Effective vertex expansion for tensorial group field theory

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Motivation

- New background independent approach of quantum gravity is required
- Random matrix model admits large N-limit with Feynman graphs of genus $g = 0$ (the planar graphs). Matrix model may be considered as quantum gravity in two dimensions $D = 2$ with the following topological expression$\int \sqrt{-\mathrm{g}}R \equiv 4\pi\chi$, $\chi = 2 - 2g = V - L + F$. The amplitude of a Feynman graph G will be $A(G) = \lambda^V N^\chi$
- The matrices are dual to the triangles on which the vertices are the triangles, the lines are the sides and the faces are vertices. This leads to dynamical triangulation of the spacetime.
- For a general structure, high dimensions quantum gravity as a generalization of matrices may be described by tensor, the large N-limit exist and the Feynman graphs are the melon with Gurau number $\omega = 0$. The computation of the amplitude is not simple !!!!
- Group field theory comes from spin foams, dynamical triangulation and loop quantum gravity. These theories are combined with random tensor models : leads to tensorial group field theory

Outline

Definition of the model

In the context of TGFT, we consider the pair of complex fields φ and $\bar{\varphi}$ which takes values of d-copies of arbitrary group G :

$$\varphi, \bar{\varphi} : G^d \to \mathbb{C} \tag{1}$$

The particular case is $G = U(1)$ the Abelian compact Lie group. The model we will be mainly considering here is a tensorial φ^4-theory on $\mathrm{U}(1)^{\times 5}$. Namely,

$$\begin{aligned} S[\bar{\varphi}, \varphi] = &\int_{\mathrm{U}(1)^5} d\mathbf{g}\, \bar{\varphi}(\mathbf{g})(-\Delta + m^2)\varphi(\mathbf{g}) \\ &+ \frac{\lambda}{2} \sum_{c=1}^{5} \int_{\mathrm{U}(1)^{20}} d\mathbf{g}\, d\mathbf{g}'\, d\mathbf{h}\, d\mathbf{h}'\ \bar{\varphi}(\mathbf{g})\varphi(\mathbf{g}')\bar{\varphi}(\mathbf{h})\varphi(\mathbf{h}')K_c(\mathbf{g}, \mathbf{g}', \mathbf{h}, \mathbf{h}') \end{aligned} \tag{2}$$

where $\Delta = \sum_{\ell=1}^{5} \Delta_\ell$ and Δ_ℓ is the Laplace-Beltrami operator on $U(1)$ acting on colour-ℓ indices, bold variables stand for 5-dimensional variables ($\mathbf{g} = (\mathrm{g}_1, \ldots, \mathrm{g}_5)$), and K_c is the vertex.

Symmetry of the model

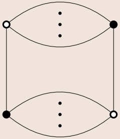


FIGURE – The φ^4 vertex in the large N-limit

The above diagram called melon in invariant under the symmetry group $U(N)$, $N \to \infty$. This symmetry implies the existence of some relations between correlation function called Ward-Takahashi identities which are considered as the quantum equivalent of the Noether theorem.

Definition of the model

The statistical physics description of the model is encoded in the partition function :

$$\mathcal{Z}[\bar{J},J]=\int D\varphi D\bar{\varphi}e^{-S[\bar{\varphi},\varphi]+J\bar{\varphi}+\varphi\bar{J}}=e^{W[\bar{J},J]}, \tag{3}$$

where $\bar{J}$ and J represent the sources and $W[\bar{J},J]$ is the generating functional for the connected Green's functions. Then the N-point connected Green functions take the form

$$G_N(\mathbf{g}_1,\cdots,\mathbf{g}_{2N})=\frac{\partial\mathcal{W}(\bar{J},J)}{\partial J_1\partial\bar{J}_1\cdots\partial J_N\partial\bar{J}_N}\Big|_{\mathbf{J}=\bar{\mathbf{J}}=0}. \tag{4}$$

Now let φ_{class} denote the classical field defined by the expectation value of φ in the presence of sources J,$\bar{J}$:

$$\varphi_{class}=\langle\varphi\rangle=\frac{\delta W[\bar{J},J]}{\delta\bar{J}},\quad\bar{\varphi}_{class}=\langle\bar{\varphi}\rangle=\frac{\delta W[\bar{J},J]}{\delta J}. \tag{5}$$

