United Nations
Educational, Scientific and
Cultural Organization

ICMPA - UNESCO Chair

University of Abomey-Calavi

Some solvable physicals systems treated in Noncommutative Phase Space

MEMAQUAN-2022

Jeannot Mensah Allognon

July 12, 2022

Introduction

Motivation

- Nair (2001) consider a case of a charged particle in a magnetic field with a harmonic oscillator (HO) potential and shows that a critical point with the density of states becoming infinite for the value of the magnetic field equal to the inverse of the noncommutativity parameter.
- The higher-dimensional HO in noncommutative phase space (NCPS) cannot simply be divided into independent 1d HO, and the calculation of the energy spectra is very complicated if using the normal methods in NCPS.
- Note that, despite the large number of models studied in the literature, anharmonicity of vibration plays an important role in several branches of physics. Its importance was first recognized in acoustics with the alteration of the pitch of overtones of mode with intensity changes were successfully accounted for as the effects of the anharmonicity of the oscillator.
- Recently, the subject has acquired a fresh interest in relation to the subject of molecular spectroscopy. The thermal expansion of crystals also owes its origin to the anharmonic nature of the vibrations inside a crystal lattice.

Introduction

- To our knowledge, the explicit expression of the energy spectra of an anisotropic HO in generalized NCPS has not been reported in the literature so far.
- All these formal aspects and relevant effects of noncommutativity, researchers used many methods: factorization method; perturbation method; eigen-operator method ..., to study a quantum exactly solvable higher-dimensional NC oscillator with quasi-harmonic behaviour.

The case of noncommutativity with: anharmonic oscillator; anisotropic harmonic oscillator received little attention, even though it deserves also to be investigated in order to analyze the properties of a wide class of physical systems in NCPS.

Scientific questions

The scientific questions that immediately arise are:

- How the noncommutativity of space removes the degeneracies of spectrum thought a system of particles plunged into anharmonic potential?
- How to compute the energy Spectra of an anisotropic HO in a Generalized NCPS by using the Invariant eigen-operator method?

Outline

A 2-d anharmonic oscillator

Let us consider a physical model for a 2-d anharmonic oscillator moving in a noncommutative plane described by the following quantum Hamiltonian:

$$H = \frac{1}{2m}(\hat{p}_1^2 + \hat{p}_2^2) + \frac{1}{2}m\omega^2(\hat{x}_1^2 + \hat{x}_2^2) + \alpha(\hat{x}_1^2 + \hat{x}_2^2)^2, \tag{1}$$

where $\hat{x}_1, \hat{x}_2, \hat{p}_1$ and $\hat{p}_2$ are the noncommutative coordinates expressed by

$$\hat{x}_i \quad = \quad x_i - \frac{\theta}{2\hbar}\epsilon_{i,j}p_j, \qquad \hat{p}_i = p_i + \frac{\eta}{2\hbar}\epsilon_{i,j}x_j \quad \text{where } \epsilon_{12} = -\epsilon_{21} = 1, \tag{2}$$

which satisfy the following canonical commutation relations:

$$[\hat{x}_i, \hat{p}_j] \quad = \quad i\hbar\widetilde{\delta}_{i,j}, \qquad [\hat{x}_i, \hat{x}_j] = i\theta_{i,j}, \qquad [\hat{p}_i, \hat{p}_j] = i\eta_{i,j} \quad \forall i, j = 1, 2 \tag{3}$$

