Algebraic Solution of the N-dimensional Harmonic Oscillator with Minimal Length Uncertainty Relations and Thermodynamic Properties

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Introduction

Standard Heisenberg algebra

The ordinary quantum mechanics is governed by the standard Heisenberg algebra gives as follows

$$[x, p] = i\hbar\mathbf{1}. \tag{1}$$

Introduction

Standard Heisenberg algebra

- Then it is not possible to simultaneously measure these two observable quantities which are said to be complementary.
- The notion of phase space disappears in quantum mechanics, and the quantum object is in fact completely described by its wave function.

Introduction

Heisenberg Uncertainty Principle

The commutation relation is directly related to the uncertainty relation through the formula

$$\Delta A \Delta B \geq |\langle [A, B] \rangle| \tag{2}$$

So, we have

$$\Delta x \Delta p \geq \frac{\hbar}{2}. \tag{3}$$

The more localized the particle, the less defined its momentum, and vice versa.

Introduction

Deformed Heisenberg algebra

- The deformed Heisenberg algebra

$$[\mathcal{X},\mathcal{P}] = i\hbar(1+\tau p^2)\mathbf{1}, \quad \tau = \frac{\beta}{\hbar m\omega}, \quad 0 \leq \beta \leq 1 \tag{4}$$

It has been rigorously studied by Kempf and his collaborators in 1995.

- The generalized uncertainty relation gives:

$$\Delta\mathcal{X}\Delta\mathcal{P} \geq \frac{\hbar}{2}\{1+\tau(\Delta\mathcal{P})^2+\tau\langle\mathcal{P}\rangle^2\}. \tag{5}$$

Generalized uncertainty relation

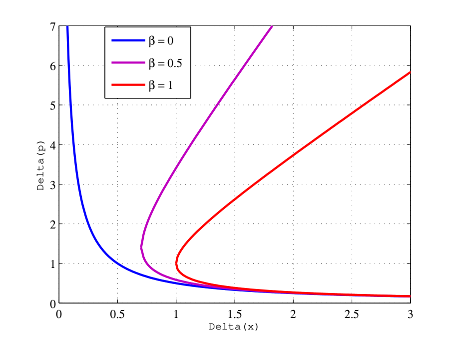

Figure 1: The generalized uncertainty relation, implying a minimal length ($\Delta\mathcal{X}_{min} = \sqrt{\beta}$) .

Introduction

Several models

The one-dimensional and multi-dimensional harmonic oscillator. The problem of a charged particle of spin $1/2$ moving in a constant magnetic field, the one, two and three-dimensional Dirac oscillator, the hydrogen atom...

Several methods

The method approximation, the Nikiforov-Uvarov method, supersymmetric quantum mechanics...

Outline

1. Deformed N-dimensional Harmonic oscillator
2. Thermodynamic properties
3. Graphs of thermodynamic properties

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Hamiltonian

In the $N-$dimensional space the deformed Hamiltonian is defined by

$$\begin{aligned} H &= \sum_{i=1}^{N}\left(\frac{1}{2m}\mathcal{P}_i\mathcal{P}_i + \frac{1}{2}m\omega^2\mathcal{X}_i\mathcal{X}_i\right) \\ &= \frac{1}{2m}\mathcal{P}^2 + \frac{1}{2}m\omega^2\mathcal{X}^2, \end{aligned} \tag{6}$$

Commutation relations

The operators $\mathcal{X}_i$ and $\mathcal{P}_i$ satisfy the following commutation relations

$$[\mathcal{X}_i, \mathcal{P}_j] = i\hbar(1 + \tau\mathcal{P}^2)\delta_{ij}\mathbb{1}, \quad \mathcal{P}^2 = \sum_{i=1}^{N} \mathcal{P}_i\mathcal{P}_i. \tag{7}$$

With the representation

$$\mathcal{X}_i = i\hbar(1 + \tau p^2)\frac{\partial}{\partial p_i}, \quad \mathcal{P}_i = p_i, \tag{8}$$

Rotational symmetry

It is a lucky circumstance that rotational symmetry is preserved. This means that isotropic systems can be reduced to a quantization of some effective one-dimensional model on the positive real line.