Definition of the model

Then the 1PI effective action Γ_{1PI} is given by the Legendre transform of $W[\bar{J}, J]$ as

$$\Gamma_{1PI} = -W[\bar{J}, J] + \int (J\bar{\varphi}_{class} + \varphi_{class}\bar{J}). \tag{6}$$

For the rest we consider only the Fourier transform of the fields φ and $\bar{\varphi}$ denoted respectively by $T_{\vec{p}}$ and $\bar{T}_{\vec{p}}$, $\vec{p} \in \mathbb{Z}^d$ written as (for $\vec{g} \in U(1)^d$, $g_j = e^{i\theta_j}$) :

$$\varphi(\vec{\theta}) = \sum_{\vec{p}\in\mathbb{Z}^d} T_{\vec{p}}\, e^{i\sum_{j=1}^{d}\theta_j p_j}, \quad \bar{\varphi}(\vec{\theta}) = \sum_{\vec{p}\in\mathbb{Z}^d} \bar{T}_{\vec{p}}\, e^{-i\sum_{j=1}^{d}\theta_j p_j}. \tag{7}$$

without all confusion we set $\varphi_{class} \equiv M$ and $\bar{\varphi}_{class} \equiv \bar{M}$ in the Fourier modes.

RG description

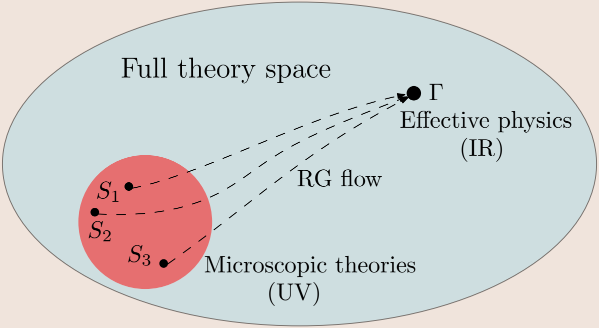


FIGURE – Renormalisation group from UV to IR

Wetterich flow equation

The Wetterich equation is a functional integro-differential equation for the effective action Γ, now taking into account the quantum fluctuations caracterized by the parameter s, and called average effective action denoted by Γ_s, $-\infty < s < +\infty$. It is the Legendre transformation of the standard free energy $\mathcal{W}_s = \ln \mathcal{Z}_s$:

$$\Gamma_s[M, \bar{M}] = \langle \bar{J}, M \rangle + \langle \bar{M}, J \rangle - \mathcal{W}_s[J, \bar{J}] - R_s[M, \bar{M}] \qquad (8)$$

where $R_s[M, \bar{M}] := Tr(M r_s \bar{M})$ and r_s is called the IR regulator. The appearance of this regulator r_s is introduced as new parameter function, which controls the scale fluctuation from IR to UV such that

$$\lim_{s \to -\infty} r_s = 0, \qquad \lim_{s \to +\infty} r_s = \infty. \qquad (9)$$

This definition ensures that Γ_s satisfies the boundary conditions $\Gamma_{s=\ln \Lambda} = S,\ \Gamma_{s=-\infty} = \Gamma$, where Λ is the UV cutoff.

Wetterich flow equation

The fields M and $\bar{M}$ are the mean values of T and $\bar{T}$ respectively and are given by

$$M = \frac{\partial \mathcal{W}}{\partial \bar{J}}, \quad \bar{M} = \frac{\partial \mathcal{W}}{\partial J} \tag{10}$$

where $\mathcal{W} := \mathcal{W}_{s=-\infty}$. In general the regulator r_s is chosen to be $r_s = Z(s)k^2 f\left(\frac{\vec{p}^{\,2}}{k^2}\right),\ k = e^s$, and such that the conditions (9) is well satified. Let $\Gamma_s^{(2)}$ is the second order partial derivative of Γ_s with respect to the mean fields M and $\bar{M}$, the Wetterich equation is then given by

$$\partial_s \Gamma_s = Tr\, \partial_s r_s (\Gamma_s^{(2)} + r_s)^{-1} \tag{11}$$