2-d anharmonic oscillator

The Hamiltonian (1) can be viewed as the unperturbed Hamiltonian H_0 of the usual 2-d harmonic oscillator

$$H_0 = \frac{1}{2M}(p_1^2 + p_2^2) + \frac{M\tilde{\Omega}^2}{2}(x_1^2 + x_2^2) - \left(\frac{M\tilde{\Omega}^2}{2}\frac{\theta}{\hbar} + \varpi\frac{\eta}{\hbar}\right)(x_1 p_2 - x_2 p_1), \tag{4}$$

perturbed by the anharmonic term $\alpha(\hat{x}_1^2 + \hat{x}_2^2)^2$, where

$$M = \frac{m}{1 + \frac{m^2\omega^2\theta^2}{4\hbar^2}}, \qquad \tilde{\Omega} = \frac{\sqrt{(m^2\theta^2\omega^2 + 4\hbar^2)(4m^2\omega^2\hbar^2 + \eta^2)}}{4m\hbar^2}$$

$$\text{and } \varpi = \frac{1}{2M}\left(\frac{4\hbar^2 - M^2\tilde{\Omega}^2\theta^2}{4\hbar^2 + \eta\theta}\right). \tag{5}$$

The lowering and raising operators are given by:

$$a_j = \sqrt{\frac{M\tilde{\Omega}}{2\hbar}}\left(x_j + i\,\frac{p_j}{M\tilde{\Omega}}\right), \qquad a_j^\dagger = \sqrt{\frac{M\tilde{\Omega}}{2\hbar}}\left(x_j - i\,\frac{p_j}{M\tilde{\Omega}}\right), \quad j = 1, 2, \tag{6}$$

and the unperturbed non-diagonal Hamiltonian (4) leads to

$$H_0 = \hbar\tilde{\Omega}(a_1^\dagger a_1 + a_2^\dagger a_2 + 1) + i(\frac{M\tilde{\Omega}^2\theta}{2} + \varpi\eta)(a_1^\dagger a_2 - a_2^\dagger a_1). \tag{7}$$

Using the following unitary transformations $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}\begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}$, the Hamiltonian (7) takes the diagonalized form and its eigenvalues are

$$E^{(0)}_{n_1,n_2} = \hbar\tilde{\Omega}(n_1 + n_2 + 1) - (\frac{M\tilde{\Omega}^2\theta}{2} + \varpi\eta)(n_1 - n_2). \tag{8}$$

These eigenvalues are non-degenerate, what is *a priori* unexpected.

Anharmonic term

In term of ladder operators, the anharmonic term yields to

$$(\hat{x}_1^2+\hat{x}_2^2)=4\tilde{\beta}^2(\hat{N}_1+\frac{1}{2})+4\tilde{\gamma}^2(\hat{N}_2+\frac{1}{2})+4i\tilde{\beta}\tilde{\gamma}(\hat{a}_1^\dagger\hat{a}_2^\dagger-\hat{a}_1\hat{a}_2) \tag{9}$$

where $\tilde{\beta}=\frac{1}{2}\left(\sqrt{\frac{\hbar}{M\tilde{\Omega}}}-\sqrt{\frac{M\tilde{\Omega}\theta^2}{4\hbar}}\right)$ and $\tilde{\gamma}=\frac{1}{2}\left(\sqrt{\frac{\hbar}{M\tilde{\Omega}}}+\sqrt{\frac{M\tilde{\Omega}\theta^2}{4\hbar}}\right)$.

In Fock space, we use the perturbation theory to compute the anharmonic term as

$$\alpha(\hat{x}_1^2+\hat{x}_2^2)^2|n_1,n_2\rangle^{(0)} = \sum_{j=-2}^{2}\tilde{C}^{(j)}_{n_1,n_2}|n_1+j,n_2+j\rangle^{(0)}, \tag{10}$$

where
$$\begin{aligned}\tilde{C}^{(-2)}_{n_1,n_2} &= -16\alpha\tilde{\beta}^2\tilde{\gamma}^2\sqrt{n_1n_2(n_1-1)(n_2-1)},\\ \tilde{C}^{(-1)}_{n_1,n_2} &= -32i\alpha\tilde{\beta}\tilde{\gamma}(\tilde{\beta}^2n_1+\tilde{\gamma}^2n_2)\sqrt{n_1n_2},\end{aligned} \tag{11}$$