N free deformed harmonic oscillators

The following N free deformed harmonic oscillators

$$\mathcal{H}_{free} = \frac{\hbar\omega}{2} \sum_{i=1}^{N} \left(\mathcal{J}_i^+ \mathcal{J}_i^- + \mathcal{J}_i^- \mathcal{J}_i^+ \right), \tag{9}$$

where $\mathcal{J}_i^{\pm}$ are the generators of the $su(1,1)$ algebra in N dimensions.

Generators of the $su(1,1)$ algebra

The $\mathcal{J}_i^{\pm}$ are the generators of the $su(1,1)$ algebra in N dimensions and satisfy the following algebra

$$[\mathcal{J}_i^-, \mathcal{J}_i^+] = 2\sqrt{\frac{\beta}{2}}\mathcal{J}_i^0, \quad [\mathcal{J}_i^0, \mathcal{J}_i^{\pm}] = \pm\sqrt{\frac{\beta}{2}}\mathcal{J}_i^{\pm}, \tag{10}$$

where

$$\mathcal{J}_i^- = \sqrt{\frac{\beta}{2}(a_i^\dagger a_i + 2\ell_N^0)}a_i, \quad \mathcal{J}_i^- = a_i^\dagger\sqrt{\frac{\beta}{2}(a_i^\dagger a_i + 2\ell_N^0)}, \tag{11}$$

$$\mathcal{J}_i^0 = \sqrt{\frac{\beta}{2}}(a_i^\dagger a_i + \ell_N^0), \quad \ell_N^0 = \frac{N}{2} + \sqrt{\frac{N^2}{4} + \frac{1}{\beta^2}}. \tag{12}$$

Representation of deformed $su(1,1)$ algebra

The action of the above realizations on the state $|\ell_N^0, n_i\rangle$ $(n_i = 0, 1, 2, \cdots)$, gives

$$\mathcal{C}|\ell_N^0, n_i\rangle = \frac{\beta}{2}\ell_N^0(\ell_N^0 - 1)|\ell_N^0, n_i\rangle, \quad (13)$$

$$\mathcal{J}_i^0|\ell_N^0, n_i\rangle = \sqrt{\frac{\beta}{2}}(n_i + \ell_N^0)|\ell_N^0, n_i\rangle, \quad (14)$$

$$\mathcal{J}_i^-|\ell_N^0, n_i\rangle = \sqrt{\frac{\beta}{2}n_i(2\ell_N^0 + n_i - 1)}|\ell_N^0, n_i - 1\rangle, \quad (15)$$

$$\mathcal{J}_i^+|\ell_N^0, n_i\rangle = \sqrt{\frac{\beta}{2}(n_i + 1)(2\ell_N^0 + n_i)}|\ell_N^0, n_i + 1\rangle. \quad (16)$$

Eigenvalue equation

We can write the eigenvalue equation,

$$\mathcal{H}_{free}|\ell_N^0, n_i\rangle = \mathcal{E}_{N,n}^{0,\beta}|\ell_N^0, n_i\rangle, \tag{17}$$

where

$$\mathcal{E}_{N,n}^{0,\beta} = \hbar\omega\beta\left[\ell_N^k(n+\frac{N}{2})+\frac{1}{2}n^2\right] \tag{18}$$

and

$$|\ell_N^0, n\rangle = \prod_{i=1}^{N}\sqrt{\frac{2^{n_i}\Gamma(2\ell_N^0)}{(\beta)^{n_i}n_i!\Gamma(n_i+2\ell_N^0)}}(\mathcal{J}_i^+)^{n_i}|\ell_N^0, 0\rangle. \tag{19}$$

N-dimensional isotropic harmonic oscillator

Now, we can write the Hamiltonian in the following form

$$H_N^k = \hbar\omega\beta \sum_{i=1}^{N} \left[\ell_N^k (a_i^\dagger a_i + \frac{N}{2}) + \frac{1}{2}(a_i^\dagger a_i)^2 \right] + \frac{1}{2}\hbar\omega\beta L^2, \qquad (20)$$

where

$$L^2 = k(k+N-2), \quad \ell_N^k = \frac{N}{2} + \sqrt{\frac{N^2}{4} + L^2 + \frac{1}{\beta^2}}. \qquad (21)$$

