholm module for the algebra $\mathcal{A}$ relative to $(\mathcal{N}, \tau)$, and suppose the parity of the Fredholm module is the same as the parity of n. Then we define the Chern character $[Ch_F]$ to be the cyclic cohomology class of the single term (b, B)-cocycle defined by

$$Ch_F^n(a_0, a_1, \ldots, a_n) := \begin{cases} \frac{\Gamma(\frac{n}{2}+1)}{n!}\tau'(\gamma a_0[F, a_1]\cdots[F, a_n]), \\ \\ \sqrt{2i}\frac{\Gamma(\frac{n}{2}+1)}{n!}\tau'(a_0[F, a_1]\cdots[F, a_n]), \end{cases}$$

If $e \in \mathcal{A}^\sim$ is a projection we define $Ch_0(e) = e \in \mathcal{A}^\sim$ and

$$Ch_{2k}(e) = (-1)^k \frac{(2k)!}{k!}(e-1/2)\otimes e\otimes\cdots\otimes e \in (\mathcal{A}^\sim)^{\otimes 2k+1}.$$

If $u \in \mathcal{A}^\sim$ is a unitary then we define

$$Ch_{2k+1}(u) = (-1)^k\, k!\, u^*\otimes u\otimes\cdots\otimes u^*\otimes u \in (\mathcal{A}^\sim)^{\otimes 2k+2}.$$

Importantly, if both $e, f \in M_k(\mathcal{A}^\sim)$ and $[e]-[f] \in K_0(\mathcal{A})$ then $Ch(e)-Ch(f)$ defines a class in the reduced cyclic homology. This follows since $K_0(\mathcal{A}) = \ker(q_* : K_0(\mathcal{A}^\sim) \to K_0(\mathbf{C}))$ where $q : \mathcal{A}^\sim \to \mathbf{C}$ is the quotient map, and this implies that $q(e) = q(f) \in \mathbf{C} = \mathcal{A}^\sim/\mathcal{A}$.

Proposition Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a nonunital semifinite spectral triple which is finitely summable with spectral dimension $p \geq 1$. Let $[(\mathcal{K}^2_{\mathcal{N}}, F_\mu)] \in KK^j(A, \mathcal{K}_{\mathcal{N}})$ be the Kasparov module associated to the double of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ ($j = 0$ if the spectral triple is $\mathbf{Z_2}$-graded and $j = 1$ otherwise). Then for $x \in K_j(\mathcal{A})$ we have

$$\langle x, (\mathcal{A}, \mathcal{H}, \mathcal{D})\rangle = \tau_*\Big(x\otimes_A[(\mathcal{K}^2_{\mathcal{N}}, F_\mu)]\Big) = c_j\, Ch_F^{\lfloor p\rfloor}\Big(Ch_{\lfloor p\rfloor}(x)\Big).$$

where the pairing on the left is the numerical index pairing , $c_0 = 1$ and $c_1 = -(2i\pi)^{-1/2}$.

Part II: The local index formula for semifinite nonunital spectral triples.

Definitions. (i) For any positive number $s > 0$, we define the weight φ_s on $\mathcal{N}$ by

$$\varphi_s(T) := \tau\Big((1+\mathcal{D}^2)^{-s/4} T (1+\mathcal{D}^2)^{-s/4}\Big).$$

(ii) Let

$$\mathcal{B}_p = \mathcal{B}_p(\mathcal{D}) := \bigcap_{s>p} \Big(\mathrm{dom}(\varphi_s)^{1/2} \cap (\mathrm{dom}(\varphi_s)^{1/2})^*\Big).$$

(iii) Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple relative to $(\mathcal{N}, \tau)$. Then we say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is smoothly summable if $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is finitely summable with spectral dimension p and for all $j \in \mathbf{N}$,

$$\delta^j(\mathcal{A}) \cup \delta^j([\mathcal{D}, \mathcal{A}]) \subset (\mathcal{B}_p)^2.$$

A. The resolvent and residue cocycles and other cochains.

Start with $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, a nonunital, semifinite, smoothly summable spectral triple, with spectral dimension $p \geq 1$ and parity $\mathcal{P}$. ($\mathcal{P} = 0$ for an even spectral triple and $\mathcal{P} = 1$ for odd triples.)

