Symplectic categories and quantum categories

Alan Weinstein
University of California, Berkeley

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Many interesting morphisms in symplectic geometry are not functions $X \leftarrow Y$ but just relations. These are the canonical relations, i.e. lagrangian submanifolds of $X \times \overline{Y}$.

Although relations between sets form a category, with nice properties (symmetric monoidal, rigid, admitting a "transposition"), various classes of nice relations, such canonical relations, do not form subcategories because they are not closed under composition without a transversality assumption (and more).

How can we talk about these "bad compositions" in a systematic way? That is the subject of these talks, where the aim is to include the canonical relations as morphisms in categories to which we try to apply "quantization functors." A central construction is that of Wehrheim and Woodward, originating from symplectic topology.

In these talks, I will suggest a way to abstract the particularities of composition of canonical relations, and then look for similar categories in the "quantum worlds" of unbounded linear operators and semiclassical operators.

Along the way, I will present some geometric ideas related to generating functions and "hyperlagrangian" submanifolds.

Begin with categories of relations as guiding example.
(The abstract version is known as "allegories" [Freyd].)

Start with sets.
A relation between X and Y is a subset $B \subseteq X \times Y$.

$$\begin{tikzcd} & B \arrow[dl] \arrow[dr] & & C \arrow[dl] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd}$$

Composition is by forming $B \times_Y C$ and taking its image $B \circ C$ in $X \times Y$.
If the projection $B \circ C \leftarrow B \times_Y C$ is injective, we call (B, C) a monic pair.

More generally, a hyperrelation (also called "span", or "correspondence") is any diagram

$$\begin{tikzcd} & B \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

B' (perhaps up to isomorphism)

Composition is by fibre product as before; the HREL product of two relations is again a relation iff the pair is monic.

Given $\begin{tikzcd}[column sep=small] & B \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$, we call the "legs" the <u>target</u> and <u>source</u>, and we introduce special kinds (already interesting in REL)

injective : target is injective
surjective : target is surjective
coinjective : source is injective ("single valued")
cosurjective : source is surjective ("everywhere defined")

Each of these properties is closed under composition.

<u>TWO SPECIAL KINDS</u> (<u>Benenti-Tulczyjew</u>)

$X \leftarrow Y$ surjective and coinjective means that X is a subquotient of Y. We write $X \twoheadleftarrow Y$ and call this a <u>reduction</u> (= epi)

$X \leftarrow Y$ injective and cosurjective means that the elements of Y all map to disjoint subsets of X. $X \leftarrowtail Y$, <u>coreduction</u>. (= mono)

A (contravariant) transposition functor exchanges properties with their "coproperties".

When X and Y have more structure, we can consider compatible relations, e.g. linear, affine, topological, smooth. We can then refine the special types above, i.e. injective ⇝ properly embedded, surjective ⇝ submersive (having local cross sections).

(In the smooth case, the refinements amount to applying the tangent functor and then imposing the original conditions.)

<u>COMPOSITIONS</u>

The class of smooth relations is not closed under composition. A useful sufficient condition is that, for
$$\begin{tikzcd}[column sep=small] & B \arrow[dl] \arrow[dr] & & C \arrow[dl] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd}$$
the map $\begin{matrix} B\times C \\ \downarrow \\ Y\times Y \end{matrix}$ is transversal to the diagonal Δ_Y. We then call (B,C) a <u>transversal pair</u>.
(Alternatively, $B\times C$ is transversal to $X\times\Delta_Y\times Z$ in $X\times Y\times Y\times Z$)

For a transversal pair of smooth relations to have smooth composition we also want $X\times Z \longleftarrow B\times_Y C$ to be a proper embedding. (Strong version of <u>monic</u>.) We call (B,C) <u>strongly transversal</u>.

[Note: No useful notion of "transversal" in set-theoretic setting. ??]

<u>Important fact 1</u>: Compositions of the form $\leftarrowtail\ \leftarrow$, $\leftarrow\ \twoheadleftarrow$ are always strongly transversal

<u>Important fact 2</u>: Composition of a ST pair of <u>canonical</u> relations is a canonical relation.

<u>Remark</u>: A canonical relation $X \twoheadleftarrow Y$ is essentially a symplectic reduction, i.e. the domain is coisotropic, and the fibres of this map are (discrete unions of) characteristic submanifolds

<u>Remark</u>: Any smooth $M \xleftarrow{f} N$ has a lift $T^*M \xleftarrow{T^*f} T^*N$; this operation is "functorial" (when compositions are nice).

(HIGHLY) - SELECTIVE CATEGORIES

A *selective category* has distinguished morphisms called *suave* and distinguished composable pairs (f,g) called *congenial*. $(f \pitchfork g)$

WE REQUIRE

0. Every identity morphism 1_X is suave.
1. $f \pitchfork g \Rightarrow f, g$, and fg are all suave
2. $X \xleftarrow{f} Y \xleftarrow{g} Z$, if one is suave and the other is a smooth isomorphism, then $f \pitchfork g$.