Spectrum of H_N^k

It is a straightforward exercise to obtain the eigenvalues

$$\mathcal{E}_{N,n}^{k,\beta} = \hbar\omega\beta\left[\ell_N^k(n+\frac{N}{2}) + \frac{1}{2}n^2 + \frac{1}{2}k(k+N-2)\right], \qquad (22)$$

associated to the eigenstates

$$|\ell_N^k, n\rangle = \prod_{i=1}^{N}\sqrt{\frac{2^{n_i}\Gamma(2\ell_N^k)}{(\beta)^{n_i} n_i!\Gamma(n_i+2\ell_N^k)}}(\mathcal{J}_i^+)^{n_i}|\ell_N^k, 0\rangle. \qquad (23)$$

Particular cases

the one-dimensional result can be reproduced from this expression by setting $N = 1$ and $L^2 = 0$,

$$\mathcal{E}_{1,n}^{0,\beta} = \hbar\omega\beta\left[\ell_1^0(n + \frac{1}{2}) + \frac{1}{2}n^2\right], \quad \ell_1^0 = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\beta^2}}. \qquad (24)$$

Particular cases

For the $N = 2$ and $N = 3$ cases, the explicit expressions are

$$\mathcal{E}_{2,n}^{k,\beta} = \hbar\omega\beta\left[\ell_2^k(n+1) + \frac{1}{2}(n^2+k^2)\right], \quad \ell_2^k = 1 + \sqrt{1 + k^2 + \frac{1}{\beta^2}}. \tag{25}$$

$$\begin{aligned}\mathcal{E}_{3,n}^{k,\beta} &= \hbar\omega\beta\left[\ell_3^k(n+\frac{3}{2}) + \frac{1}{2}[n^2 + k(k+1)]\right], \\ \ell_3^k &= 1 + \sqrt{\frac{9}{4} + k(k+1) + \frac{1}{\beta^2}}.\end{aligned} \tag{26}$$

Outline

Vibratory partition function

We study the thermodynamic properties of N identical and independent deformed harmonic oscillators (one-dimensional)

$$\begin{aligned} Z(\alpha) &= \sum_{n=0}^{[\lambda]} e^{-\alpha \mathcal{E}_{N,n}^{0,\beta}} \\ &= \left[\sum_{n=0}^{[\lambda]} e^{-\alpha\hbar\omega\beta\left[\ell_1^0 (n+\frac{1}{2})+\frac{1}{2}n^2\right]} \right]^N , \end{aligned} \tag{27}$$

where $\alpha = \frac{1}{k_B T}$, k_B is the Boltzmann constant and $\ell_1^0 = \ell = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\beta^2}}$.

Classical limit

In the classical limit, at high temperature T for large $[\lambda]$, the sum can be replaced by the following integral, $[\lambda] = \lambda - 1$.

Vibratory partition function

$$
\begin{aligned}
Z(\alpha) &= \left[\int_0^{\lambda} e^{-\alpha\hbar\omega\beta\left[\ell_1^0(n+\frac{1}{2})+\frac{1}{2}n^2\right]}dn\right]^N \\
&= \left[\sqrt{\frac{\pi}{2c\alpha}}e^{\alpha\tilde{c}}\left(\mathrm{erf}\left[\sqrt{\alpha}\tilde{a}\right]-\mathrm{erf}\left[\sqrt{\alpha}\tilde{b}\right]\right)\right]^N \qquad (28)
\end{aligned}
$$

where

$$c=\hbar\omega\beta, \quad \tilde{c}=\frac{1}{2}c\,\ell(\ell-1), \quad \tilde{a}=\sqrt{\frac{1}{2}c}\,(\lambda+\ell), \quad \tilde{b}=\sqrt{\frac{1}{2}c}\,\ell, \qquad (29)$$

$$\mathrm{erf}(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt. \qquad (30)$$

The vibrational mean energy U

$$\begin{aligned} U(\alpha) &= -\frac{\partial}{\partial\alpha}\ln Z(\alpha) \\ &= N\left(\frac{1}{2\alpha} - \tilde{c} - \frac{\Lambda}{\sqrt{\pi\alpha}\Omega}\right), \end{aligned} \tag{31}$$

where

$$\Omega = \text{erf}\left(\sqrt{\frac{1}{2}c\alpha}(\lambda+\ell)\right) - \text{erf}\left(\sqrt{\frac{1}{2}c\alpha}\ell\right), \quad \Lambda = \tilde{a}e^{-\tilde{a}^2\alpha} - \tilde{b}e^{-\tilde{b}^2\alpha}. \tag{32}$$