$$\tilde{C}^{(0)}_{n_1,n_2} = 16\alpha[\tilde{\beta}^4\left(n_1+\frac{1}{2}\right)^2+\tilde{\gamma}^4\left(n_2+\frac{1}{2}\right)^2+2\tilde{\beta}^2\tilde{\gamma}^2\left(2n_1n_2+n_1+n_2+\frac{3}{4}\right)]$$
$$\tilde{C}^{(1)}_{n_1,n_2} = 32i\alpha\tilde{\beta}\tilde{\gamma}\sqrt{(n_1+1)(n_2+1)}\left[\tilde{\beta}^2(n_1+1)+\tilde{\gamma}^2(n_2+1)\right],$$
$$\tilde{C}^{(2)}_{n_1,n_2} = -16\alpha\tilde{\beta}^2\tilde{\gamma}^2\sqrt{(n_1+1)(n_2+1)(n_1+2)(n_2+2)}.$$

Proposition 1:

To the first order of the correction, the n-th energy level of the system are given by

$$E_{n_1,n_2} \simeq \hbar\tilde{\Omega}(n_1+n_2+1)-\left(\frac{M\tilde{\Omega}^2\theta}{2}+\varpi\eta\right)(n_1-n_2)+16\alpha\left[\tilde{\beta}^2\left(n_1+\frac{1}{2}\right)+\tilde{\gamma}^2\left(n_2+\frac{1}{2}\right)\right]^2+16\alpha\tilde{\beta}^2\tilde{\gamma}^2\left[n_1n_2+(n_1+1)(n_2+1)\right]. \quad (12)$$

In limit $\theta,\eta\to 0$, they reduce to the degenerate form $E^{(0)}_{n_1,n_2}\to\hbar\omega(n_1+n_2+1)$.

Proposition 2:

First and second order the n-th level excited state are given by

$$|n_1, n_2\rangle \simeq |n_1, n_2\rangle^{(0)} + \alpha \sum_{j=-2 (j\neq 0)}^{2} \frac{\tilde{C}^{(j)}_{n_1,n_2}}{-2j\hbar\tilde{\Omega}} |n_1 + j, n_2 + j\rangle^{(0)}, \quad (13)$$

$$|n_1, n_2\rangle \simeq |n_1, n_2\rangle^{(0)} + \alpha |n_1, n_2\rangle^{(1)} + \alpha^2 \sum_{\mathbf{k} \neq \mathbf{n}}^{\infty} \sum_{\mathbf{k}' \neq \mathbf{n}}^{\infty} \left\{ \frac{H_{1\mathbf{k}\mathbf{k}'} H_{1\mathbf{k}'\mathbf{n}}}{\Delta E^0_{\mathbf{n}\mathbf{k}} \Delta E^0_{\mathbf{n}\mathbf{k}'}} |\varphi^0_{\mathbf{k}}\rangle - \frac{|\varphi^0_{\mathbf{k}}\rangle \left(H_{1\mathbf{n}\mathbf{n}} - \frac{1}{2} H_{1\mathbf{n}\mathbf{k}}\right) H_{1\mathbf{k}\mathbf{n}}}{(\Delta E^0_{\mathbf{n}\mathbf{k}})^2} \right\}, \quad (14)$$

where $\mathbf{k} = (k_1, k_2)$, $|\varphi^0_{\mathbf{n}}\rangle \equiv |n_1, n_2\rangle^{(0)}$, $H_{1\mathbf{n}\mathbf{k}} \equiv \langle\varphi^0_{\mathbf{k}}|H_1|\varphi^0_{\mathbf{n}}\rangle$ and $\Delta E^0_{\mathbf{n}\mathbf{k}} \equiv E^0_{\mathbf{n}} - E^0_{\mathbf{k}}$.

