Statistical operators analysis in quantum Hilbert spaces and some applications

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Abstract

This work gives first value to the importance of Hilbert-Schmidt operators Hilbert space in the study of the thermal state related to von Neumann algebras with coherent states construction to elaborate the density operator diagonal expansion using the Glauber-Sudarshan (GS)-P-representation. Then, relevant statistical quantities like the Q-Husimi distribution and the Wehrl entropy are investigated. Next, in the underlying Hilbert space of the generalized hypergeometric coherent states constructed for photon-added quantum systems, the thermal density operator is elaborated with its P-representation also provided. Then, quantum optical properties are investigated using the thermal expectation values of observables derived from the density operator.

Introduction-Physical motivations

- Hilbert spaces, which are pre-Hilbert spaces complete with respect to a norm, at the mathematical side, realize the skeleton of quantum theories.
- Let $\mathfrak{H}$ denotes a Hilbert space, assumed separable, infinite or finite dimensional, and endowed with the scalar product in the "bra-ket" notation, $\langle\phi|\psi\rangle$.
- For $\phi, \psi \in \mathfrak{H}$, the rank one operator $T = |\phi\rangle\langle\psi|$ is defined to be

$$T|\chi\rangle = \langle\psi|\chi\rangle|\phi\rangle. \tag{1}$$

- The convergence of the operator integral given by

$$I = \int_X f(x)d\mu(x) \tag{2}$$

is obtained as follows

$$\int_X \langle\phi|f(x)|\psi\rangle d\mu(x) < \infty, \qquad \phi, \psi \in \mathfrak{H}. \tag{3}$$

Introduction-Physical motivations

• Consider an ensemble of quantum states that is prepared as a classical statistical mixture.
• Let the elements of the ensemble be the states $|\psi_1\rangle, |\psi_2\rangle, \dots |\psi_n\rangle$ and the probability of finding each state given by $p_1, p_2, \dots, p_n$.

The density operator representing a collection of states $|\psi_1\rangle, |\psi_2\rangle, \dots |\psi_n\rangle$ is

$$\rho = \sum_{i=1}^{n} p_i |\psi_i\rangle\langle\psi_i| \tag{4}$$

A *mixed state* is a collection of different pure states, each occurring with a given probability.

If the state of a system is known exactly, then it is a pure state.

Introduction-Physical motivations

That is the density operator for a pure state $|\psi\rangle$ can be written as

$$\rho = |\psi\rangle\langle\psi|. \tag{5}$$

This is a projection operator. And in the case of a pure state, the density operator satisfies

$$\rho^2 = \rho. \tag{6}$$

Properties of the density operator

- The density operator is Hermitian, $\rho = \rho^\dagger$
- The trace of any density matrix is equal to one, $Tr(\rho) = 1$
- For a pure state, since $\rho^2 = \rho$, $Tr(\rho^2) = 1$
- For a mixed state $Tr(\rho^2) < 1$
- The eigenvalues of a density operator satisfy $0 \leq \lambda_i \leq 1$

Introduction-Physical motivations

For the usual treatment of the equilibrium in statistical mechanics using the Gibbs's canonical distribution , the *thermal* normalized density operator is given by[a]

$$\rho = \frac{1}{Z} e^{-\beta H}, \tag{7}$$

where $Z = \mathrm{Tr}(e^{-\beta H})$ is the partition function, $\beta = 1/k_B T$; T is the temperature, and k_B the Boltzmann constant, which, in SI units, has the value $1.3806503 \times 10^{-23} J/K$.

[a] S. Curilef, F. Pennini, and A. Plastino, *Fisher information, Wehrl entropy, and Landau Diamagnetism*, Phys. Rev. B **71**, 024420 (2005); D. Popov, N. Pop, M. Popov, and S. Şimon, *The Information-Theoretical Entropy of Some Quantum Oscillators*, AIP Conf. Proc. **1634**, 192 (2014)

Introduction - Density operator diagonal representation 7

The diagonal expansion of the density operator, known as the Glauber-Sudarshan (GS)-*P*-representation, was introduced independently by Glauber [1] and Sudarshan [2] for the harmonic oscillator coherent states (CSs). It is given in the coherent states denoted $|z\rangle$ as follows:

$$\rho = \int_{\mathcal{D}} d\nu(z) |z\rangle P(|z|^2) \langle z| \tag{8}$$

with P a quasiprobability distribution function (i.e., if the quantum system has a classical analog, e.g. a coherent state or thermal radiation, then P is non-negative everywhere like an ordinary probability distribution.)

$$\int_{\mathcal{D}} d\nu(z)\, P(|z|^2) = 1. \tag{9}$$

[1] Glauber, R. J.: Coherent and Incoherent States of the Radiation Field, Phys. Rev. **131**, 2766 (1963).

[2] Sudarshan, E. C. G.: Equivalence of Semiclassical and Quantum Mechanical Description of Statistical Light Beams. Phys. Rev. Lett. **10**, 277 (1963).