Vibrational specific heat C

$$\begin{aligned} C(\alpha) &= -k_B\alpha^2\frac{\partial U}{\partial\alpha} \\ &= -Nk_B\alpha^2\left[-\frac{1}{2\alpha^2}+\frac{1}{\sqrt{\pi\alpha}\Omega}\left(\tilde{a}^3e^{-\tilde{a}^2\alpha}-\tilde{b}^3e^{-\tilde{b}^2\alpha}\right)\right. \\ &+ \left.\frac{\Lambda}{\sqrt{\pi}\Omega}\left(\frac{1}{2\alpha^{\frac{3}{2}}}+\frac{\Lambda}{\sqrt{\pi\alpha}\Omega}\right)\right]. \end{aligned} \quad (33)$$

Vibrational mean free energy F

$$F(\alpha) = -\frac{N}{\alpha}\ln Z(\alpha)$$
$$= -\frac{N}{\alpha}\left(\ln\sqrt{\frac{\pi}{2\alpha c}} + \tilde{c}\alpha + \ln\Omega\right). \quad (34)$$

Vibrational entropy S

$$\begin{aligned} S(\alpha) &= Nk_B \ln Z(\alpha) - Nk_B\alpha\frac{\partial}{\partial\alpha}\ln Z(\alpha) \\ &= Nk_B\left(\ln\sqrt{\frac{\pi}{2\alpha c}} + \tilde{c}\alpha + \ln\Omega\right) \\ &+ Nk_B\alpha\left(\frac{1}{2\alpha} - \tilde{c} - \frac{\Lambda}{\sqrt{\pi\alpha\Omega}}\right). \end{aligned} \tag{35}$$

Outline

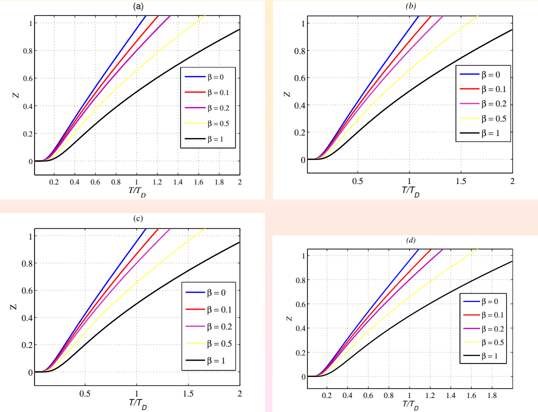

Figure 2: Vibrational partition function (27) as a function of $\rho = T/T_D$ for different values of β. In (a): $\lambda = 50$, in (b): $\lambda = 100$, in (c) :$\lambda = 150$, and in (d): $\lambda = 200$.

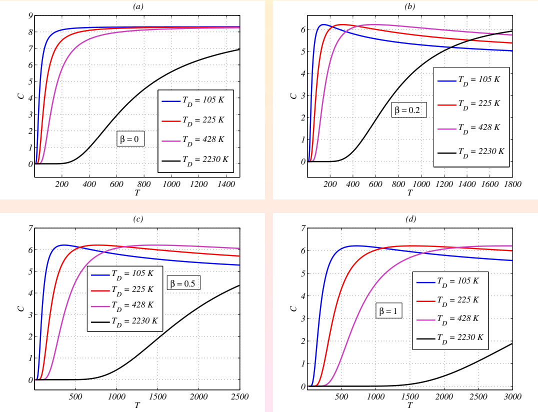

Figure 3: Specific heat capacity of lead(Pb, $T_D = 105K$), silver(Ag, $T_D = 225K$), aluminum(Al, $T_D = 428K$) and diamond($T_D = 2230K$) as a function of temperature.

Conclusion

- The solutions of the deformed N-dimensional harmonic oscillator in the presence of a minimal length have been obtained by the algebraic method.
- The hidden symmetry $su(1,1)$ has been identified for the N free deformed harmonic oscillators. This latter is considered as a one-dimensional deformed crystal of N identical and independent atoms.

Conclusion

- These maxima, obtained at the level of the curvatures of the curve representing the specific heat, neither show nor indicate the existence of a phase transition.
- In the limit $\beta \longrightarrow 0$, the specific heat of a crystalline body is independent of the temperature and of the body considered for large values of the temperature: *Dulong-Petit law*

THANKS!