Proposition 3:

The arbitrary excited wave function of the unperturbed problem are given by

$$\Psi^{1,2,(0)}_{n_1,n_2} = (-1)^{n_2} \frac{\sqrt{n_1!n_2!}}{2^{(n_1+n_2)}} \Psi^{1,2,(0)}_{00} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \frac{i^{k_1+k_2} H^1_{n_1-\Delta k} H^2_{n_2+\Delta k}}{k_1!k_2!(n_1-k_1)!(n_2-k_2)!} \tag{15}$$

where $\Delta k := k_1 - k_2$, $\Psi^{1,2}_n := \Psi_n(x_1, x_2)$, $H^i_n := H_n(\sqrt{\delta} x_i)$ represents Hermite polynomial of degree n and $\delta = \frac{M\tilde{\Omega}}{\hbar}$.
The first order correction to the ground state is given by

$$\Psi^{1,2,(1)}_{00} = \frac{8\tilde{\beta}\tilde{\gamma}}{\hbar\Omega} \left[-2i(\tilde{\beta}^2 + \tilde{\gamma}^2)\Psi^{1,2,(0)}_{11} + \tilde{\beta}\tilde{\gamma}\Psi^{1,2,(0)}_{22} \right] \tag{16}$$

and the first order approximation to the ground state leads to

$$\Psi^{1,2}_{00} = \Psi^{1,2,(0)}_{00} + \alpha\Psi^{1,2,(1)}_{00}. \tag{17}$$

3-d anisotropic harmonic oscillator (HO)

Invariant eigen operator method

Let us compute energy-level gap for an 3-d anisotropic HO in NCPS whose Hamiltonian $\widehat{H} = \sum_{i=1}^{3}(\frac{\hat{p}_i^2}{2m} + \frac{1}{2}m\omega_i^2\hat{x}_i^2)$ does not depend explicitly on time and assume that $\widehat{F} = \sum_{i=1}^{3}(a_i\hat{x}_i + b_i\hat{p}_i)$ is invariant under the action $i\hbar\frac{d}{dt}$. With $|\Psi_m\rangle$, $|\Psi_n\rangle$ two adjacent eigenstates of $\widehat{H}$, the Heisenberg equation motion $\frac{d}{dt}\widehat{F} = \frac{i}{\hbar}[\widehat{F}, \widehat{H}]$

$$[\widehat{F}, \widehat{H}] = \lambda\,\widehat{F}, \tag{18}$$

where $\lambda = (E_n - E_m)$ denote the energy gap.

The key point of the method relies on finding the appropriate invariant eigen-operators for the Hamiltonian. If, for some cases, finding invariant eigen-operators via a single commutator calculation fails, we should try the calculation of double commutators through the equation $\left(i\hbar\frac{d}{dt}\right)^2\widehat{F} = \left[[\widehat{F}, \widehat{H}], \widehat{H}\right] = \lambda\widehat{F}$.

In this case the energy gap between the two adjacent levels is $\sqrt{\lambda}$. This method is called invariant eigen-operator method. Once the invariant eigen-operator $\widehat{F}$ is found, some information about the eigenvalues can be obtained.

Noncommutative phase space

Let us consider a NCPS described by the operators $\hat{x}_i$ and $\hat{p}_i$

$$\hat{x}_i = x_i - \frac{\theta_{i,j}}{2\hbar} p_j \qquad \hat{p}_i = p_i + \frac{\eta_{i,j}}{2\hbar} x_j. \tag{19}$$

They satisfy the following extended Heisenberg algebra:

$$[\hat{x}_i, \hat{p}_j] = i\hbar\widetilde{\delta}_{i,j}, \qquad [\hat{x}_i, \hat{x}_j] = i\theta_{i,j}, \qquad [\hat{p}_i, \hat{p}_j] = i\eta_{i,j} \tag{20}$$

where $\theta_{i,j} = \epsilon_{i,j,k}\theta_k$, $\eta_{i,j} = \epsilon_{i,j,k}\eta_k$, $\widetilde{\delta}_{i,j} = \frac{\hbar^{eff}}{\hbar}\delta_{i,j} - \frac{\theta_i\eta_j}{4\hbar^2}$, $\hbar^{eff} = \hbar\left(1 + \frac{\vec{\theta}\cdot\vec{\eta}}{4\hbar^2}\right)$ and θ_i and η_i are positive real parameters.