Introduction - Density operator diagonal representation 8

The Q-Husimi function is in fact the diagonal element of the density operator in the CSs representation. It can be connected with the normalized P-quasidistribution function as follows. Given a state $|\xi\rangle (\xi \in \mathbb{C})$,

$$Q(|z|^2) = \langle z|\rho|z\rangle = \int_{\mathcal{D}} d\nu(z) |\langle z|\xi\rangle|^2 P(|\xi|^2), \tag{10}$$

where the relation assuring the normalization to unity of the Q-Husimi function $Q(|z|^2)$ is given by

$$\int_{\mathcal{D}} d\nu(z)\ Q(|z|^2) = 1. \tag{11}$$

The normalization of the density operator in the CSs $\{|z\rangle\}$ basis leads to

$$\mathrm{Tr}\rho = \int_{\mathcal{D}} d\nu(z)\, \langle z|\rho|z\rangle = 1. \tag{12}$$

Introduction-Physical motivations

In quantum information, for an ensemble, e.g., of qubits, the density operator is used to describe the informational content [3]. The quantum-mechanical phase-space distributions of the harmonic oscillator CSs are used in different situations [4]. In [5], the concepts of Husimi distribution [6] and Wehrl entropy [7] in generalized Fisher and Shannon information measures [8] are discussed.

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[3] N. Pop, D. Popov, and M. Davidovic, *Density Operator in Terms of Coherent States Representation with Applications in the Quantum Information,* Int. J. Theor. Phys. **52**, 2275 (2013)

[4] F. Pennini, G. Ferri, and A. Plastino, *Fisher Information and Semiclassical Treatments, Entropy* **11**, 972 (2009).

[5] A. Anderson and J. J. Halliwell, *Information-theoretic measure of uncertainty due to quantum and thermal fluctuations,* *Phys. Rev. D* **48**, 2753 (1993).

[6] K. Husimi, *Some Formal Properties of the Density Matrix, Proc. Phys. Math. Soc. Japan* **22**, 264 (1940).

[7] A. Wehrl, *On the relation between classical and quantum-mechanical entropy,* *Rep. Math. Phys.* **16**, 353 (1979)

[8] S. Curilef, F. Pennini, and A. Plastino, *Fisher information, Wehrl entropy, and Landau Diamagnetism,* Phys. Rev. B **71**, 024420 (2005).

- Coherent states (CSs) were introduced for the first time by Schrödinger[9] in 1926,

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \qquad \alpha \in \mathbb{C} \tag{13}$$

for solving the eigenvalue problem of the quantum harmonic oscillator $H_{Osc} = \frac{1}{2}\left(-\hbar\frac{\partial^2}{\partial x^2} + x^2\right)$, where $\{|n\rangle\}_{n=0}^{\infty}$ span the corresponding Hilbert space.

- These CSs minimize the Heisenberg uncertainty relation $\Delta Q \Delta P \geq \dfrac{\hbar}{2}$ through

$$\langle \Delta Q \rangle_{|\alpha\rangle} \langle \Delta P \rangle_{|\alpha\rangle} = \frac{\hbar}{2} \tag{14}$$

where the position and momentum operators Q and P are related to the annihilation and creation operators $a, a^\dagger$ as follows:

$$a = \frac{1}{\sqrt{2}}(Q + iP), \quad a^\dagger = \frac{1}{\sqrt{2}}(Q - iP) \quad \text{with} \quad [a, a^\dagger] = \mathbb{I}. \tag{15}$$

[9] E. Schrödinger, *Der stetige Übergang von der Mikro-zur Makromechanik, Naturwissenschaften* **14** 664 (1926)

From

$$|z\rangle = e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle, \qquad z \in \mathbb{C} \tag{16}$$

taking $z = \frac{q+ip}{\sqrt{2}}$, q, p being coordinates in the phase space, and considering the position and momentum operators $Q = \frac{1}{\sqrt{2}}(a^\dagger + a)$, $P = \frac{i}{\sqrt{2}}(a^\dagger - a)$, the expectation values corresponding to the average of Q and P in the CSs $|z\rangle$ are obtained in terms of the coordinates as

$$\langle z|Q|z\rangle = q, \quad \langle z|P|z\rangle = p, \tag{17}$$

$$\langle z|Q^2|z\rangle = q^2 + \frac{1}{2}, \quad \langle z|P^2|z\rangle = p^2 + \frac{1}{2} \tag{18}$$

establishing that the CSs $|z\rangle$ are the closest possible to their classical counterparts.

- The CSs can be defined over complex domains with $\mathfrak{H} = span\{\phi_m, m \in \mathbb{N}\}$ as linear superposition of the states or functions ϕ_m:

$$|z\rangle = (\mathcal{N}(|z|))^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{z^m}{\sqrt{\rho(m)}} |\phi_m\rangle, \qquad z = re^{i\theta} \tag{19}$$

  where

- $\{\rho(m)\}_{m=0}^{\infty}$ is a sequence of non-zero positive numbers chosen so as to ensure the convergence of the sum in a non-empty open subset $\mathfrak{D}$ of the complex plane,
- $\mathcal{N}(|z|)$ is the normalization factor ensuring that $\langle z|z\rangle = 1$.

There are a number of ways to define a set of CSs. For various generalizations, approaches and their properties, see[10].