The average effective action is chosen to be of the form

$$\Gamma_s = Z(s) \sum_{\vec{p}\in\mathbb{Z}^d} T_{\vec{p}}(\vec{p}^{\,2} + e^{2s}\bar{m}^2(s))\bar{T}_{\vec{p}} + \sum_n Z(s)^{\frac{n}{2}} \bar{\lambda}_n V_n(T, \bar{T}) \tag{12}$$

Wetterich flow equation

In the case of quartic melonic interaction and by taking the standard modified Litim's regulator :

$$r_s(\vec{p}) = Z(s)(e^{2s} - \vec{p}^{\,2})\Theta(e^{2s} - \vec{p}^{\,2}) \tag{13}$$

the Wetterich equation can be solved analytically and the phase diagram may be given. The flow equations are

$$\left\{ \begin{array}{ll} \dot{m}^2 & = -2d\lambda I_2(0) \\ \dot{Z}(s) & = -2\lambda I_2'(q=0) \\ \dot{\lambda} & = 4\lambda^2 I_3(0) \end{array} \right. \qquad I_n(q) = \sum_{\vec{p}\in\mathbb{Z}^{(d-1)}} \frac{\dot{r}_s}{(Z(s)\vec{p}^{\,2} + Zq^2 + m^2 + r_s)^n}. \tag{14}$$

with the renormalization condition

$$m^2(s) = \Gamma_s^{(2)}(\vec{p} = \vec{0}), \quad \lambda(s) = \frac{1}{4}\Gamma_s^{(4)}(\vec{0},\vec{0},\vec{0},\vec{0}), \tag{15}$$

$$Z := \frac{d}{d\vec{p}^{\,2}}\Gamma^{(2)}(\vec{p} = \vec{0}\,)\,, \qquad \eta := \frac{1}{Z}\frac{\partial Z}{\partial s}\,.. \tag{16}$$

Wetterich flow equation

Explicitly using the integral representation of the above sum and with $d = 5$, $\eta = \dot{Z}/Z$ we get

$$I_n(0) = \frac{\pi^2 e^{6s-2ns}}{6Z(s)^{n-1}(\bar{m}^2+1)^n}(\eta+6), \quad I_n'(0) = -\frac{\pi^2 e^{4s-2ns}}{2Z(s)^{n-1}(\bar{m}^2+1)^n}(\eta+4). \quad (17)$$

In term of dimensionless parameter $\lambda = Z^2\bar{\lambda}$, $m^2 = e^{2s} Z \bar{m}^2$ the system (14) becomes

$$\begin{cases} \beta_m & = -(2+\eta)\bar{m}^2 - 2d\bar{\lambda}\, \frac{\pi^2}{(1+\bar{m}^2)^2}\, \left(1+\frac{\eta}{6}\right)\ , \\ \beta_\lambda & = -2\eta\bar{\lambda} + 4\bar{\lambda}^2\, \frac{\pi^2}{(1+\bar{m}^2)^3}\, \left(1+\frac{\eta}{6}\right)\ , \end{cases} \quad (18)$$

where $\beta_m := \dot{\bar{m}}^2$, $\beta_\lambda = \dot{\bar{\lambda}}$ and :

$$\eta := \frac{4\bar{\lambda}\pi^2}{(1+\bar{m}^2)^2 - \bar{\lambda}\pi^2}\ . \quad (19)$$

Wetterich flow equation

The solutions of the system (18) is given analytically :

$$p_{\pm} = \Big(\bar{m}_{\pm}^2 = -\frac{23 \mp \sqrt{34}}{33}, \bar{\lambda}_{\pm} = \frac{328 \mp 8\sqrt{34}}{11979\pi^2}\Big). \tag{20}$$

Numerically

$$p_{+} = (-0.52, 0.0028), \quad p_{-} = (-0.87, 0.0036). \tag{21}$$

Apart from the fact that we have a singularity line around the point $\bar{m}^2 = -1$ in the flow equation (14), another second singularity arise from the anomalous dimension denominator, and corresponds to a line of singularity, with equation :

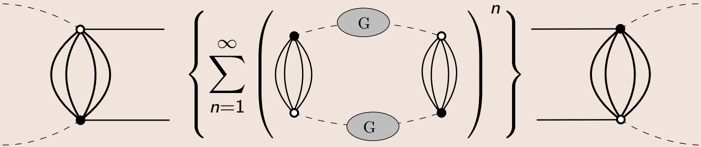
$$\Omega(\bar{m}, \bar{\lambda}) := (\bar{m}^2 + 1)^2 - \pi^2 \bar{\lambda} = 0 \tag{22}$$

This line of singularity splits the two dimensional phase space of the truncated theory into two connected regions characterized by the sign of the function Ω.

