Renormalizable noncommutative Quantum Fields I
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Harald Grosse

Faculty of Physics, University of Vienna

Introduction

- Motivation:
  Improve QFT in 4 dimensions, add ”gravity” effects
- Renormalizable Quantum Fields (not summmable)
- Space-Time
- Renormalizable Noncommutative Quantum Fields
  formulation, IR/UV mixing
- Main Results H G + R Wulkenhaar ...
  - Renormalizable ncQFTs: Scalar Higgs model
  - Taming the Landau Ghost summable almost solvable, nontrivial !
  - Gauge models, Fermions
  - properties: Wedge locality
- Outlook

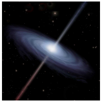
Two Pillars:

QUANTUM physics
Quantize Phase Space
$\hat{x} = x, \hat{p} = -i\hbar\frac{d}{dx}$

- $\nu_{i,j} = \nu_i - \nu_j$
  $\nu_{i,j} = \nu_{i,k} + \nu_{k,j}$
  $x_{i,j}(t) = x_{i,j}(0)e^{2\pi i\nu_{i,j}t}$
  gives matrix product
- $i\hbar\frac{d}{dt}\hat{x} = [H, \hat{x}]$ is derivation, example of nc geometry
- $[\hat{x}, \hat{p}] = i\hbar 1$

generalize to relativistic ...
renormalizable QFT

classical GRAVITY
Einstein Equations
$R_{i,j} - \frac{1}{2}Rg_{i,j} = 8\pi G T_{i,j}$



- Riemannian geometry
- $(ds)^2 = g_{ik}dx^i dx^k, \delta\int ds = 0$
  geodesic line
  Christoffel symbol Γ^i_{kl}
- Riemann curvature R^r_{ikl}
- $G^i_k = R^i_k - \frac{1}{2}Rg^i_k$

"quantize" gravity
not renormalizable

incompatible

Program

- Wightman QFT — Euclidean QFT
  - UV, IR, convergence problem, Landau ghost, triviality
  - NC QFT ideas, formulation, regularization, IR/UV mixing
  - Euclidean NC QFT
  - gauge model-matrix model (Japan,...)
  - deformed Wightman QFT fulfills Wedge locality
- Euclidean NC QFT
  - Renormalization
  - taming Landau ghost, β function vanishes
  - gauge Higgs model-matrix model
- Construction
  - Ward identity
  - Schwinger-Dyson equation
  - integral equation for renormalized N pt functions
  - Fermions - Spectral triple
- Outlook

Requirements

Quantum mechanical properties

- states are vectors of a separable Hilbert space H
- Space-time translations are symmetries:
  spectrum $\sigma(P_\mu)$ in closed forward light cone
  Ground state $\Omega \in H$ invariant under $e^{ia_\mu P^\mu}$
- $\Phi(f)$ on D dense, $\Phi(f) = \int d^4x \Phi(x) f^*(x)$,
  Ω is cyclic

Relativistic properties

- $U_{(a,\Lambda)}$ unitary rep. of Poincaré group on H, Covariance
- Locality
  $[\phi(f), \phi(g)]_\pm \Psi = 0 \qquad \text{for} \qquad suppf \subset (suppg)'$

Define Wighman functions $W_N(f_1 \otimes ... \otimes f_N) :=< \Omega\phi(f_1) \cdots \phi(f_N)\Omega >$

Euclidean

Schwinger distributions: $S_N(x_1, ..., x_N) = \int \phi(x_1)...\phi(x_N) d\nu(\phi)$

$$S_N(x_1 \dots x_N) = \frac{1}{Z} \int [d\phi]\, e^{-\int dx \mathcal{L}(\phi)} \prod_i^N \phi(x_i)$$

extract free part $\quad d\mu(\phi) \propto [d\phi]\, e^{-\frac{m^2}{2} \int \phi^2 - \frac{a}{2} \int (\partial_\mu \phi)(\partial^\mu \phi)}$