[10] J.R. Klauder, B. S. Skagerstam, *Coherent states. Applications in physics and mathematical physics.* World Scientific Publishing Co., Singapore (eds) (1985); S. T. Ali, J. P. Antoine, and J. P. Gazeau, *Coherent States, Wavelets and their Generalizations 2nd edition, Theoretical and Mathematical Physics*, Springer, New York (2014).

- The resolution of the identity is given by

$$\int_{\mathfrak{D}} |z\rangle\langle z| d\mu = I_{\mathfrak{H}}, \tag{20}$$

where $d\mu$ is an appropriate measure and $I_{\mathfrak{H}}$ the identity operator on $\mathfrak{H}$.

- For $z_1 \neq z_2$, taking $|z\rangle = e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle$,

$$\langle z_1|z_2\rangle = e^{-\frac{1}{2}|z_1|^2 - \frac{1}{2}|z_2|^2} e^{\bar{z}_1 z_2} \neq 0 \Rightarrow |z_1\rangle, |z_2\rangle \text{ are not orthogonal} \tag{21}$$

- From (20) and (21), the CSs $|z\rangle$ constitute an overcomplete family of vectors in the Hilbert space $\mathfrak{H}$.
- From (20), for all $\Phi, \Psi \in \mathfrak{H}$,

$$|\Psi\rangle = \int_{\mathfrak{D}} \langle z|\Psi\rangle |z\rangle \; d\mu, \quad \int_{\mathfrak{D}} \langle \Phi|z\rangle\langle z|\Psi\rangle d\mu = \langle \Phi|\Psi\rangle. \tag{22}$$

Outline

Hilbert space of Hilbert-Schmidt operators

- The theory of Hilbert-Schmidt operators plays a key role in the formulation of the noncommutative quantum mechanics. In the past three decades, the von Neumann algebras[11] underwent a vigorous growth after the discovery of a natural infinite family of pairwise nonisomorphic factors and the advent of Tomita-Takesaki theory [12].
- The modular theory of von Neumann algebras has been created by M. Tomita[13] in 1967 and perfectioned by M. Takesaki around 1970 [14].

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[11] J. v. Neumann, On rings of operators. III., *Ann. Math.* **41**, 94-161 (1940); F. J. Murray and J. v. Neumann, On rings of operators, *Ann. Math.* **37**, 116-229 (1936).

[12] M. Takesaki *Tomita's Theory of Modular Hilbert Algebras and Its Applications* (New York: Springer, 1970); M. Tomita, *Standard forms of von Neumann algebras*, V-th Functional Analysis Symposium of the Mathematical Society of Japan, Sendai (1967).

[13] M. Tomita, *Standard forms of von Neumann algebras*, V-th Functional Analysis Symposium of the Mathematical Society of Japan, Sendai (1967).

[14] M. Takesaki *Tomita's Theory of Modular Hilbert Algebras and Its Applications* (New York: Springer, 1970).

Hilbert space of Hilbert-Schmidt operators

Definition (Hilbert-Schmidt operator)

Given a bounded operator, having the decomposition $A = \sum_k |\Phi_k\rangle\lambda_k\langle\Phi_k|$, where $\{\Phi_k\}$ is an othonormal basis of $\mathfrak{H}$, and $\lambda_1, \lambda_2, \ldots$ positive numbers, A is called a *Hilbert-Schmidt operator* if[a]

$$\mathrm{Tr}[AA^*] = \sum_k \langle\Phi_k|A^* A\Phi_k\rangle = \sum_k \lambda_k^2 < +\infty. \tag{23}$$

[a] E. Prugovečki, Quantum Mechanics in Hilbert Spaces, 2nd edn. (Academic Press, New York, 1981).

Remark

If A is any operator of trace class, then A^* is also of trace class:

$$\mathrm{Tr}A^* = \sum_k \langle e_k|A^* e_k\rangle = \sum_k \langle e_k|Ae_k\rangle^* = (\mathrm{Tr}A)^*. \tag{24}$$

Hilbert space of Hilbert-Schmidt operators

Definition

Let $\mathcal{B}_2(\mathfrak{H})$, $\mathcal{B}_2(\mathfrak{H}) \subset \mathcal{L}(\mathfrak{H})$ the set of all bounded operators on $\mathfrak{H}$, be the Hilbert space of Hilbert-Schmidt operators on $\mathfrak{H} = L^2(\mathbb{R})$, with the scalar product

$$\langle X|Y\rangle_2 = Tr[X^*Y] = \sum_k \langle \Phi_k|X^*Y\Phi_k\rangle, \tag{25}$$

where $\{\Phi_k\}_{k=0}^{\infty}$ is an orthonormal basis of $\mathfrak{H}$.

Definition (Hilbert-Schmidt norm)

For any Hilbert-Schmidt operator A, the quantity

$$||A||_2 = \sqrt{\mathrm{Tr}[A^*A]} \tag{26}$$

exists, and is called *Hilbert-Schmidt norm* of A.

Hilbert space of Hilbert-Schmidt operators

Definition

Let A and B two operators on $\mathfrak{H}$. The operator $A \vee B$, is such that

$$A \vee B(X) = AXB^*, \; X \in B_2(\mathfrak{H}). \tag{27}$$

For A and B, both bounded operators, $A \vee B$ defines a linear operator on $B_2(\mathfrak{H})$.