   (implies: $X \xleftarrow{f} Y$ suave $\Rightarrow 1_X \pitchfork f$, $f \pitchfork 1_Y$.)
3. [optional?] If $f \pitchfork g$ and $(fg) \pitchfork h$, then $g \pitchfork h$ and $f \pitchfork (gh)$.

A *selective functor* is one which preserves suavity and congeniality.

EXAMPLES

1. All relations, everything suave, everything congenial.
2. As above, but only monic pairs congenial.
3. On manifolds: suave = smooth, congenial = str. trans. (SREL)
4. Canonical relations, as in 3.
5. Cotangent lift is a selective functor.

The Wehrheim-Woodward construction

Given a selective category $\mathcal{C}$, we build $WW(\mathcal{C})$ containing all the suave morphisms, with all congenial compositions as before.

The construction is "dynamical": a *path* is a diagram of suave morphisms,

$$\cdots X_{-2} \xleftarrow{f_{-1}} X_{-1} \xleftarrow{f_0} X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \cdots \quad \text{in } \mathcal{C}$$

with all but finitely many f_i's identity morphisms. ("compactly supported"). If you prefer, a "word".

Two paths are <u>equivalent</u> if they are related by "moves" of the following two kinds:

(1) Insertion (or deletion) of an identity morphism

(2) Replacement of congenial $X_{j-1} \xleftarrow{f_j} X_j \xleftarrow{f_{j+1}} X_{j+1}$

by $X_{j-1} \xleftarrow{1_{X_{j-1}}} X_{j-1} \xleftarrow{f_j \circ f_{j+1}} X_{j+1}$ or $X_{j-1} \xleftarrow{f_j \circ f_{j+1}} X_{j+1} \xleftarrow{1_{X_{j+1}}} X_{j+1}$.

(The identities aren't really necessary, but convenient when you look at tensor products.)

In the path language, you may think of these moves as "homotopies."

Then we have:

TARGET
SOURCE
COMPOSITION
UNITS

$$X \xleftarrow{\text{identities}} \cdots \leftarrow X_a \leftarrow \cdots \xleftarrow{f_b} X_b \leftarrow \cdots \xleftarrow{\text{idens.}} \cdots \leftarrow Y$$

$$Y \xleftarrow{\text{idens.}} \cdots \leftarrow Y_p \leftarrow \cdots \leftarrow Y_q \leftarrow \cdots \xleftarrow{\text{identities}} Z$$

The result is the category $WW(\mathcal{C})$. Notice that we have a functor $\mathcal{C} \longleftarrow WW(\mathcal{C})$ which is "the identity" on morphisms $\overset{\leftarrow}{\cdots} X_0 \leftarrow X_1 \overset{\leftarrow}{\cdots}$ given by a single nonidentity arrow. The functor is given by composition in $\mathcal{C}$ ("scattering operator"?), and its existence shows that the morphisms in $\mathcal{C}$ are not "collapsed" by the equivalence relation.

In fact, we have a cross section

$\mathcal{C}_{\text{suave}} \longrightarrow WW(\mathcal{C})$ multiplicative on congenial pairs.

(universal property).

A selective category is <u>highly-selective</u> if it contains subcategories of suave morphisms called <u>reductions</u> $X \twoheadleftarrow Y$ and <u>coreductions</u> $X \leftarrowtail Y$ such that

(1) Every suave isomorphism is a reduction and a coreduction.

(2) A composition $X \leftarrow Y \leftarrow Z$ is congenial if one of the morphisms is decorated at Y (or if both are).

(3) Every suave morphism $X \leftarrow Y$ may be factored as a congenial product $X \twoheadleftarrow Q \leftarrowtail Y$.

<u>EXAMPLES.</u>

The category SREL of relations between symplectic manifolds is highly selective, with suave morphisms being canonical relations, and reductions and coreductions as for smooth relations.

And the category of relations between manifolds is, too.

The key issue is <u>factorization</u>. For maps of manifolds, it's easy using the "<u>graph</u>"

$$\begin{tikzcd} X & & Y \arrow[ll, "f"'] \arrow[dl, tail, "f\times 1_Y"] \\ & X\times Y \arrow[ul, two heads, "p_X"] & \end{tikzcd}$$

For relations it's more complicated. We have morphisms:

$$X\times\bar{X} \xleftarrow{\delta_X} pt \quad \text{and} \quad pt \xleftarrow{\varepsilon_X} \bar{X}\times X.$$

The graph of any $X \xleftarrow{L} Y$ can be thought of as $X\times\bar{Y} \xleftarrow{\gamma_L} pt$, and this γ_L has the congenial composition form:

$$X\times\bar{Y} \xleftarrow{L\times 1_{\bar{Y}}} Y\times\bar{Y} \xleftarrow{\delta_Y} pt.$$

We then have the congenial factorization

$$\begin{tikzcd} X & & Y \arrow[ll, "L"'] \arrow[dl, tail, "\gamma_L\times 1_Y"] \\ & X\times\bar{Y}\times Y \arrow[ul, two heads, "1_X\times\varepsilon_Y"] & \end{tikzcd}$$

where isomorphisms $X \cong X\times pt$ and $Y \cong pt\times Y$ are "implied".