Diagram representation given the structure equation

Melons are the trees in the intermediate field representation

$$-\Gamma^4 \quad =$$



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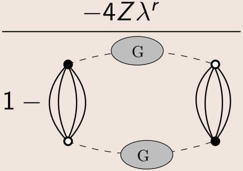


$$= \quad \frac{-4Z\lambda^r}{1+2Z\lambda^r\mathcal{A}_s}, \quad \mathcal{A}_s = \sum_{\vec{p}_\perp}[G_s(\vec{p}_\perp)]^2 = \sum_{\vec{p}_\perp}\frac{1}{[\Gamma_s^{(2)}(\vec{p}_\perp)+r_s(\vec{p}_\perp)]^2} \tag{23}$$

$\vec{p}_\perp := (0, p_1, \cdots, p_d)$ and $G_s^{-1}(\vec{p}) = Z_{-\infty}\vec{p}^{\,2} + m^2 + r_s(\vec{p}) - \Sigma_s(\vec{p})$.

Flow equations using the EVE

Let us consider the flow equation for $\dot{\Gamma}^{(2)}$, obtained from (11) deriving with respect to M and $\bar{M}$:

$$\dot{\Gamma}^{(2)}(\vec{p}) = -\sum_{\vec{q}} \Gamma^{(4)}_{\vec{p},\vec{p},\vec{q},\vec{q}}\, G_s^2(\vec{q}\,)\dot{r}_s(\vec{q}\,)\,, \tag{24}$$

where we discard all the odd contributions, vanishing in the symmetric phase. Deriving on both sides with respect to p_1^2, and setting $\vec{p}=\vec{0}$, we get :

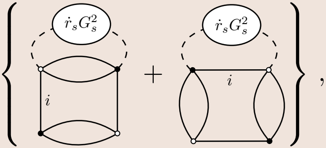
$$\dot{Z} = -\sum_{\vec{q}} \Gamma^{(4)\,\prime}_{\vec{0},\vec{0},\vec{q},\vec{q}}\, G_s^2(\vec{q}\,)\dot{r}_s(\vec{q}\,) - \Gamma^{(4)}_{\vec{0},\vec{0},\vec{q},\vec{q}}\, G_s^2(\vec{q}\,)\dot{r}_s(\vec{q}\,)\,, \tag{25}$$

where the "prime" designates the partial derivative with respect to p_1^2. In the deep UV ($k \gg 1$) the argument used in the T^4-truncation to discard non-melonic contributions holds, and we keep only the melonic diagrams as well.

Flow equations using the EVE

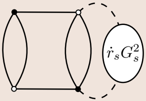
Deriving twice the exact flow equation, we get schematically :

$$\partial_s \Gamma^{(2)} = -2\lambda \sum_{i=1}^{d}$$



(26)

the diagrams being computed with the effective propagator $\dot r_s G_s^2$, the “dot” means that the derivative is with respect to s. Explicitly :



$$= \sum_{\vec{p}\in(\mathbb{Z}^D)^{(d-1)}} \frac{\dot r_s}{(Z\vec{p}^{\,2\alpha} + Zq^{2\alpha} + m^{2\alpha} + r_s)^2} \qquad (27)$$

where q denotes the external momenta running through the effective loop. Also we get

$$\dot Z = 4\lambda^2 \mathcal{A}'_s(0)\, I_2(0) - 2\lambda I'_2(0)\,. \qquad (28)$$