We also let $N := \lfloor (p+\mathcal{P}+1)/2 \rfloor$ and $M := 2N-\mathcal{P}$, the greatest integer of parity $\mathcal{P}$ in $[0, p+1]$. In particular, $M = p$ when p is an integer of parity $\mathcal{P}$. The grading degree allows us to define a graded commutator by

$$[S,T]_\pm := ST - (-1)^{\deg(S)\deg(T)} TS\,.$$

In the following we are working with the double of a given spectral triple. The trace on $M_2(\mathcal{N})$ is $\tau \circ \mathrm{Tr}_2$.

The residue cocycle. For a multi-index $k \in \mathbf{N}^m$, we define

$$\alpha(k)^{-1} := k_1! \cdots k_m! (k_1+1)(k_1+k_2+2)\cdots(|k|+m),$$

and we let $\sigma_{n,j}$ be the non-negative rational numbers defined by the identities

$$\prod_{j=0}^{n-1}(z+j+\tfrac{1}{2}) = \sum_{j=0}^{n} z^j\,\sigma_{n,j}, \quad \mathcal{P}=1,$$

$$\prod_{j=0}^{n-1}(z+j) = \sum_{j=1}^{n} z^j \sigma_{n,j}, \quad \mathcal{P}=0.$$

Definition. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a QC^∞ finitely summable nonunital spectral triple of spectral dimension p. We say that the spectral dimension is isolated, if for any element $b \in \mathcal{N}$, of the form

$b = a_0\, da_1^{(k_1)} \cdots da_m^{(k_m)} (1+\mathcal{D}^2)^{-|k|-m/2},\ a_0, \cdots, a_m \in \mathcal{A}$, the zeta function $\zeta_b(z) := \tau\Big(b(1+\mathcal{D}^2)^{-z}\Big)$, has analytic continuation to a deleted neighbourhood of $z = (1-p)/2$. In this case, we define the numbers

$$\tau_j(b) := \mathrm{res}_{z=(1-p)/2}\, z^j\, \zeta_b(z).$$

Definition. Assume that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ has isolated spectral dimension. For $m = \mathcal{P}$ mod 2, with τ_j and for a multi-index k setting $h = |k| + (m - \mathcal{P})/2$, the m-th component of the **residue cocycle** $\phi_m : \mathcal{A}^\sim \otimes \mathcal{A}^{\otimes m} \to \mathbf{C}$ is defined by

$$\phi_m(a_0, \cdots, a_m) =$$

$$(\sqrt{2i\pi})^{\mathcal{P}} \sum_{|k|=0}^{M-m} (-1)^{|k|} \alpha(k)$$

$$\sum_{j=A}^{h} \sigma_{h,j}\, \tau_{j-A}\Big(\gamma a_0\, da_1^{(k_1)} \cdots da_m^{(k_m)} (1+\mathcal{D}^2)^{-|k|-m/2}\Big).$$

By definition of isolated spectral dimension, we see that for $m > 0$ the components of the residue cocycle take finite values on $\mathcal{A}^{\sim} \otimes \mathcal{A}^{\otimes m}$. (There is a slight issue with the zeroth term in the even case but it still works)