Two point correlation: $\langle \phi(x_1)\phi(x_2) \rangle = C(x_1, x_2)$

$\tilde{C}(p_1, p_2) = \delta(p_1 - p_2) \frac{1}{p_1^2 + m^2}$

$$\int d\mu(\phi)\phi(x_1) \dots \phi(x_N) = \sum_{\text{pairings}} \prod_{l \in \gamma} C(x_{i_l} - x_{j_l})$$

add $\frac{\lambda}{4!}$ Φ^4 interaction, expand

Φ^4 Interaction

$$S_N(x_1 \dots x_N) = \sum_n \frac{(-\lambda)^n}{n!} \int d\mu(\phi) \prod_j^N \phi(x_j) \left(\int dx \, \frac{\phi^4(x)}{4!} \right)^n$$

$$= \sum_{\text{graph } \Gamma_N} \frac{(-\lambda)^n}{Sym_{\Gamma_N}(G)} \int_V \prod_{l \in \Gamma_N} C_\kappa(x_l - y_l) \sim \Lambda^{\omega_D(G)}$$

put cutoffs, e.g.: $\tilde{C}_\kappa(p) = \int_{\kappa=1/\Lambda^2}^\infty d\alpha e^{-\alpha(p^2+m^2)}$

degree of divergence given by $\omega_D(G) = (D-4)n + D - \frac{D-2}{2}N$

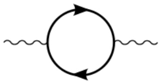
$\omega_2(G) = 2 - 2n,$ $\omega_4(G) = 4 - N$

n order # of vertices, N # of external lines, $(4n+N)!!$ is nr of Feynman graphs

use Stirling and $\frac{1}{n!}$ from exponential large order behavior $K^n n!$ ... no Taylor (Borel) convergence

Renormalization

For QED: polarization + higher order terms....



Leads to Vacuum fluctuations, Casimir effect, Lambshift, pair production virtual particles...

$$\tilde{G}_2(p) = \frac{1}{p^2+m^2} + \frac{1}{p^2+m^2}\Sigma\frac{1}{p^2+m^2} + \ldots = \frac{1}{p^2+m^2-\Sigma}$$

Impose renormalization conditions need 3

$$G_2(p^2=0) = \frac{1}{m^2_{phys}},\ \frac{d}{dp^2}G_2(p^2=0) = -\frac{a^2}{m^4_{phys}},\ G_4(p^2=0) = \lambda_{phys}$$

if NO NEW interactions are generated.... model is renormalizable

Space-Time, History

- Interacting models in $D = 2, 3$ are constructed
- $D = 4$ use renormalized pertubation theory RG FLOW
- add "Gravity" or quantize Space-Time

Project

merge general relativity with quantum physics through noncommutative geometry

Limited localisation in space-time $D \geq R_{ss} = G/c^4 hc/\lambda \geq G/c^4 hc/D$
$D \geq l_p \quad 10^{-35} m$

- Riemann, Schrödinger, Heisenberg,...
- 1947 Snyder,...
- 1986 Connes NCG...
- 1992 H G and J Madore, (Japan,...)
- 1995 Filk Feynman rules,...
- 1999 Schomerus: obtained nc models from strings,...

Algebra, fields, diff. calculus,...

- Space-Time
  - Gelfand - Naimark theorem Gelfand - isomorphism,...
  - Deform Algebra T^2_Θ, S^2_N, CP^N (Japan,...),R^4_Θ
  - Obtain associative star product, e.g. Moyal-Weyl product
  - Lie algebra $[x_k, x_l] = f^m_{kl} x_m$,
  - quantum group $x_k x_l = R^{mn}_{kl} x_m x_n$
- Fields
  - sections of bundles,...modules over A
  - (Serre-Swan) projective modules
- Differential Calculus
  - e g on $A_N = Mat(N, \mathbb{C}) = \{\mathbf{1}, \lambda_j \Big|_{j=1}^{N^2-1}\}$
  - with vector fields e_i inner derivations on A_N:
    $e_i(\mathbf{1}) = 0,\ e_i(\lambda_j) = [\lambda_i, \lambda_j] = f_{ij}{}^k \lambda_k$
  - Differential forms given by duality: $d\mathbf{1} = 0$ $(d\lambda^j)(e_k) := e_k(\lambda^j)$