Flow equations using the EVE

Keeping only the melonic contributions, we get finally the following autonomous system by using the Litim's regulation :

$$\left\{ \begin{array}{ll} \beta_m & = -(2+\eta)\bar{m}^2 - 2d\bar{\lambda}\,\frac{\pi^2}{(1+\bar{m}^2)^2}\,\left(1+\frac{\eta}{6}\right)\,, \\ \beta_\lambda & = -2\eta\bar{\lambda} + 4\bar{\lambda}^2\,\frac{\pi^2}{(1+\bar{m}^2)^3}\,\left(1+\frac{\eta}{6}\right)\left[1-\frac{1}{2}\pi^2\bar{\lambda}\left(\frac{1}{(1+\bar{m}^2)^2}+\left(1+\frac{1}{1+\bar{m}^2}\right)\right)\right]. \end{array} \right. \tag{29}$$

where the anomalous dimension is then given by :

$$\eta = 4\bar{\lambda}\pi^2\frac{(1+\bar{m}^2)^2-\frac{1}{2}\bar{\lambda}\pi^2(2+\bar{m}^2)}{(1+\bar{m}^2)^2\Omega(\bar{\lambda},\bar{m}^2)+\frac{(2+\bar{m}^2)}{3}\bar{\lambda}^2\pi^4}\,. \tag{30}$$

The other solutions are :

$$p_0 = (\bar{m}^2 = -1.28, \bar{\lambda} = 0.025), \quad p_1 = (\bar{m}^2 = 1.96, \bar{\lambda} = 1.10), \tag{31}$$

Constraint from WI

In the symmetric phase, the zero-momenta 4-point fonction satisfies :

$$\pi_{00}^{(1)} Z_{-\infty} \mathcal{L}_s = -\frac{\partial}{\partial p_1^2} \left(\Gamma_s^{(2)}(\vec{p}_\perp) - Z_{-\infty} \vec{p}^{\,2} \right) \Bigg|_{\vec{p}=0} , \tag{32}$$

where we defined the loop $\mathcal{L}_s$ as :

$$\mathcal{L}_s := \sum_{\vec{p}_\perp} \left(1 + \frac{\partial \tilde{r}_s(\vec{p}_\perp)}{\partial p_1^2} \right) [G_s(\vec{p}_\perp)]^2 , \qquad r_s =: Z_{-\infty} \tilde{r}_s . \tag{33}$$

implies

$$\beta_\lambda = -\eta \bar{\lambda} \left(1 - 2\bar{\lambda} \frac{\Omega_4}{(1+\bar{m}^2)^2} \right) + 4\bar{\lambda}^2 \frac{\Omega_4}{(1+\bar{m}^2)^3} \beta_m . \tag{34}$$

$\Omega_m = \frac{\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2}+1)}$.

WI violation

We get $\beta_m(p) = 0 = \beta_\lambda(p) = 0$. Then the constraint (34) implies that a the point p

$$\eta\bar{\lambda}\Big(1 - \frac{\bar{\lambda}\pi^2}{(1+\bar{m}^2)^2}\Big)(p) = 0. \tag{35}$$

The particular solution $\bar{\lambda} = 0$ correspond to the Gaussian fixed point. For $\bar{\lambda} \neq 0$ we have only

$$\eta = 0, \text{ or } \frac{\bar{\lambda}\pi^2}{(1+\bar{m}^2)^2} = 1 \tag{36}$$

It is clear that the fixed point $p_+ = (-0.55, 0.0025)$, $\eta \approx 0.7$ violate these constraints. Finally all the fixed point discovered from EVE or EV methods violate the Ward identities.

To sum up, in this presentation

- We show that the IR fixed point obtained in the FRG applications for TGFT lack an important constraint coming from Ward identities. This constraint reduces the physical region of the phase space to a one-dimensional subspace without fixed point, suggesting that the phase transition scenario abundantly cited in the TGFT literature may be an artifact of an incomplete method.
- This suggestion is improved with a more sophisticated method, taking into account the momentum dependence of the effective vertex, and providing a maximal extension of the symmetric region. Despite with this improvement, the resulting numerical fixed point does not cross any of the physical lines provided from the Ward constraint.
- In the literature, the quartic truncation has been largely investigated, for various group manifold and dimensions. We shown from our analysis that none of these models modify our conclusions, except for TGFT including closure constraint as a gauge symmetry.

Thank you for your attention