BACK TO GENERAL THEORY.

THEOREM 1

Let $\mathcal{C}$ be highly selective.

Any $WW(\mathcal{C})$ morphism

$$X \;\cdots\leftarrow X_0 \leftarrow \cdots\; Y$$

can be represented by a 2-step sequence

$$\leftarrow X \twoheadleftarrow R \hookleftarrow Y \leftarrow$$

where $X \twoheadleftarrow R$ is a reduction
and $R \hookleftarrow Y$ is a coreduction

e.g. Chains of canonical relations or, as we will see in the quantum case, Schwartz kernels.

For the next result, the most general setting is that of "extra special rigid monoidal categories", which have products and duals, but just think $\mathcal{C} = \mathrm{SREL}$.

HYPERRELATIONS AND HYPERLAGRANGIAN SUBMANIFOLDS

A lagrangian submanifold of X is just a canonical relation $X \leftarrow pt$. It's natural to look at a WW morphism $X \leftarrow pt$ as a generalized l.s. By Theorem 1, we can write it is $X \twoheadleftarrow Q \hookleftarrow pt$, i.e. as a reduction of some lagrangian submanifold of a (possibly) bigger manifold Q.

We can apply this idea to more general relations $X \leftarrow Y$, as follows:

THEOREM 2

There is an isomorphism of categories between $WW(\mathcal{C})$ and the category $HR(\mathcal{C})$ whose objects are the same and whose morphisms are "hyperrelations"

$$[X \times \bar{Y} \twoheadleftarrow Q \hookleftarrow pt]$$

with composition given by:

$$\begin{tikzcd} pt \arrow[d] \arrow[dd, bend left] \\ pt \times pt \arrow[d] \\ Q \times Q' \arrow[d] \arrow[dd, bend right] \\ X \times \bar{Y} \times Y \times \bar{Z} \arrow[d] \\ X \times \bar{Z} \end{tikzcd}$$

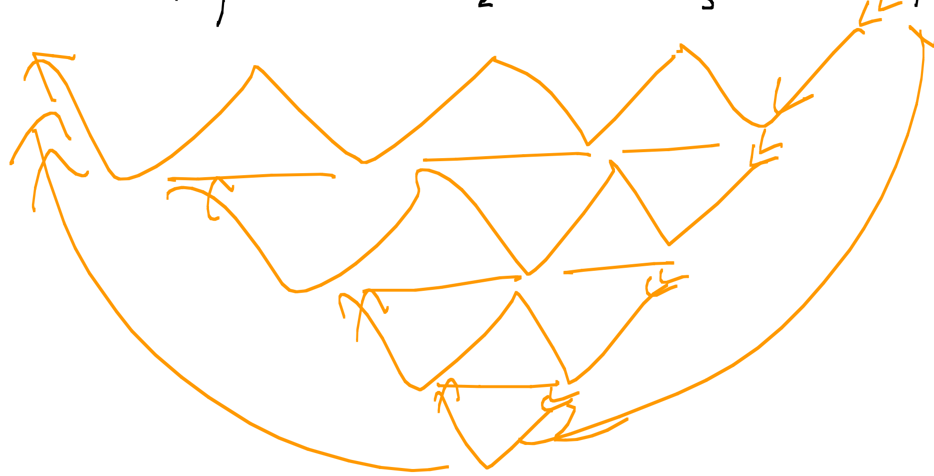
$$\boxed{\begin{array}{c} X \times \bar{X} \xleftarrow{\delta_X} pt \\ pt \xleftarrow{\varepsilon_X} \bar{X} \times X \end{array}}$$

MORE DETAILS for Theorem 1

The factorization
$$\begin{tikzcd} X & & Y \arrow[ll] \arrow[dl] \\ & Q \arrow[ul, two heads] & \end{tikzcd}$$
in the definition of a highly selective category is the lowest case of Theorem 1, but is the basis for the entire proof.

Let's look at $n=4$.

$$\cdots \leftarrow X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow X_4 \leftarrow \cdots$$



See ArXiv 1012.0105

Related to { simple homotopy theory (Whitehead)?
waldhausen categories?

More on Theorem 2

The idea of hyperlagrangian $X \twoheadleftarrow Q \leftarrow pt$ comes from exposition by Guillemin-Sternberg of the Maslov-Hormander theory of generating functions for lagrangian submanifolds of T^*P (eg. $T^*(M \times N)$).
Similar ideas occur in the work of Benenti (recent book).