    $\Omega^0(A_N) = A_N,\ \Omega^1(A_N) = \{fdg | f, g \in A_N\}$,...
    Gives differential complex $(\Omega^\star, d)$, $d^2 = 0$.

nc Scalar field model

Formulation e g : $D = 2$ $[c\hat{T}, \hat{X}] = i\Theta$

ϕ^4 on nc $\mathbb{R}^4$, $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}1$ antisymmetric, $u_p = e^{ip\hat{x}}$ or equivalently star product, $\partial_\mu u_p = ip_\mu u_p = i[\tilde{x}_\mu, u_p]$ $\tilde{x}_\mu := (\theta^{-1})_{\mu\nu}\hat{x}^\nu$

$$\Phi = \int dpe^{ip\hat{x}}\Phi_p$$

ϕ^4 action

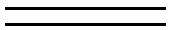
$$S = \frac{1}{2}Tr(-[\tilde{x}_\mu, \Phi][\tilde{x}^\mu, \Phi] + m^2\Phi\Phi + \frac{\lambda}{2}\Phi^4)$$

yields Schrödinger equation:

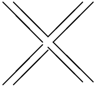
$$[\tilde{x}_\mu, [\tilde{x}^\mu, \Phi]] + m^2\Phi + \lambda\Phi^3 = E\Phi$$

Feynman rules

since $e^{ip\hat{x}} e^{iq\hat{x}} = e^{\frac{i}{2} p\Theta q} e^{i(p+q)\hat{x}}$



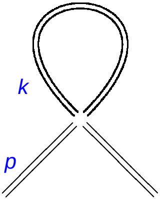
$\frac{1}{p^2+m^2}$



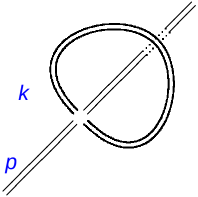
$\lambda e^{-\frac{i}{2}\sum_{i<j} p_i^\mu p_j^\nu \theta_{\mu\nu}}$

cyclic order of momenta yields ribbon graphs model not renormalizable

planar regular, nonregular and nonplanar contributions:



$$= \frac{\lambda}{6} \int dk \frac{1}{k^2+m^2}$$



$$= \frac{\lambda}{12} \int dk \frac{\mathrm{e}^{ip^\mu k^\nu \theta_{\mu\nu}}}{k^2+m^2} \; p \overset{\sim}{\to} 0 \; \frac{1}{\tilde{p}^2}$$

planar graphs: renormalize BPHZ ... nonplanar graphs "finite"

Theorem

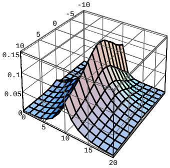
Main Result H. G. and R. Wulkenhaar ϕ^4 model modified,
IR/UV mixing: short and long distances related
Theorem: Action

$$S = \frac{1}{2} Tr(-[\tilde{x}_\mu, \Phi][\tilde{x}^\mu, \Phi] + \Omega^2 [\tilde{x}_\mu, \Phi]_+ [\tilde{x}^\mu, \Phi]_+ + m^2 \Phi\Phi + \lambda \Phi^4)$$

for $\tilde{x}_\mu := (\theta^{-1})_{\mu\nu} \hat{x}^\nu$
is perturbativly renormalizable to all orders in λ, 3 proofs,

- Action has position-momentum duality
- Power counting: $\Lambda^{4-N+4(1-B-2g)}$
- 4 rel/marginal operators! Propagator:

Do RG- FLOW



RG FLOW

Wilson RG-Flow

divide covariance for free Euclidean scalar field into slices

$$\Phi_m = \sum_{j=0}^{m} \phi_j, \quad C_j = \int_{M^{-2j}}^{M^{-2(j-1)}} d\alpha \frac{e^{-m^2\alpha - x^2/4\alpha}}{\alpha^{D/2}}$$

integrate out degrees of freedom

$$Z_{m-1}(\Phi_{m-1}) = \int d\mu_m(\phi_m) e^{-S_m(\phi_m + \Phi_{m-1})}$$

$$Z_{m-1}(\Phi_{m-1}) = e^{-S_{m-1}(\Phi_{m-1})}$$

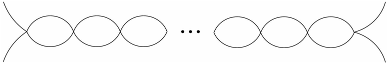
to evaluate: use loop expansion

Landau Ghost

- superficial degree of divergence for Feynman graph G

$$D = 4 \quad \omega(G) = 4 - N(G)$$

- BPHZ Theorem: renormalizability generate no new terms
- but: certain chain of finite subgraphs with m bubbles grows like



$$\int \frac{d^4q}{(q^2 + m^2)^3} (\log |q|)^m \simeq C^m m!$$

- not Borel summable

$$\lambda_j \simeq \frac{\lambda_0}{1 - \beta\lambda_0 j}$$

- sign of β positive: Landau ghost, triviality

TamingLandaughost, calculate β function

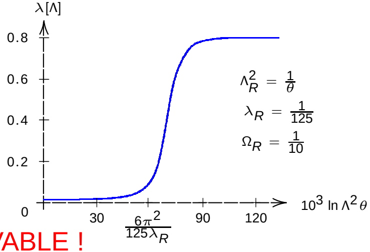
evaluate β function, H. G. and R. Wulkenhaar,

$$\Lambda \frac{d\lambda}{d\Lambda} = \beta_\lambda = \lambda^2 \frac{(1-\Omega^2)}{(1+\Omega^2)^3} + \mathcal{O}(\lambda^3)$$

flow bounded, L. ghost killed!
Due to wave fct. renormalization
$\Omega = 1$ betafunction vanishes
to all orders

$$\Omega^2[\Lambda] \leq 1$$

($\lambda[\Lambda]$ diverges in comm. case)



construction possible! Almost SOLVABLE !

Gauge model

H G + M. Wohlgenannt + ...

Couple scalar field to "external" gauge field

$$S[A] = Tr\left(\phi[X^\mu,[X_\mu,\phi]] + \Omega^2\phi\{X^\mu,\{X_\mu,\phi\}\} + \mu^2\phi^2 + \frac{\lambda}{4}\phi^4\right)$$

where $X^\mu = \tilde{x}^\mu + A^\mu$ are covariant coordinates.

Gauge transformation

$$X^\mu \to U^\dagger X^\mu U,\, A^\mu \to -iU^\dagger[\tilde{x}^\mu, U] + U^\dagger A^\mu U\,.$$

One-loop regularized effective action

$$\Gamma_\epsilon[A] = -\frac{1}{2}\int_\epsilon^\infty \frac{dt}{t} Tr\left(e^{-tH} - e^{-tH_0}\right)\,.$$

Use Duhamel expansion

$$e^{-tH} = e^{-tH_0} - \int_0^t dt_1 e^{-t_1 H_0} V_A e^{-(t-t_1)H_0} + \ldots$$

Identify: $\frac{\delta^2 S}{\delta\phi\delta\phi} = H_0 + V_A$ expansion up to order V^4 gives

Action, Fermions

$$S[A] = -\frac{1}{4}\int F^2 + \alpha\int((\tilde{X}_\nu \star \tilde{X}^\nu)^{\star 2} - (\tilde{x}^2)^2) + \beta\int((\tilde{X}_\nu \star \tilde{X}^\nu) - (\tilde{x}^2))$$

$F^{\mu,\nu} = [X^\mu, X^\nu] - i(\Theta^{-1})^{\mu,\nu}$ yields matrix models IKKT

Blaschke, Wohlgenannt, Steinacker, Schweda,....

Generalize BRST complex to nc gauge models with oscillator:
BRST invariant renormalization? tadpole ?