Thermal state and von Neumann algebras

- Let α_i, $i = 0, 2, \ldots, \infty$ be a sequence of non-zero, positive numbers, satisfying :

$$\sum_{i=1}^{\infty} \alpha_i = 1. \tag{28}$$

- Consider the state[a]

$$\Phi = \sum_{i=0}^{\infty} \alpha_i^{\frac{1}{2}} \mathbb{P}_i = \sum_{i=1}^{\infty} \alpha_i^{\frac{1}{2}} X_{ii} \in \mathcal{B}_2(\mathfrak{H}) \quad \text{with} \quad X_{ii} = |\zeta_i\rangle\langle\zeta_i|. \tag{29}$$

[a] Ali., S. T., and Bagarello, F., Some physical appearances of vector coherent states and coherent states related to degenerate Hamiltonians, J. Math. Phys. 46, 053518 (2005).

Thermal state and von Neumann algebras

Let the two von Neumann algebras given by

$$\mathfrak{A}_l = \{A_l = A \vee I | A \in \mathcal{L}(\mathfrak{H})\}, \tag{30}$$

$$\mathfrak{A}_r = \{A_r = I \vee A | A \in \mathcal{L}(\mathfrak{H})\}, \tag{31}$$

where $\mathfrak{A}_l$ corresponds to the operators given with A in the left, and $\mathfrak{A}_r$ corresponds to the operators given with A in the right of the identity operator I on $\mathfrak{H}$, respectively.

As important result, since the state Φ satisfies (28) and (29), is then cyclic for both algebras $\mathfrak{A}_l$ and $\mathfrak{A}_r$, see[15].

[15] I. Aremua, E. Baloïtcha, M. N. Hounkonnou and K. Sodoga, On Hilbert-Schmidt operator formulation of noncommutative quantum mechanics, *Mathematical Structures and Applications (Springer Nature Switzerland AG, 2018), T. Diagana, B. Toni (eds.); S. T. Ali, F. Bagarello, and G. Honnouvo, Modular structures on trace class operators and applications to Landau levels, J. Phys. A: Math. Theor. 43, 105202 (2010).*

- From, (28) and (29), that is

$$\Phi = \sum_{i=1}^{N} \alpha_i^{\frac{1}{2}} \mathbb{P}_i = \sum_{n=1}^{N} \alpha_n^{\frac{1}{2}} X_{nn} \in \mathcal{B}_2(\mathfrak{H}) \tag{32}$$

Φ is related to the thermal equilibrium state at inverse temperature β,

$$\Phi_\beta = \left[1 - e^{-\omega\beta}\right]^{\frac{1}{2}} \sum_{n=0}^{\infty} e^{-\frac{n}{2}\omega\beta} |\Phi_n\rangle\langle\Phi_n|, \tag{33}$$

with the identifications

$$\alpha_n = \left[1 - e^{-\omega\beta}\right] e^{-n\omega\beta}, \qquad X_{nn} = |\Phi_n\rangle\langle\Phi_n|. \tag{34}$$

- Φ_β corresponds to the harmonic oscillator Hamiltonian $H_{OSC} = \frac{1}{2}(P^2 + Q^2)$, with $H_{OSC}\Phi_n = \omega(n + \frac{1}{2})\Phi_n$, $n = 0, 1, 2, \ldots,$ the density matrix

$$\rho_\varphi = \frac{e^{-\beta H_{OSC}}}{\mathrm{Tr}[e^{-\beta H_\varphi}]} = (1 - e^{-\omega\beta}) \sum_{n=0}^{\infty} e^{-n\omega\beta} |\Phi_n\rangle\langle\Phi_n|. \tag{35}$$

In the sequel, the related coherent states will be defined on the Hilbert space $\tilde{\mathfrak{H}} = L^2(\mathbb{R}^2, dxdy)$.

1. Consider the Wigner unitary transform, given by

$$\mathcal{W} : \mathcal{B}_2(\mathfrak{H}) \to L^2(\mathbb{R}^2, dxdy)$$

$$(\mathcal{W}X)(x,y) = \frac{1}{(2\pi)^{1/2}} Tr[(U(x,y))^* X], \tag{36}$$

where $X \in \mathcal{B}_2(\mathfrak{H}), x, y \in \mathbb{R}$ and $U(x,y)$ a unitary operator.