The resolvent cocycle. Initially we do not assume that our spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ has isolated spectral dimension. We must however use the double, so there exists $\mu > 0$ such that $\mathcal{D}^2 \geq \mu$. We let $\mathcal{D}_u := \mathcal{D}|\mathcal{D}|^{-u}$ for $u \in [0, 1]$ (which is well defined precisely because $|\mathcal{D}|$ is invertible) and $d_u(a) := [\mathcal{D}_u, a]$. Note that this derivation interpolates between the two natural notions of differential in quantised calculus, that is $d_0(a) = da = [\mathcal{D}, a]$ and $d_1(a) = [F, a]$. We also set $\dot{\mathcal{D}}_u := -\mathcal{D}_u \log |\mathcal{D}|$, the formal derivative of $\mathcal{D}_u$ with respect to the parameter u. We finally introduce the short-hand notations:

$$R_{s,t,u}(\lambda) := (\lambda - (t + s^2 + \mathcal{D}_u^2))^{-1}\,, \tag{1}$$

$$R_{s,t}(\lambda) := R_{s,t,0}(\lambda)\,, \quad R_{s,u}(\lambda) := R_{s,0,u}(\lambda) \tag{2}$$

$$R_s(\lambda) := R_{s,1,0}(\lambda)$$

The range of the parameters is $0 < \Re(\lambda) < \mu/2$, $s \in \mathbf{R}$ and $t, u \in [0, 1]$. For a multi-index $k \in \mathbf{N}^{m+1}$, we set $|k| := k_0 + k_1 + \cdots + k_m$.

Definition. For $a \in (0, \mu/4)$, let ℓ be the vertical line $\ell = \{a + iv : v \in \mathbf{R}\}$. Given $m \in \mathbf{N}$, $s \in \mathbf{R}^+$, $r \in \mathbf{C}$ and operators $A_0, ..., A_m \in \mathrm{OP}^{k_i}$, such that $|k| - 2m < 2\Re(r)$, we define

$$\langle A_0, \cdots, A_m \rangle_{m,r,s,t,u} :=$$

$$\frac{1}{2\pi i} \tau\Big(\gamma \int_\ell \lambda^{-r-p/2} A_0\, R_{s,t,u}(\lambda) \cdots A_m\, R_{s,t,u}(\lambda)\, d\lambda\Big),$$

Here γ is the $\mathbf{Z}_2$-grading in the even case and the identity operator in the odd case. When $|k| - 2m - 1 < 2\Re(r)$, we set

$$\langle\langle A_0, \cdots, A_m \rangle\rangle_{m,s,r,t,u} :=$$

$$\sum_{j=0}^{m} (-1)^{\deg(A_j)} \langle A_0, \cdots, A_j, \mathcal{D}, A_{j+1}, \cdots, A_m \rangle_{m+1,s,r,t,u} .$$

We now state the definition of the resolvent cocycle in terms of the expectations $\langle \cdot, \ldots, \cdot \rangle_{m,r,s,t,u}$.

Definition. For $m \in \mathbf{N}$ and $m \equiv \mathcal{P} \bmod 2$, we introduce the constants η_m by

$$\eta_m = \left(-\sqrt{2i}\right)^{\mathcal{P}} 2^{m+1} \frac{\Gamma(m/2+1)}{\Gamma(m+1)}.$$

Then for $t \in [0,1]$ and $\Re(r) > (1-m)/2$, the m-th component of the **resolvent cocycles** $\phi^r_m, \phi^r_{m,t} : \mathcal{A}^\sim \otimes \mathcal{A}^{\otimes m} \to \mathbf{C}$ are defined by $\phi^r_m := \phi^r_{m,1}$ and

$$\phi^r_{m,t}(a_0, \ldots, a_m) := \eta_m \int_0^\infty s^m \langle a_0, da_1, \ldots, da_m \rangle_{m,r,s,t,0}\, ds\,,$$

Definition. For t, $u \in [0,1]$ and $\Re(r) > (1-m)/2$, the m-th component, $m = A, A+2, M+1$, of the **transgression cochains** $\Phi^r_{m,t,u} : \mathcal{A} \otimes \mathcal{A}^{\otimes m} \to \mathbf{C}$ are defined by

$$\Phi^r_{m,t,u}(a_0, \ldots, a_m)$$

$$:= \eta_{m+1} \int_0^\infty s^{m+1} \langle\langle a_0, d_u(a_1), \ldots, d_u(a_m) \rangle\rangle_{m,r,s,t,u}\, ds.$$

By specialising the parameters t, u to 1, 0 respectively we define $\Phi^r_m := \Phi^r_{m,1,0}$, $\Phi^r_{m,u} := \Phi_{m,1,u}$, and $\Phi^r_{m,t} = \Phi^r_{m,t,0}$.