Roughly, we use generating families
$P \overset{\pi}{\twoheadleftarrow} B \xrightarrow{S} \mathbb{R}$ which might not be "Morse";
to get $T^*P \twoheadleftarrow T^*B \leftarrow pt$.

This turns out to be the general local model.

<u>Functor from WW to HL.</u> We associate to each generator $X \xleftarrow{L} Y$ its graph $X \times \bar{Y} \twoheadleftarrow X \times \bar{Y} \leftarrow pt$
Then show that each nice composition goes to the composition of graphs (up to equivalence).

<u>Functor from HL to WW.</u> Assign to $X \times \bar{Y} \overset{C}{\twoheadleftarrow} Q \xleftarrow{L} pt$,
the WW morphism

$$X \overset{1_X \times \varepsilon_Y}{\twoheadleftarrow} X \times \bar{Y} \times Y \overset{C \times 1_Y}{\twoheadleftarrow} Q \times Y \xleftarrow{L \times 1_Y} Y$$

(unit objects and associativity implied).

Then show functoriality and inverse property.

A good way to analyze hypercanonical relations (in particular, hyperlagrangian submanifolds) is via

TRAJECTORIES

Given a path

$$X \cdots \leftarrow X_{-2} \xleftarrow{f_{-1}} X_{-1} \xleftarrow{f_0} X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \leftarrow \cdots \quad Y$$

of relations,

a *trajectory* is a sequence

$$x_{-2}, x_{-1}, x_0, x_1, \cdots \cdots$$

such that $(x_{i-1}, x_i) \in f_i$.

Given a path $f = \cdots f_{-1}, f_0, f_1, \cdots$ of relations, we have a trajectory space $\mathcal{T}(f)$ and a map

$X \times Y \xleftarrow{\tau(f)} \mathcal{T}(f)$ whose image is the composed relation.

The fibres of $X \times Y \xleftarrow{\tau(f)} \mathcal{T}(f)$ are an <u>invariant</u> of the WW morphism represented by the sequence. In particular, for $Y = pt$ in SREL, we find that for

$$X \overset{C}{\twoheadleftarrow} Q \overset{L}{\leftarrowtail} pt$$

the fibres of the projection $X \leftarrow C \cap L$ are an invariant of a hyperlagrangian submanifold of X. It is trivial when the hyperlagrangian is lagrangian.

(Resemblance to the isotropy of stacks?)

Lecture 2. Quantum categories

Recall: (Highly) Selective categories $\mathcal{C}$

Selective: Suave morphisms
Congenial pairs
$\rightsquigarrow WW(\mathcal{C})$ solves universal problem

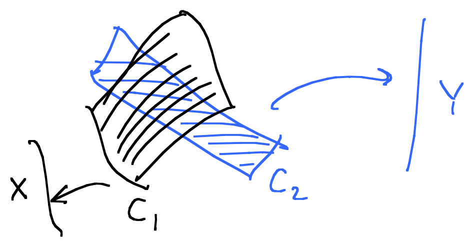
$$\begin{tikzcd} \mathcal{C}_{suave} \arrow[r] \arrow[dr] & \mathcal{D} \\ & WW(\mathcal{C}) \end{tikzcd}$$

Highly selective: Reductions
Coreductions
Factorization

THEOREM. Can reduce any element of $WW(\mathcal{C})$ to two steps

$$X \twoheadleftarrow Q \leftarrowtail Y$$

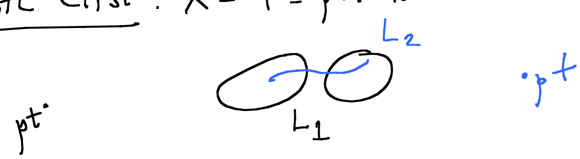
In SREL:



Can reduce further when each C_1 fibre meets each C_2 fibre in at most one point, transversally.

e.g. for symplectomorphism $X \xleftarrow{f} Y$, take $Q = X \times \overline{Y} \times Y$
$C_1 = X \times \Delta_Y$
$C_2 = \Gamma_f \times Y$
$C_1 \cap C_2 = \{(x, y_1, y_2) \mid x = f(y_1), y_1 = y_2\}$.

SPECIAL CASE: $X = Y = pt$. How to classify morphisms?



$\approx$ $\cdot \Lleftarrow : \Lleftarrow \cdot pt$

The ~~set~~ $L_1 \cap L_2$ is invariant. How much more?
e.g. if L_2 is connected, what can we say?