Take Dirac operator on Hilbert space $L^2(R^4) \otimes C^{16}$

$$D = (i\Gamma^\mu\partial_\mu + \Omega\Gamma^{\mu+4}\tilde{x}_\mu)$$

$\mu = 1, ...4$, Γ_k generate 8-dim Clifford algebra $\{\Gamma_k\Gamma_l\} = 2\delta_{kl}$

$$D^2 = (-\Delta + \Omega^2||\tilde{x}||^2)1 - i\Omega\Theta^{-1}_{\mu\nu}[\Gamma^\mu, \Gamma^{\nu+4}]$$

gives a spectral triple

QFT on noncommutative Minkowski space

- H G + Lechner; extended by Buchholz, Summers
  algebra generated by selfadjoint $\hat{x}_\mu$, $[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}$

Definition of quantum fields on NC Minkowski space DFR 95

$$\phi_\otimes(x) := \int dp\, e^{ip\cdot\hat{x}} \otimes e^{ip\cdot x}\, \tilde{\phi}(p)$$

$\phi_\otimes(f)$ act on $\mathcal{V}\otimes\mathcal{H}$, build polynomial algebra $\mathcal{F}^\otimes$

- Vacuum $\omega_\theta := \nu \otimes \langle\Omega, .\Omega\rangle$ independent of ν
- GNS representation of $\mathcal{F}^\otimes$ w.r.t. ω_θ yields $\mathcal{H}_\theta = \mathcal{H}$, $\Omega_\theta = \Omega$, $\pi_\theta(\phi_\otimes(f)) =: \phi_\theta(f)$
  $\phi^\theta(f)\Psi_n(G) = \Psi_{n+1}(f \otimes_\theta G)$

Definition (Moyal tensor product) $f_n \in S(R^{nd}), g \in S(R^{md})$
$(f_n \otimes_\theta g_m)(x,y) = \int d^d\xi \int d^dk\, f_n^\xi(x)\, g_m^{\theta k}(y)\, e^{-2i\xi\cdot k}$

Obtain deformation of Wightman functions

$\tilde{W}_\theta(p_1,\dots,p_N) := \langle\Omega, \tilde{\phi}^\theta(p_1)\cdots\tilde{\phi}^\theta(p_N)\Omega\rangle = \tilde{W}_0(p_1,\dots,p_N)\cdot\prod_{l<r} e^{-\frac{i}{2}p_l\theta p_r}$ continuous commutative limit $\theta \to 0$

Wedges and Wedge local QF

- We relate the antisymmetric matrices to Wedges: $W_1 = \left\{ x \in \mathbb{R}^D | x_1 > |x_0| \right\}$ act on standard wedge by proper Lorentz transformations $i_\Lambda(W) = \Lambda W$.
- Stabilizer group is $SO(1,1)xSO(2)$, which corresponds to boosts and rotations.

$$\mathcal{W}_0 = \mathcal{L}_+^\uparrow W_1$$

- Reflections: $j_\mu : x_\mu \mapsto -x_\mu$
  $\mathcal{W}_0$ with $\mathcal{L}$ -action $i_\Lambda : \mathcal{W}_0 \mapsto \Lambda \mathcal{W}_0$ is $\mathcal{L}_+$ - homogenous space.
- Get isomorphism $(\mathcal{W}_0, i_\Lambda) \cong (\mathcal{A}, \gamma_\Lambda)$
  $\mathcal{A} = \{\gamma_\Lambda(Q_1) | \Lambda \in \mathcal{L}_+\}$ $Q(\Lambda W_1) := \gamma_\Lambda(Q_1)$
- Gives correspondence of set of wedges to antisymmetric matrices.
- Define wedge local fields: $\phi = \{\phi_W | W \subset \mathcal{W}_0\}$ get family of fields,
- covariance and localization in wedges.

Wedge locality

With this isomorphism define $\Phi_W(x) := \Phi_{\Theta(W)}(x)$. Transformation properties

$$U_{y,\Lambda}\Phi_W(x)U^\dagger_{y,\Lambda} = \Phi_{\gamma_\Lambda(\Theta(W))}(\Lambda x + y)$$

Theorem

Let $\kappa_e \geq 0$ the family $\Phi_W(x)$ is a wedge local quantum field on Fockspace:

$$[\phi_{W_1}(f), \phi_{-W_1}(g)](\psi) = 0,$$

for $supp(f) \subset W_1$, $supp(g) \subset -W_1$.