2. Using the isometry $\mathcal{W}$, given $\phi_{nl} = |\phi_n\rangle\langle\phi_l| \in \mathcal{B}_2(\mathfrak{H})$

A vector $\Psi_{jk} \in L^2(\mathbb{R}^2, dxdy)$ is given by

$$\mathcal{W}\phi_{jk} = \mathcal{W}(|\phi_j\rangle\langle\phi_k|) = \Psi_{jk}. \tag{37}$$

CSs from the harmonic oscillator thermal state

- The CSs, denoted $|z, \bar{z}, \beta\rangle^{\text{KMS}}$, built from the thermal state Φ_β, are given by

$$|z, \bar{z}, \beta\rangle^{\text{KMS}} = U_1(z)|\Phi_\beta\rangle = e^{zA_1^\dagger - \bar{z}A_1}|\Phi_\beta\rangle. \tag{38}$$

with $U_1(z) := U_1(x, y) = e^{zA_1^\dagger - \bar{z}A_1}$, where the actions of the annihilation and creation operators, A_1 and $A_1^\dagger$ are given on $\tilde{\mathfrak{H}} = L^2(\mathbb{R}^2, dxdy)$ by

$$A_1^\dagger|\Psi_{nl}\rangle = \sqrt{n+1}\Psi_{n+1,l}, \qquad A_1|\Psi_{nl}\rangle = \sqrt{n}\Psi_{n-1,l}. \tag{39}$$

Their expressions are given as follows:

$$\begin{aligned} |z, \bar{z}, \beta\rangle^{\text{KMS}} &= U_1(z)|\Phi_\beta\rangle = (2\pi)^{\frac{1}{2}} \sum_{i,j=0}^{\infty} \lambda_i^{\frac{1}{2}} \overline{\Psi_{ji}(x, y)} |\Psi_{ji}\rangle, \\ {}^{\text{KMS}}\langle z, \bar{z}, \beta| &= \langle \Phi_\beta | U_1(z)^* = (2\pi)^{\frac{1}{2}} \sum_{l,k=0}^{\infty} \langle \Psi_{lk} | \lambda_k^{\frac{1}{2}} \Psi_{lk}(x, y). \end{aligned} \tag{40}$$

Proposition

Using the fact that for any normalized vector $\Phi \in \mathfrak{H} = L^2(\mathbb{R})$, the vectors $U(z)\Phi, z \in \mathbb{C}$, where $U(z) := U(x, y)$ satisfy

$$\frac{1}{2\pi}\int_{\mathbb{C}} |U(z)\Phi\rangle\langle U(z)\Phi| dxdy = I_{\mathfrak{H}} \tag{41}$$

from the isometry $\mathcal{W}$, the CSs $|z, \bar{z}, \beta\rangle^{\text{KMS}}$ satisfy the resolution of the identity condition

$$\frac{1}{2\pi}\int_{\mathbb{C}} |z, \bar{z}, \beta\rangle^{\text{KMS}\,\text{KMS}}\langle z, \bar{z}, \beta| dxdy = I_{\tilde{\mathfrak{H}}}, \quad \tilde{\mathfrak{H}} = L^2(\mathbb{R}^2, dxdy). \tag{42}$$

Proof. See[16].

□

[16] I. Aremua, E. Baloïtcha, M. N. Hounkonnou and K. Sodoga, On Hilbert-Schmidt operator formulation of noncommutative quantum mechanics, *Mathematical Structures and Applications (Springer Nature Switzerland AG, 2018), T. Diagana, B. Toni (eds.)*.

Proposition

The components of the KMS CSs $|z,\bar{z},\beta\rangle^{\mathrm{KMS}}$[a] and the densty operator are

$$|\langle\Phi_n|z,\bar{z},\beta\rangle^{\mathrm{KMS}}|^2 = \left[1-e^{-\omega\beta}\right]\sum_{s,t=0}^{\infty}\sum_{i,j=0}^{\infty} e^{-(t+j)\frac{\omega\beta}{2}}\left\{2\pi\,\langle\Phi_n|\Psi_{ji}\overline{\Psi_{ji}(x,y)}\Psi_{st}(x,y)\Psi_{ts}|\Phi_n\rangle\right\}, \quad (43)$$

$$\begin{aligned} {}^{\mathrm{KMS}}\langle z,\bar{z},\beta|\rho_\beta|z,\bar{z},\beta\rangle^{\mathrm{KMS}} &= 2\pi\left[1-e^{-\omega\beta}\right]\sum_{n=0}^{\infty} e^{-n\omega\beta}\sum_{s,t=0}^{\infty}\sum_{i,j=0}^{\infty}\lambda_j^{1/2}\lambda_t^{1/2} \\ &\times\left\{\langle\Phi_n|\Psi_{ji}\overline{\Psi_{ji}(x,y)}\Psi_{st}(x,y)\Psi_{ts}|\Phi_n\rangle\right\}. \end{aligned} \quad (44)$$

[a] I. Aremua, M. N. Hounkonnou and E. Baloıtcha, Density operator formulation for mag- netic systems: Physical and mathematical aspects, J. Math. Phys. 62, 013503 (2021); doi: 10.1063/5.0012588