EXAMPLE
$$pt \xleftarrow{L_1} Q \xleftarrow{L_2} pt$$
$$\approx \; pt \xleftarrow{\varepsilon_Q} \overline{Q} \times Q \xleftarrow{L_1 \times L_2} pt$$
$$\approx \; pt \xleftarrow{\overline{L}_2} Q \xleftarrow{\overline{L}_1} pt$$

$\underline{Y = pt}$. Local models from generating functions

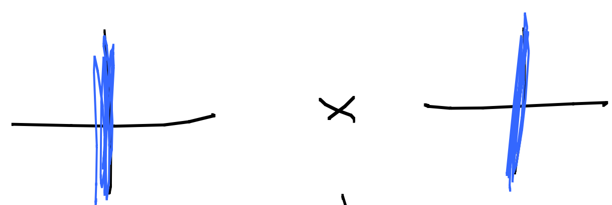
$$T^*M \xleftarrow{T^*p} T^*B \xleftarrow{dS} pt \qquad M \xleftarrow{p} B$$

or

$$T^*M \xleftarrow{(T^*i)^t} T^*B \xleftarrow{dS} pt \qquad M \xhookrightarrow{i} B$$

THE QUANTUM CASE

GUIDING EXAMPLE



$$T^*\mathbb{R} \xleftarrow{L} T^*\mathbb{R} \quad \rightsquigarrow \quad f(0)\cdot\delta_0(x) \mapsfrom f(y)$$

$$? \xleftarrow{\mathrm{Quan}(L)} ?$$

What should the structure on the target of $\rightsquigarrow$ be?

FIRST SOLUTION

Objects are rigged Hilbert spaces (also called Gelfand triples) $\mathcal{H}_\infty \subseteq \mathcal{H} \subseteq \mathcal{H}_{-\infty}$; e.g. $C_c^\infty(M) \subseteq L^2(M) \subseteq \mathcal{D}'(M)$ or $\mathcal{S}(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n) \subseteq \mathcal{S}'(\mathbb{R}^n)$

Morphisms:

$$\begin{array}{ccc} \underset{\sim}{\mathcal{G}} & & \mathcal{G}_\infty \subseteq \mathcal{G} \subseteq \mathcal{G}_{-\infty} \\ \downarrow & & \mathcal{G}_0 \nearrow \searrow \\ \underset{\sim}{\mathcal{H}} & & \mathcal{H}_\infty \subseteq \mathcal{H} \subseteq \mathcal{H}_{-\infty} \end{array}$$

Composition: Use largest (?) possible domain, with suitable topology.

Suave if $\mathcal{G}_\infty \hookrightarrow \mathcal{G}_0 \hookrightarrow \mathcal{G}_{-\infty}$ $\Longleftrightarrow$ given by a "Kernel" in $\mathcal{G}_{-\infty} \otimes \mathcal{H}_{-\infty}$

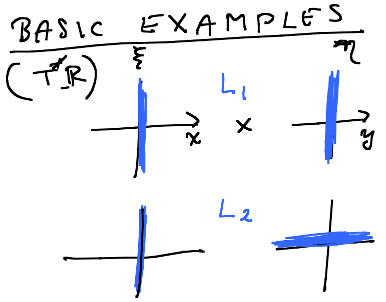
Guillemin – Sternberg (1978) construct "highly selective quantization functor" on linear canonical relations.

Also Tulczyjew – Zakrzewski on "Fresnel kernels" (1984).

A pair $\underset{\sim}{\mathcal{G}} \xleftarrow{\sigma} \underset{\sim}{\mathcal{H}} \xleftarrow{\tau} \underset{\sim}{\mathcal{K}}$ of suave morphisms is congenial if $\mathcal{H}_0 \supseteq \tau(\mathcal{K}_\infty)$.

A morphism $\underset{\sim}{\mathcal{G}} \xleftarrow{\sigma} \underset{\sim}{\mathcal{H}}$ is a reduction if $\mathcal{G}_\infty \supseteq \sigma(\mathcal{H}_\infty)$ and a coreduction if $\mathcal{H}_0 = \mathcal{H}_{-\infty}$, i.e. defined on all of $\mathcal{H}_{-\infty}$.
(Duality exchanges these two conditions.)

BASIC EXAMPLES



$(L^2(\mathbb{R}))$ $\quad Q(L_1)$

$\rightsquigarrow \quad f(0)\,\delta_0(x) \longleftarrow\!\!| \; f(y)$

$Q(L_2)$

$\rightsquigarrow \quad \left[\int_{-\infty}^{\infty} f(y)\,dy\right] \delta_0(x) \longleftarrow\!\!| \; f(y)$

$L_1 \circ L_1 = L_1$ (not monic)
$L_2 \circ L_2 = L_2$ (monic)
$L_1 \circ L_2 = L_2$ (not monic)
$L_2 \circ L_1 = L_1$ (monic)

EXAMPLE. If $M \subseteq N$, restriction $\underset{\sim}{L^2(M)} \longleftarrow \underset{\sim}{L^2(N)}$ is a reduction and extension $\underset{\sim}{L^2(N)} \longleftarrow L^2(M)$ is a coreduction.
(Some normal data is needed to define these.)