Condition for the existing of nonzero solutions

Using the Weyl Heisenberg algebra (20), the eigenvector-like equation (18) transform to the system of linear homogeneous algebraic equations of the variables a_1, a_2, a_3, b_1, b_2 and b_3 respectively for 1-d and 3-d anisotropic HO. Then, the necessary and sufficient condition for the existing of nonzero solutions are

$$\begin{vmatrix} i\lambda & -m\omega^2\hbar^{eff} \\ \frac{\hbar^{eff}}{m} & i\lambda \end{vmatrix} = 0 \text{ and } \begin{vmatrix} \frac{i\lambda}{m\omega_1^2} & -\theta & 0 & -\hbar^{eff} & 0 & 0 \\ \theta & \frac{i\lambda}{m\omega_2^2} & 0 & 0 & -\hbar^{eff} & 0 \\ 0 & 0 & \frac{i\lambda}{m\omega^{\,2}} & 0 & 0 & -\hbar \\ \hbar^{eff} & 0 & 0 & im\lambda & -\eta & 0 \\ 0 & \hbar^{eff} & 0 & \eta & im\lambda & 0 \\ 0 & 0 & \hbar & 0 & 0 & im\lambda \end{vmatrix} = 0 \quad (21)$$

Energy-level gaps

Then, the energy-level gaps for $\omega_1 = \omega_2 \neq \omega$ are given by

$$\lambda = \pm\hbar^{eff}\omega \quad \text{and} \quad \lambda = \begin{cases} \lambda_1^- = -\hbar\,\tilde{\omega} - \varepsilon, & \lambda_2^- = \hbar\,\tilde{\omega} - \varepsilon \\ \lambda_1^+ = -\hbar\,\tilde{\omega} + \varepsilon & \lambda_2^+ = \hbar\,\tilde{\omega} + \varepsilon \\ \lambda_3^- = -\hbar\omega\,, & \lambda_3^+ = \hbar\omega \end{cases} \tag{22}$$

where $\tilde{\omega} = \omega_1\sqrt{\frac{\theta^2\eta^2}{16\hbar^4} + \frac{m^2\omega_1^2\theta^2}{4\hbar^2} + \frac{\eta^2}{4m^2\hbar^2\omega_1^2} + 1}$ and $\varepsilon = \frac{1}{2}(\frac{\eta}{m} + m\,\omega_1^2\theta)$.

Proposition

Since the energy-level gaps do not depend on the quantum numbers, the energy spectra derived from the 3-d anisotropic HO may be unique and given by

$$E_{n_1,n_2,n_3} \quad = \quad \hbar\omega\,(\,n_1 + \frac{1}{2}) + \hbar\,\tilde{\omega}(n_2 + n_3 + 1) + \varepsilon(n_3 - n_2). \tag{23}$$

The minimal energy level of the system is given by $E_0 = \frac{\hbar}{2}(\omega + 2\tilde{\omega})$.

Wavefunction

In cylindrical coordinates $(r\,,\phi\,,z)$, following a similar method as in a commutative space case, the Schrödinger equation of the system translates to two equations witch the radial equation describes a system of 2-d particle moving in the perpendicular homogeneous field in the circular gauge.

$$\Psi(s,\phi,\tilde{z}) = \sqrt{\frac{c\,\delta^{\frac{1}{2}}\jmath!}{\pi^{\frac{3}{2}}2^{\,n}\,n!(\jmath+|\ell|)!}}\,s^{\frac{|\ell|}{2}}\,\mathrm{e}^{-\frac{s+\tilde{z}^2-2\,i\,|\ell|\phi}{2}}\,{}_1F_1(\,\jmath;\,|\ell|+1;\,s)\,\mathbf{H}_{\,n}(\tilde{z}). \quad (24)$$