Density operator - Husimi distribution

Proof. Starting with (42) and the definition of Glauber-Sudarshan P-distribution of the density operator i.e.,

$$\rho_\beta = \frac{1}{2\pi}\int_{\mathbb{C}} dxdyP(|z|^2)|z,\bar{z},\beta\rangle^{\text{KMS}\,\text{KMS}}\langle z,\bar{z},\beta| \tag{45}$$

we get

$$\begin{aligned}
\langle\Phi_n|\rho_\beta|\Phi_n\rangle &= \frac{1}{\bar{n}_0+1}\left(\frac{\bar{n}_0}{\bar{n}_0+1}\right)^n = (1-e^{-\omega\beta})e^{-n\omega\beta} \\
&= \frac{1}{2\pi}\int_{\mathbb{C}} dxdyP(|z|^2)|\langle\Phi_n|z,\bar{z},\beta\rangle^{\text{KMS}}|^2 \\
&= \left[1-e^{-\omega\beta}\right]\sum_{s,t=0}^{\infty}\sum_{i,j=0}^{\infty} e^{-(t+j)\frac{\omega\beta}{2}} \\
&\quad\times\left\{\int_{\mathbb{C}} dxdyP(|z|^2)\langle\Phi_n|\Psi_{ji}\overline{\mathcal{W}(|\Phi_j\rangle\langle\Phi_i|)(x,y)}\mathcal{W}(|\Phi_s\rangle\langle\Phi_t|)(x,y)\Psi_{ts}|\Phi_n\rangle\right\},
\end{aligned} \tag{46}$$

where the thermal occupancy for the harmonic oscillator $\bar{n}_0 = \left[e^{\omega\beta}-1\right]^{-1}$ with the angular frequency ω has been introduced.

Density operator diagonal-Husimi distribution

Using the Q-Husimi distribution expression (44), the Wehrl entropy is deduced as

$$\begin{aligned} W &= -\frac{1}{2\pi}\int_{\mathbb{C}} dxdy\ {}^{\mathrm{KMS}}\langle z,\bar{z},\beta|\rho_\beta|z,\bar{z},\beta\rangle^{\mathrm{KMS}} \ln\left\{{}^{\mathrm{KMS}}\langle z,\bar{z},\beta|\rho_\beta|z,\bar{z},\beta\rangle^{\mathrm{KMS}}\right\} \\ &= -\frac{1}{2\pi}\int_{\mathbb{C}} dxdy \left(2\pi\left[1-e^{-\omega\beta}\right]\right. \\ &\quad \times \sum_{n=0}^{\infty} e^{-n\omega\beta}\sum_{s,t=0}^{\infty}\sum_{i,j=0}^{\infty}\lambda_j^{1/2}\lambda_t^{1/2}\left.\left\{\langle\phi_n|\Psi_{ji}\overline{\Psi_{ji}(x,y)}\Psi_{st}(x,y)\Psi_{ts}|\phi_n\rangle\right\}\right) \\ &\quad \times \ln\left(2\pi\left[1-e^{-\omega\beta}\right]\right. \\ &\quad \left.\sum_{n=0}^{\infty} e^{-n\omega\beta}\sum_{s,t=0}^{\infty}\sum_{i,j=0}^{\infty}\lambda_j^{1/2}\lambda_t^{1/2}\left\{\langle\phi_n|\Psi_{ji}\overline{\Psi_{ji}(x,y)}\Psi_{st}(x,y)\Psi_{ts}|\phi_n\rangle\right\}\right). \end{aligned} \tag{47}$$

Density operator for generalized hypergeometric CSs

The generalized photon-added associated hypergeometric type coherent states (GPAH-CSs) denoted by $|z,m\rangle_p$, are obtained by repeating the action of the raising operator a_m^+ on the generalized associated hypergeometric type CSs $|z,m\rangle$ as[17]:

$$|z,m\rangle_p \equiv (a_m^+)^p |z,m\rangle, \tag{48}$$

where p is a positive integer standing for the number of added quanta (or photons). Their expression is given by

$$|z,m\rangle_p = \mathcal{N}_p(|z|^2,m) \sum_{n=0}^{\infty} \frac{z^n}{K_n^p(m)} |n+p\rangle \tag{49}$$

and they satisfy on the quantum Hilbert space $\mathfrak{H}_{m,p} = span\{|n+p\rangle\}_{n\geq 0}$, the completeness relation

$$\int_{\mathbb{C}} d^2z\ |z;m\rangle_p \omega_p(|z|^2;m)_p\langle z;m| = \mathbb{I}_{\mathfrak{H}_{m,p}} \equiv \sum_{n=0}^{\infty} |n+p\rangle\langle n+p|. \tag{50}$$

[17] K. Sodoga, I. Aremua, and M. N. Hounkonnou, "Generalized photon-added associated hypergeometric coherent states: Characterization and relevant properties," Eur. Phys. J. D 72, 172 (2018)

Density operator for generalized hypergeometric CSs

Consider a quantum gas of the system in the thermodynamic equilibrium with a reservoir at temperature T, which satisfies a quantum canonical distribution. The corresponding normalized density operator is given as

$$\rho^{(p)} = \frac{1}{Z}\sum_{n=0}^{\infty} e^{-\beta e_n}|n+p\rangle\langle n+p| \tag{51}$$

where in the exponential e_n is the eigen-energy, and the partition function Z is taken as the normalization constant.