FACTORIZATION comes from the Schwartz kernel theorem; each suave $\underset{\sim}{\mathcal{G}} \xleftarrow{\sigma} \underset{\sim}{\mathcal{H}}$ has a kernel $K_\sigma \in \mathcal{G}_{-\infty} \otimes \mathcal{H}_{-\infty}$ which we may consider as a suave morphism $\underset{\sim}{\mathcal{G} \otimes \mathcal{H}} \longleftarrow \mathbb{C}$.

Then we have the factorization:

$$\int g(x,y,y)\,dy \longleftarrow\!\!| \; g(x,y',y)$$

$$\mathcal{G} = \mathcal{G} \otimes \mathbb{C} \xleftarrow{1_{\mathcal{H}} \otimes E_{\mathcal{H}}} \underset{\sim}{\mathcal{G}} \otimes \underset{\sim}{\mathcal{H}} \otimes \underset{\sim}{\mathcal{H}} \xleftarrow{K_\sigma \otimes 1_{\mathcal{H}}} \mathbb{C} \otimes \underset{\sim}{\mathcal{H}} = \underset{\sim}{\mathcal{H}}$$

$$K_\sigma(x,y')\,f(y) \longleftarrow\!\!| \; 1 \otimes f(y) = f(y)$$

Theorem 2 and the surrounding discussion give rise to notions of hyperdistribution which can be manipulated more freely than distributions. (e.g. hyperdistribution on M is embedding $M \longrightarrow N$ together with a distribution on N, up to some equivalence).

Application: multiplication of (hyper) distributions.

Symplectically: $T^*M \xleftarrow{\mu} T^*M \times T^*M$

fibre addition = cotangent lift of codiagonal $M \xleftarrow{\Delta^t} M \times M$ (A groupoid structure!)

Given $L_1, L_2 \in T^*M$, we form

$$\begin{tikzcd} T^*M & T^*M \times T^*M \arrow[l, "\mu"'] & pt \times pt \arrow[l, "L_1 \times L_2"'] & pt \arrow[l] \arrow[lll, bend left, "L_1 + L_2"] \end{tikzcd}$$

Not necessarily a congenial composition.

Quantum version:

$$\begin{tikzcd} \underbrace{L^2(M)}_{} & \underbrace{L^2(M \times M)}_{} \arrow[l, "\text{restrict to diagonal}"'] & \mathbb{C} \otimes \mathbb{C} \arrow[l, "\varphi_1 \otimes \varphi_2"'] & \mathbb{C} \arrow[l] \arrow[lll, bend left, "\varphi_1 \cdot \varphi_2"] \end{tikzcd}$$

Well defined as a "hyperdistribution".)

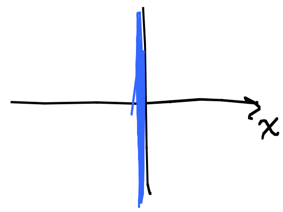
Even more basic is pairing of distributions

$$\mathbb{C} \xleftarrow{\varphi^t} \underset{\sim}{L^2(M)} \xleftarrow{\varphi_2} \mathbb{C}$$

Always well-defined as a "hypernumber" $\mathbb{C} \leftarrow \mathbb{C}$.

What is the structure of the hypernumbers?

SECOND SOLUTION: (Semiclassical operators)

For  , insert a compactly supported amplitude:

$$f(x,\hbar) = \int_{-\infty}^{\infty} e^{i\xi x/\hbar}\, a(\xi)\, d\xi$$

If $a \equiv 1$, this is $2\pi\hbar\delta$. Order -1 in $(1/\hbar)$.

If a has compact support, we have

$$\int_{-\infty}^{\infty} e^{i\xi x/\hbar}\, a(\xi)\, d\xi \overset{\xi = \hbar\eta}{=\!=} \hbar \int_{-\infty}^{\infty} e^{i\eta x}\, a(\hbar\eta)\, d\eta$$

Testing against a smooth function of x with compact support, we have

$$\hbar \int_{-\infty}^{\infty} e^{i\eta x}\, a(\hbar\eta)\, u(x)\, dx\, d\eta$$

$$= \hbar \int_{-\infty}^{\infty} a(\hbar\eta)\, \hat{u}(-\eta)\, d\eta, \text{ which is}$$

bounded by $\hbar \int_{-\infty}^{\infty} \|a\|_\infty |\hat{u}(-\eta)|\, d\eta = O(\hbar)$, so we have something smooth, of order -1

