Un voyage dans les univers de de Sitter autour des notions de système élémentaire aux sens classique (Kostant-Kirillov-Souriau) & quantique (Wigner, théorie quantique des champs)
Lecture 3:
Un aperçu sur la théorie quantique des champs en espace-temps de de Sitter

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Porto Novo, Bénin, 11-16 Juillet 2022

It is worth noting that most of the available treatments of quantum field theory (QFT) in curved spacetime either are oriented strongly toward mathematical issues (and deal, e.g., with C^*-algebras, KMS states, etc.) or are oriented toward a concrete physical problem (and deal, e.g., with particular mode function expansions of a quantum field in a certain spacetime).

R. M. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics*, Chicago Lectures in Physics (1994).

Besides the Λ evidence, the interest of setting up a QFT in de Sitter spacetime stems from

- dS is *maximally symmetric*
- It provides a (dS covariant) infrared cut-off for Minkowskian QFT's
- Its symmetry is a one-parameter (curvature) deformation of minkowskian symmetry
- It is so an excellent laboratory for both, mathematical or concrete, approaches to QFT in curved spacetime
- As soon as a constant curvature is present (like the currently observed one!), one loses some of our so familiar conservation laws like energy-momentum conservation.
- Then, what is the physical meaning of a scattering experiment ("space" in dS is like the sphere S^3, let alone the fact that time is ambiguous)?
- Which relevant "physical" quantities are going to be considered as (asymptotically? contractively?) experimentally available?

A baby step in dS QFT

A baby step[4] in dS QFT by considering a single free massive scalar field operator on a fixed background de Sitter spacetime.

- Consider a scalar field in d-dimensional dS with the action

$$S = -\frac{1}{2}\int \mathrm{d}^d x\sqrt{-g}\left[((\nabla\phi)^2 + m^2\phi^2\right],$$

where m is viewed here as masslike parameter.

- All information is encoded in the Wightman two-point function of operator ϕ:

$$\mathcal{W}(x,y) = \langle\Omega|\phi(x)\phi(y)|\Omega\rangle\,, \quad |\Omega\rangle \equiv \text{ vacuum}$$

which obeys the free field equation

$$(\Box - m^2)\mathcal{W}(x,y) = 0.$$

$\Box = \nabla\cdot\nabla$ is the Laplacian on dS, acting, say, with respect to x.

[4] Les Houches Lectures on de Sitter Space by Marcus Spradlin, Andrew Strominger and Anastasia Volovich, hep-th/0110007

dS Quantum Field Theory: a preamble (continued)

- There are other two-point functions : retarded, advanced, Feynman, Hadamard and so on, but these can all be obtained from the Wightman function, for example by taking the real or imaginary part, and/or by multiplying by a step function in time.

- One assumes that the "vacuum" state $|\Omega\rangle$ is invariant under the $SO_0(1,d)$ de Sitter group. Then $\mathcal{W}(x,y)$ will be de Sitter invariant, and so at generic points can only depend on the de Sitter invariant length $\mathcal{S}(x,y)$ between x and y.

- Writing $\mathcal{W}(x,y) = G(\mathcal{S}(x,y))$, dS Klein-Gordon reduces to a hypergeometric differential equation in $\mathcal{Z} = \dfrac{1+\mathcal{S}}{2}$, and so

$$G(\mathcal{S}) = c_m\, {}_2F_1\left(h_+, h_-; \frac{d}{2}; \mathcal{Z}\right),\ h_\pm = \frac{1}{2}\left[(d-1) \pm \sqrt{(d-1)^2 - 4m^2}\right]$$

- There is singularity at $\mathcal{Z} = 1$ (i.e. when the points x and y are separated by a null geodesic) and branch cut for $1 < \mathcal{Z} < \infty$.

- At short distances the scalar field is insensitive to the fact that it is in de Sitter space and the form of the singularity is precisely the same as that of the propagator in flat d-dimensional Minkowski space. This fixes the normalization constant to be

$$c_m = \frac{\Gamma(h_+)\Gamma(h_-)}{(4\pi)^{d/2}\Gamma(\frac{d}{2})}$$

- Clearly, if $G(\mathcal{S})$ is a solution then $G(-\mathcal{S})$ is also a solution. A second linearly independent solution is therefore ${}_2F_1(h_+, h_-; \frac{d}{2}; \dfrac{1-\mathcal{S}}{2})$, with singularity at $\mathcal{S} = -1$, which corresponds to x being null separated from the antipodal point to y .

- This singularity sounds rather unphysical at first, but we should recall that antipodal points in de Sitter space are always separated by a horizon: the second Green function can be thought of as arising from an image source behind the horizon, and is nonsingular everywhere within an observer's horizon. Hence the "unphysical" singularity can not be detected by any experiment.

dS Quantum Field Theory: a preamble (continued)

- De Sitter space therefore has a one parameter family of de Sitter invariant Green functions G_α corresponding to a linear combination of both solutions described above.

- Corresponding to this one-parameter family of Green functions is a one-parameter family of de Sitter invariant vacuum states $|\alpha\rangle$ such that

$$\mathcal{W}_\alpha(x,y) = \langle\alpha|\phi(x)\phi(y)|\alpha\rangle$$

- Let us write an expansion ("mode expansion") for the scalar field, viewed as an operator in a Fock space in terms of creation and annihilation operators :

$$\phi(x) = \sum_k a_k u_k(x) + a_k^\dagger u_k^*(x)$$

where a_k and $a^\dagger$ satisfy "canonical" commutation rules (CCR)

$$[a_k, a_l^\dagger] = \delta_{kl}$$

dS Quantum Field Theory: a preamble (continued)

- The modes $u_k(x)$ satisfy the wave equation

$$(\Box_{\mathrm{dS}} - m^2)u_k(x) = 0.$$

- They are normalized with respect to the invariant Klein-Gordon inner product

$$\langle u_k | u_l \rangle_{\mathrm{KG}} = -i \int \mathrm{d}\Sigma^\mu \, (u_k \overleftrightarrow{\partial}_\mu u_l^*) = \delta_{kl}$$

  where the integral is taken over a complete spherical spacelike slice in dS and the result is independent of the choice of this slice.