The parameter t is used to pass from the resolvent cocycle to the transgression cocycle. The parameter u gives a homotopy from the transgression cocycle to the Chern character as per the Connes-Moscovici-Higson argument.

Theorem (1) In the (b, B)-bicomplex with coefficients in the set of holomorphic functions on a right half plane $\Re(r) > 1/2$, the resolvent cocycle $(\phi_m^r)_{m=P}^M$ is cohomologous to $(r - (1-p)/2)^{-1}\mathrm{Ch}_F^M$, modulo cochains with values in the set of holomorphic functions on a right half plane containing the critical point $r = (1-p)/2$.

(2) If moreover, the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ has isolated spectral dimension, then the residue cocycle $(\phi_m)_{m=\mathcal{P}}^M$ is cohomologous to the Chern character Ch_F^M.

B. The local index formula

Setting $X = PuP$, $P = (1+F)/2$ and $u \in \mathcal{A}^\sim$ unitary for an odd triple, and $X = pF_+p$, $F_+ = \frac{1}{4}(1-\gamma)F(1+\gamma)$, $p \in \mathcal{A}^\sim$ a projection, and with x standing for u or p depending on the context, we can summarize our results as follows.

NB: one may eliminate the doubled spectral triple and the claims follow with the original spectral triple.

More precisely, the double is only needed for the transgression argument where invertibility of $\mathcal{D}$ is essential. There is a passage back to the undoubled picture both at the Fredholm module and residue cocycle levels.

Theorem (1) The Chern character in cyclic homology computes the numerical index pairing, so

$$\mathrm{Index}_\tau(X) = c_{\mathcal{P}}\mathrm{Ch}_F^M\Big(\mathrm{Ch}^M(x)\Big)\,,$$

$$c_{\mathcal{P}} = 1 \text{ if } \mathcal{P} = 0, \quad c_{\mathcal{P}} = \frac{-1}{\sqrt{2\pi i}} \text{ if } \mathcal{P} = 1.$$

(2) The index can also be computed with the resolvent cocycle via

$$\mathrm{Index}_\tau(X) = c_{\mathcal{P}}\mathrm{res}_{r=(p-1)/2}\sum_{m=\mathcal{P}}^{M}\phi_m^r\Big(\mathrm{Ch}_m(x)\Big)\,,$$

and in particular $\sum_{m=P}^{M}\phi_m^r(\mathrm{Ch}_m(x))$ analytically continues to a deleted neighborhood of the critical point $r = (1-p)/2$ with at worst a simple pole at that point.

(3) If moreover the triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ has isolated spectral dimension, then the index can also be computed with the residue cocycle, via

$$\mathrm{Index}_\tau(X) = c_{\mathcal{P}} \sum_{m=\mathcal{P}}^{M} \phi_m\Big(\mathrm{Ch}_m(x)\Big).$$

Application: Index theorems for Dirac type operators on manifolds of bounded geometry.
For a manifold of bounded geometry the asymptotic expansion of the heat kernel for small time is identical to that for compact manifolds. Consequently the pole structure for the zeta functions appearing in the local index formula is identical to the compact case and we can use the argument of Ponge, for example, to obtain a local formula for the index of Dirac type operators which is similar to the relative index theorem of Gromov-Lawson. There is an L^2 index version as well.

Future work.
(i) NCG for pseudo-Riemannian manifolds.
(ii) Noncommutative applications: index theorems of Lesch-Phillips-Raeburn type for cross products by real actions.
(iii) Other noncommutative situations (beyond the Moyal plane).