Multiplying this by itself, we have

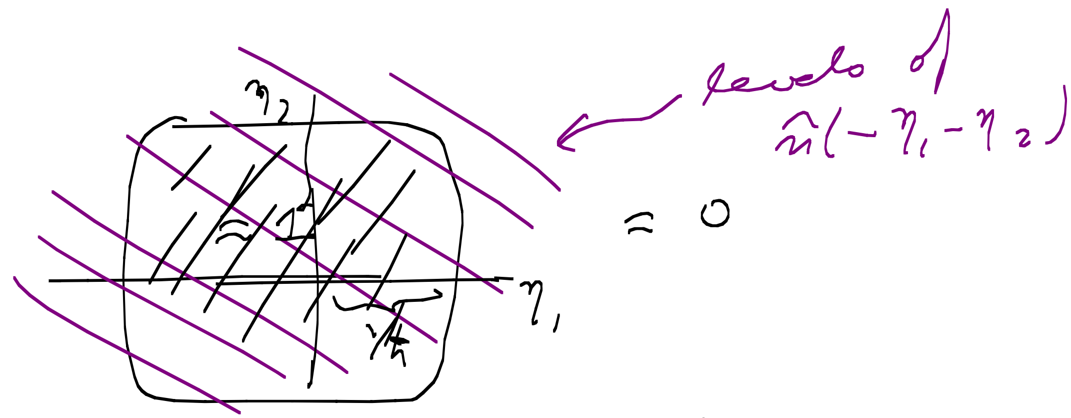
$$\int_{-\infty}^{\infty} e^{ix\xi_1/\hbar} a(\xi_1)\, d\xi_1 \cdot \int_{-\infty}^{\infty} e^{ix\xi_2/\hbar} a(\xi_2)\, d\xi_2 \text{; testing gives}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ix(\xi_1+\xi_2)/\hbar} a(\xi_1) a(\xi_2) u(x)\, dx\, d\xi_1\, d\xi_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ix(\eta_1+\eta_2)} a(\hbar\eta_1) a(\hbar\eta_2) u(x)\, dx\, \hbar^2\, d\eta_1\, d\eta_2$$

$$= \hbar^2 \iint a(\hbar\eta_1) a(\hbar\eta_2)\, \hat{u}(-\eta_1,-\eta_2)\, d\eta_1\, d\eta_2$$

Integration in \\\\ direction gives something of order $1/\hbar$, so the final result is $O(\hbar)$ rather than $O(\hbar^2)$.



We interpret this by saying that the pair (f,f) has multiplicative defect equal to 1.

We can interpret the example above as the composition of the operators $(1 \otimes f) \circ (f \otimes 1)$

$$\mathbb{C} \longleftarrow C_0^\infty(\mathbb{R}) \longleftarrow \mathbb{C}.$$

In general, to each canonical relation

$$T^*M \xleftarrow{\Lambda} T^*N,$$

we can associate a class of operators with smooth compactly supported kernels; each operator has an <u>order</u>. Given $T^*M \xleftarrow{\Lambda_1} T^*N \xleftarrow{\Lambda_2} T^*P$,

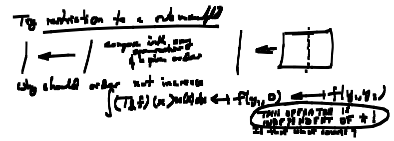
we can define a <u>defect</u> to be

$$\sup_{a_1, a_2} \left[\operatorname{ord} Op(\Lambda_1, a_1) \circ Op(\Lambda_2, a_2) - \operatorname{ord} Op(\Lambda_1, a_1) - \operatorname{ord} Op(\Lambda_2, a_2) \right];$$

we should be able to prove that the defect is <u>zero</u> for transversal pairs and positive for others.

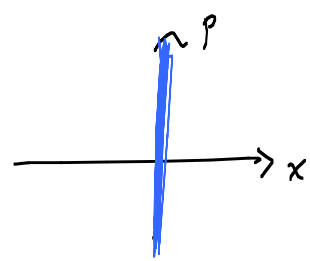
We could define congenial pairs as those whose defect is zero. (Could it even be negative — corresponding to empty composition of canonical relations?)

What are the reductions and coreductions?

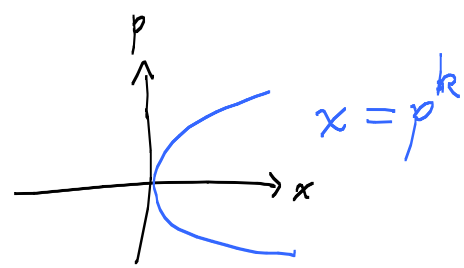


Order ≤ 0 with everything on one side?

Another example



and



order $\frac{1}{2} - \frac{1}{k+1}$ (e.g. $k=2$ gives $\frac{1}{6}$.)

QUANTITATIVE MEASURES OF UNCONGENIALITY

LINEAR RELATIONS

The badness of pairs of linear relations is measured by the defect space $\tilde{\delta}(B,C) = Y \times Y / \Delta_Y + \mathrm{Im}(B \times C)$ and the codefect space $\tilde{\delta}^*(B,C) = (B \times C) \cap (0_X \times \Delta_Y \times 0_Z)$ whose dimensions are the defect $\delta(B,C)$ and codefect $\delta^*(B,C)$.

$\tilde{\delta}$ and $\tilde{\delta}^*$ are each invariant under transposition.