The diagonal elements of $\rho^{(p)}$ which are key ingredients for our purpose, also known as the Q-distribution or Husimi's distribution, are derived in the GPAH-CSs basis as

$$_p\langle z; m|\rho^{(p)}|z; m\rangle_p = \frac{\mathcal{N}_p^2(|z|^2; m)}{Z}\sum_{n=0}^{\infty}\frac{|z|^{2n}}{|K_n^p(m)|^2}e^{-\beta e_n}. \tag{52}$$

Density operator for generalized hypergeometric CSs

The normalization of the density operator leads to

$$\mathrm{Tr}\rho^{(p)} = \int_{\mathbb{C}} d^2z\, \omega_p(|z|^2; m)\, {}_p\langle z; m|\rho^{(p)}|z; m\rangle_p = 1. \tag{53}$$

The diagonal expansion of the normalized canonical density operator over the GPAH-CSs projector is

$$\rho^{(p)} = \int_{\mathbb{C}} d^2z\, \omega_p(|z|^2; m)|z; m\rangle_p P(|z|^2)\, {}_p\langle z; m| \tag{54}$$

where the P-distribution function $P(|z|^2)|$ satisfying the normalization to unity condition

$$\int_{\mathbb{C}} d^2z\, \omega_p(|z|^2; m)P(|z|^2) = 1 \tag{55}$$

must be determined.

Density operator for generalized hypergeometric CSs

Thus, given an observable $\mathcal{O}$, one obtains:

The expectation value, i.e., the thermal average is given by

$$\langle\mathcal{O}\rangle_p = Tr(\rho^{(p)}\mathcal{O}) = \int_{\mathbb{C}} d^2z\, \omega_p(|z|^2; m) P(|z|^2)\, {}_p\langle z; m|\mathcal{O}|z; m\rangle_p. \tag{56}$$

The pseudo-thermal expectation value of the number operator N_m, and of its square N_m^2, given by $\langle N_m\rangle^{(p)} = Tr(\rho^{(p)}N_m)$ and $\langle N_m^2\rangle^{(p)} = Tr(\rho^{(p)}N_m^2)$, respectively, allow to obtain the thermal intensity correlation function as follows:

$$(g^{(2)})^{(p)} = \frac{\langle N_m^2\rangle^{(p)} - \langle N_m\rangle^{(p)}}{\left(\langle N_m\rangle^{(p)}\right)^2}. \tag{57}$$

The correlation between pairs of fields $g^{(2)}$ typically is used to find the statistical character of intensity fluctuations.

Density operator for generalized hypergeometric CSs

- Then, the thermal analogue of the Mandel parameter[18], given by

$$Q^{(p)} = \langle N_m \rangle^{(p)} \left[(g^{(2)})^{(p)} - 1 \right] \tag{58}$$

is deduced In the illustrated examples, given an appropriate function $K_n^p(m)$, this formalism will be applied to determine the concrete expressions.

- The intensity correlation, highlights the bunching(i.e., the light field shows super-Poissonian photon statistics corresponding to a photon number distribution $(\Delta N)^2 = \langle N^2 \rangle - (\langle N \rangle)^2$ for which $(g^{(2)})^{(p)} > 1$) and antibunching (i.e., the light field shows sub-Poissonian photon statistics corresponding to a photon number distribution $(\Delta N)^2 = \langle N^2 \rangle - (\langle N \rangle)^2$ for which $(g^{(2)})^{(p)} < 1$) effects of the quantum states.
- The Poissonian statistics is obtained for $(g^{(2)})^{(p)} \simeq 1$.

[18] L. Mandel and E. Wolf: Optical Coherence and Quantum Optics, Cambridge University Press, Cambridge 1995.

Density operator for generalized hypergeometric CSs

- The expansion coefficient is given by

$$K_n^p(m) = \frac{\Gamma(n+1)}{\sqrt{\Gamma(n+p+1)}} \frac{1}{c^n}. \tag{59}$$

The explicit form of the GPAH-CSs relative to Hermite and Laguerre polynomials, defined for any finite $|z|$, are :

$$|z;m\rangle_p = \frac{1}{\sqrt{\Gamma(p+1)\ {}_1F_1\left(p+1\ ;\ 1\ ;\ |cz|^2\right)}} \sum_{n=0}^{\infty} \sqrt{\Gamma(n+p+1)}\ \frac{c^n z^n}{n!} |n+p\rangle \tag{60}$$

$$Q = (m+p)\left[\frac{{}_3\mathcal{F}_3(|z|^2;m,p)}{{}_2\mathcal{F}_2(|z|^2;m,p)} - \frac{{}_2\mathcal{F}_2(|z|^2;m,p)}{{}_1\mathcal{F}_1(|z|^2;m,p)}\right] - 1.$$

$$g^{(2)} = {}_1\mathcal{F}_1(|z|^2;m,p)\frac{(m+p)_3\mathcal{F}_3(|z|^2;m,p) -_2\mathcal{F}_2(|z|^2;m,p)}{(m+p)_2\mathcal{F}_2(|z|^2;m,p)^2}.$$

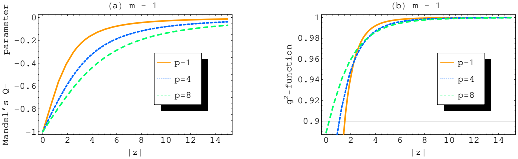

Figure 1: *Plots of: (a) the Mandel Q-parameter and (b) the second-order correlation function of the GPAH-CSs (60) versus $|z|$ with fixed derivative order parameter $m = 1$ and various values of the photon-added number p.*

The Mandel Q-parameter is strictly negative, increases with the amplitude $|z|$, asymptotically tends to 0, and decreases with increasing p. The second order correlation function is such that $0 < g^{(2)} < 1$ and asymptotically tends to 1.