Recurrence relation

Let construct ladder operators with the property $\hat{\mathcal{L}}_{\pm}|\jmath\rangle = c_{\pm}|\jmath\pm1\rangle$ for the solution of the radial euation as generalized Laguerre functions $|\jmath\rangle \equiv \sqrt{\frac{\ell!\,2c}{\jmath!(\jmath+\ell)!}}\,s^{\frac{\ell}{2}}\mathrm{e}^{-\frac{1}{2}\,s}\mathbf{L}_{\jmath}^{(\ell)}(\,s)$, and the recurrence relation is given by

$$c_{+}\Omega_{\jmath+1}(\,s) + c_{-}\Omega_{\jmath-1}(\,s) - (2\jmath+\ell+1-\,s)\Omega_{\jmath}(\,s) = 0. \quad (25)$$

Dynamic symmetry group SU(1, 1)

The ladder operators $\hat{\mathcal{L}}_+ = s\frac{d}{d\,s} - \frac{s}{2} + \frac{\ell}{2} + \hat{\jmath} + 1$ and $\hat{\mathcal{L}}_- = -\,s\frac{d}{d\,s} - \frac{s}{2} + \frac{\ell}{2} + \hat{\jmath}$ are obtained with its realizations in the basis of the functions $\Omega_\jmath(\,s)$.
We obtain $\Omega_\jmath(\,s) = \frac{1}{\sqrt{\jmath!(\ell+1)_\jmath}}\prod_{k=0}^{\jmath-1}\hat{\mathcal{L}}_+(\,s, \jmath - 1 - k)\Omega_0(\,s)$ and the operator $\hat{\mathcal{L}}_-$ annihilates the ground state $|0\rangle$. Thus $[\hat{\mathcal{L}}_-,\, \hat{\mathcal{L}}_+]|\jmath\rangle = (2\jmath + \ell + 1)|\jmath\rangle$.
Note that in the basis spanned by the generalized Laguerre functions $|\jmath\rangle$ the operators $\hat{\mathcal{L}}_\pm$ and $\hat{\mathcal{L}}_0 := \hat{\jmath} + \frac{\ell+1}{2}$ satisfy the commutation relations of the $\mathfrak{su}(1,\,1)$ algebra, which is isomorphic to an $\mathfrak{so}(2,\,1)$ algebra as follows:

$$[\hat{\mathcal{L}}_-,\, \hat{\mathcal{L}}_+] = 2\hat{\mathcal{L}}_0, \quad [\hat{\mathcal{L}}_0,\, \hat{\mathcal{L}}_\pm] = \pm\hat{\mathcal{L}}_\pm. \tag{26}$$

The UIRs of the $\mathfrak{so}(2,\,1)$ having unbounded $\hat{\mathcal{L}}_0$ eigenvalues spectra are from the general class $D(\mathcal{J}(\mathcal{J}+1), \mathcal{M}_0)$ given by $D^\pm(\mathcal{J})$.
The Casimir operator can be written as $\widehat{\mathcal{C}} = \hat{\mathcal{L}}_0(\hat{\mathcal{L}}_0 - 1) - \hat{\mathcal{L}}_+\hat{\mathcal{L}}_-$, with the property $\widehat{\mathcal{C}}|\jmath\rangle = \frac{1}{4}(\ell^2 - 1)|\jmath\rangle$.

Concluding remarks

(i) The energy spectra (8) are non-degenerate, this could have an important impact on our conception of the quantum structure of the nature of physical systems. In the absence of other perturbing potentials, the system is invariant under the 2-d translation group and the conserved charge can be identified with the center of the Landau orbit.

(ii) Energy spectra of an anisotropic HO have been derived in various situations in a NCPS by using the eigen-operator technic. In the limit that the two noncommutativity parameters vanish, the results (**??**) conform to one's expectations of the behaviour of a NC system.

(iii) Using the factorization method, the associated dynamic $\mathfrak{su}(1,1)$ Lie algebra generators have been constructed thanks to their representations in the radial wave functions basis solutions of the radial equation.

(v) In addition, the Landau problem resembles to a 2-d HO on NCS by Sayipjamal *et al* (2008) and in the situation that $\omega_1 = \omega_2 = \omega$, the results (23) reduce to 3-d isotropic HO, Hounkonnou et al (2010).

Thank you for your attention