More interesting is the following:

Each relation $X \leftarrow Y$ has a dual $Y^* \leftarrow X^*$. This is a contravariant functorial involution (in finite dimensions) and we have $\tilde{\delta}(C^*, B^*) = [\tilde{\delta}^*(B,C)]^*$, so $\delta(C^*, B^*) = \delta^*(B,$

Canonical relations are "anti-self-dual", which implies that $\delta = \delta^*$ for them.

For linear relations, we can form WW-type categories by condition

(D) $\delta = 0$

(D^*) $\delta^* = 0$

or (DD^*) $\delta = 0$ and $\delta^* = 0$

THEOREM 3

There is a well defined total (co)defect for linear WW morphisms $[X_0 \leftarrow \text{-----} \leftarrow X_n]$. This follows from properties such as:

$$\delta(L, L', L'') = \delta(L \circ L', L'') + \delta(L, L')$$
$$= \delta(L, L' \circ L'') + \delta(L', L'').$$

(Equating the two right-hand sides gives a cocycle relation.)

MORE ON THEOREM 3

Given relations $X_0 \xleftarrow{B_1} X_1 \leftarrow \cdots \xleftarrow{B_n} X_n$,

define $$\tilde{\delta}(B_1,\dots,B_n) = \frac{X_0\times X_1\times X_1\times X_2\times\cdots\times X_{n-1}\times X_n}{(B_1\times\cdots\times B_n)+(X_0\times\Delta_{X_1}\times\cdots\times\Delta_{X_{n-1}}\times X_n)}$$

$$\tilde{\delta}^*(B_1,\dots,B_n) = (B_1\times\cdots\times B_n)\cap(0_{X_0}\times\Delta_{X_1}\times\cdots\times\Delta_{X_{n-1}}\times 0_{X_n}),$$

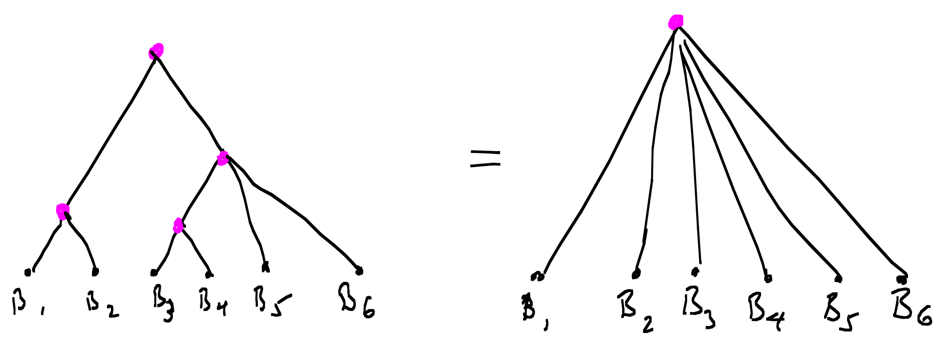
δ and δ^* their dimensions.

Then we have exact sequences like

$$0\longrightarrow\tilde{\delta}(B_j,B_{j+1})\longrightarrow\tilde{\delta}(B_1,\dots,B_n)\longrightarrow\tilde{\delta}(B_1,\dots,B_jB_{j+1},\dots,B_n)\longrightarrow 0.$$

This leads to various "cocycle identities" for δ (and δ^*, by duality); in particular, for a "composition tree", if we label nodes by defects, the sum is an invariant.

(Operadic interpretation?)



In particular, there can be no "cancellation of defects."

On the quantum side, defect should be related to failure of operator composition to respect grading by order.

Composition of "hyperpoints"

$$
\begin{tikzcd}
pt & Q \arrow[l, "\overline{L}_1"'] & pt \arrow[l, "L_2"'] & R \arrow[l, "\overline{L}_3"'] & pt \arrow[l, "L_4"']
\end{tikzcd}
$$

$$\parallel$$

$$
\begin{tikzcd}
pt & Q \arrow[l, "\overline{L}_1"'] & & R \arrow[ll, "L_2 \times \overline{L}_3"'] \arrow[dl, "L_2 \times 1_R"] \arrow[dl, phantom, bend right=20, "="] & pt \arrow[l, "L_4"'] \\
& & Q \times R \arrow[ul, "1_Q \times \overline{L}_3"] & &
\end{tikzcd}
$$

$$\parallel$$

$$
\begin{tikzcd}
pt & & Q \times R \arrow[ll, "\overline{L}_1 \times \overline{L}_3"'] & & pt \arrow[ll, "L_2 \times L_4"']
\end{tikzcd}
$$

$$\parallel$$

$$
\begin{tikzcd}
pt & Q \arrow[l, "\overline{L}_1"'] & pt \arrow[l, "L_2"'] \\
& \times & \\
pt & R \arrow[l, "\overline{L}_3"'] & pt \arrow[l, "L_4"']
\end{tikzcd}
$$