Density operator for generalized hypergeometric CSs

The relative normalized density operator and the partition function are provided as

$$\begin{aligned}\rho^{(p)} &= \frac{1}{Z}\sum_{n=0} e^{-\beta n}|n+p\rangle\langle n+p| \\ &= \frac{1}{\bar{n}_o+1}\sum_{n=0}\left(\frac{\bar{n}_o}{\bar{n}_o+1}\right)^n |n+p\rangle\langle n+p|,\ Z=\frac{1}{1-e^{-\beta}} := \bar{n}_o+1 \end{aligned} \tag{61}$$

respectively, with $\bar{n}_o = (e^{\beta}-1)^{-1}$ corresponding to the thermal expectation value of the number operator for oscillators with angular frequency $\omega = 1$, where $\hbar = 1$. From the integral relation

$$\mathrm{Tr}\rho^{(p)} = \frac{1}{Z}\int_0^{\infty} dx\, x^p\, G^{1,1}_{1,2}\left(x \left| \begin{matrix} -p & ; & \\ 0 & ; & 0 \end{matrix}\right.\right) G^{1,1}_{1,2}\left(-xe^{-\beta} \left| \begin{matrix} -p & ; & \\ 0 & ; & 0 \end{matrix}\right.\right). \tag{62}$$

we get the partition function expression

$$Z = \frac{1}{1-e^{-\beta}} = \bar{n}_o + 1, \qquad \bar{n}_o = (e^{\beta}-1)^{-1}. \tag{63}$$

Density operator and thermal expectation values

From (54), using the matrix element of the density operator

$$\langle n+p|\rho^{(p)}|n+p\rangle = \frac{1}{\bar{n}_o+1}\left(\frac{\bar{n}_o}{\bar{n}_o+1}\right)^n \tag{64}$$

we obtain the following integration equality

$$\begin{aligned}&\frac{1}{\bar{n}_o+1}\left(\frac{\bar{n}_o}{\bar{n}_o+1}\right)^n \frac{1}{|c|^{2(n+p+1)}}\frac{\Gamma(n+1)^2}{\Gamma(n+p+1)}\\ =& \int_0^\infty dx\, x^{n+p} P(x) G^{2,0}_{1,2}\left(|c|^2 x\middle| \begin{matrix} ; & 0 \\ -p,-p & ; \end{matrix}\right).\end{aligned} \tag{65}$$

After performing the exponent change $n+p = s-1$ of $x = |z|^2$, in order to get to the Stieltjes moment problem, we arrive at the P-function obtained as

$$P(|z|^2) = \frac{1}{\bar{n}_o}\left(\frac{\bar{n}_o+1}{\bar{n}_o}\right)^p \frac{G^{2,0}_{1,2}\left(\frac{\bar{n}_o+1}{\bar{n}_o}|cz|^2\middle| \begin{matrix} ; & 0 \\ -p,-p & ; \end{matrix}\right)}{G^{2,0}_{1,2}\left(|cz|^2\middle| \begin{matrix} ; & 0 \\ -p,-p & ; \end{matrix}\right)}. \tag{66}$$

Density operator and thermal expectation values

$$\rho^{(p)} = \frac{1}{\bar{n}_o}\left(\frac{\bar{n}_o+1}{\bar{n}_o}\right)^p \int_{\mathbb{C}} d^2 z\, \omega_p(|z|^2;m)|z;m\rangle_p \frac{G^{2,0}_{1,2}\left(\frac{\bar{n}_o+1}{\bar{n}_o}|cz|^2 \Big| \begin{matrix} ; & 0 \\ -p,-p & ; \end{matrix}\right)}{G^{2,0}_{1,2}\left(|cz|^2 \Big| \begin{matrix} ; & 0 \\ -p,-p & ; \end{matrix}\right)} {}_p\langle z;m|.$$

The pseudo-thermal expectation values of the number operator and of its square are obtained as

$$\begin{aligned} \langle N_m\rangle^{(p)} &= Tr(\rho^{(p)}N_m) = \frac{1}{\bar{n}_o+1}(m+p)\sum_{n=0}^{\infty}\left(1+\frac{n}{m+p+1}\right)\left(\frac{\bar{n}_o}{\bar{n}_o+1}\right)^n \\ \langle N_m^2\rangle^{(p)} &= Tr(\rho^{(p)}N_m^2) = \frac{1}{\bar{n}_o+1}(m+p)^2\sum_{n=0}^{\infty}\left(1+\frac{n}{m+p+1}\right)^2\left(\frac{\bar{n}_o}{\bar{n}_o+1}\right)^n. \end{aligned}$$

Thereby, the thermal analogue of the Mandel parameter (58) is

$$Q^{(p)} = \frac{\bar{n}_o^2(m+p) - \left[(m+p+1)^2+\bar{n}_o\right]}{(m+p+1)\left[(m+p+1)+\bar{n}_o\right]}. \qquad (68)$$

Thank you for your attention