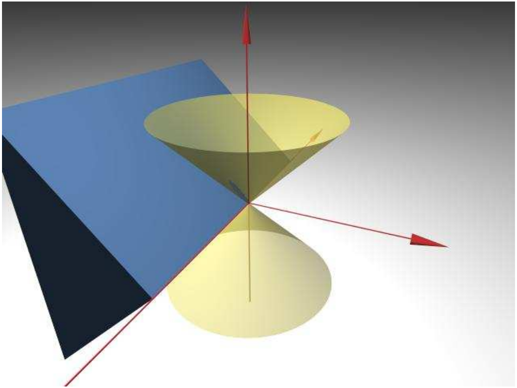
Show that

$$[a_{W_1}(f^-), a^\dagger_{-W_1}(g^+)] + [a^\dagger_{W_1}(f^+), a_{-W_1}(g^-)] = 0$$

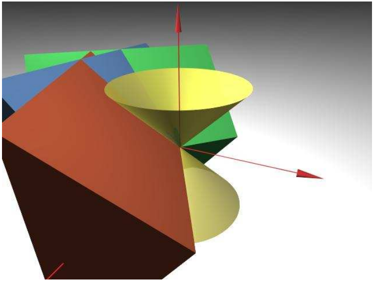
$\vartheta = \sinh^{-1}\left(p_1/(m^2 + p_2^2 + p_3^2)^{1/2}\right)$
Use analytic continuation from R to $R + i\pi$ in ϑ.

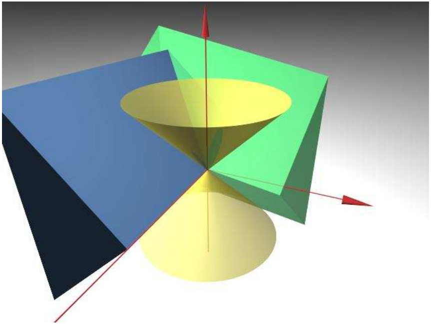
Wedges

Facts: $W_1 \subset W_2 \Leftrightarrow W_1 = W_2$, and $W(\theta)' = W(-\theta)$



Wedges





- Need well localized states for asymptotics $t \to \pm\infty$ ("in/out")
- Wedge-locality is good enough for two particle scattering
- Find for two-particle scattering with $p_1^1 > p_2^1$, $q_1^1 > q_2^1$:
  ${}^{\theta_1}_{\text{out}}\langle p_1, p_2 | q_1, q_2 \rangle^{\theta_1}_{\text{in}} = e^{\frac{i}{2} p_1 \theta p_2} e^{\frac{i}{2} q_1 \theta q_2} {}_{\text{out}}\langle p_1, p_2 | q_1, q_2 \rangle_{\text{in}}$ non-trivial S-matrix, **deformation induces interaction!**
- measurable effects of the noncommutativity (time delay)

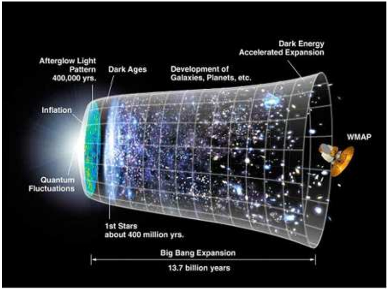
Outlook

Obtained renormalized II summable nc QFT. III locality weakened to Wedge locality

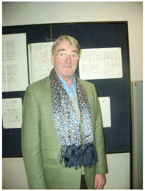
Learn Principles
Learn to obtain
ren. sum. nc Standard Model?
ren. sum. Quantum Gravity?

- implications for cosmology?
  cosmological constant problem ?
  dark matter, dark energy?

  information paradox,....? Inflation?
- implications for experiments???



Photos from Japan



arigatou gozaimasu