- The vacuum state is defined by saying that it is annihilated by all annihilation operators :

$$a_k|\Omega\rangle = 0, \ \forall\, k.$$

- The question is, **which modes** do we associate with the above creation operators and which do we associate with annihilation operators?

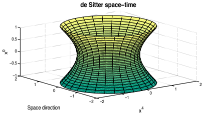
dS Quantum Field Theory: a preamble (continued)

- In Minkowski space we take positive and negative frequency modes,
- In a general curved spacetime **there is no canonical choice of a time variable** with respect to which one can classify modes as being positive or negative frequency. If we make a choice of time coordinate, we can get a vacuum state $|\Omega\rangle$ and then the state $a_n^\dagger|\Omega\rangle$ is said to have n particles in it. But if we had made some other choice of time coordinate then we would have a different vacuum $|\Omega'\rangle$ which we could express as a linear combination of the $|n\rangle$'s.
- Hence the question "How many particles are present?" is not well-defined independently of a choice of coordinates.
- This is an important and general feature of quantum field theory in curved spacetime.
- However, sound arguments (based on analyticity) favor a certain vacuum

DS_4 manifold and its causal structure

- Remind that dS space-time is visualized as a one-sheeted hyperboloid embedded in a $1+4$-dimensional Minkowski spacetime $\mathbb{R}^5$ and is topologically $\mathbb{R}^1 \times \mathbb{S}^3$

$$\underline{M}_R \equiv \left\{ x = (x^0,\ \dots, x^4) \in \mathbb{R}^5\ ;\ (x)^2 \equiv x\cdot x = \eta_{AB} x^A x^B = -R^2 \right\},$$



- Global causal ordering of $\underline{M}_R$ is induced by that of $\mathbb{R}^5$.

$$\underline{V}^+ \equiv \left\{ x \in \mathbb{R}^5\ ;\ (x)^2 = x\cdot x \geqslant 0,\ x^0 > 0 \right\}.$$

For two "events" $x, x' \in \underline{M}_R$, x' is future connected to x if $x' - x \in \underline{V}^+$, i.e., $(x'-x)^2 \geqslant 0$

- The (pseudo-)distance $d(x,x')$ on $\underline{M}_R$ is defined as:

$$\cosh\left(\frac{d(x,x')}{R}\right) = -\frac{x\cdot x'}{R^2}, \quad \text{for } x \text{ and } x' \text{ timelike separated},$$

$$\cos\left(\frac{d(x,x')}{R}\right) = -\frac{x\cdot x'}{R^2}, \quad \text{for } x \text{ and } x' \text{ spacelike separated such that } |x\cdot x'| < R^2 .$$

Quantum field theory in de Sitter space: the "massive" case I

For free fields whose the one-particle sector is determined by a given de Sitter UIR in the principal and the complementary series, the two-point Wightman function is required to satisfy:

(i) Covariance with respect to the given UIR.

(ii) Locality/(anti-)commutativity, wich respect to the causal de Sitter structure.

(iii) Positive definiteness (Hilbertian Fock structure).

(iv) Normal (maximal?) analyticity.

Then the field itself can be reobtained from the Wightman function via a Gel'fand-Naïmark-Segal (G.N.S.) type construction.

Condition (iv) will play the role of a spectral condition in the absence of a global energy-momentum interpretation in de Sitter. This condition implies a "thermal-KMS" interpretation.

Quantum field theory in de Sitter space: the “massive” case II

- For “generalized” free fields, the theory is still encoded entirely by a two-point function: all truncated n-point functions, $n > 2$, vanish, as does the “1-point” function. The axiomatic imposes the 2-point functions to obey the same conditions (i)-(iv), apart from the fact that a certain not necessarily irreducible unitary representation is now involved.
- However, the Plancherel content of this involved UR should be restricted to the principal series, and this decomposition allows a Källen-Lehman type representation of the 2-point function.
- Finally, for interacting fields in dS, the set of n-point functions is assumed to satisfy
  - (i) Covariance with respect to a certain UR.
  - (ii) Locality/(anti-)commutativity.
  - (iii) Positive definiteness.
  - (iv) “Weak” spectral condition in connection with some analyticity requirements.

J. Bros and U. Moschella, *Two-point functions and Quantum fields in de Sitter Universe*, Rev. Math. Phys., Vol.8, No.3 (1996)327-391.

De Sitter plane waves I

The whole quantum field construction rests upon dS plane wave solutions which are eigendistributions for eigenvalues of the Casimir operators (principal and complementary series).

- Principal series : $\Upsilon_{p=s,\sigma=\nu^2+\frac{1}{4}}$, $\nu \equiv \zeta_{\mathrm{dS}}$:

$$[Q_s^{(1)} - (\nu^2 + \frac{9}{4} - s(s+1))]\, \psi(x) = 0,$$

  where $\nu \geq 0$ for $s = 0, 1, 2, \cdots$, and $\nu > 0$ for $s = \frac{1}{2}, \frac{3}{2}, \cdots$.

- Complementary series : $\Upsilon_{p=s,\sigma}$, $\sigma \equiv \zeta_{\mathrm{dS}}^2 + \frac{1}{4}$:

$$[Q_s^{(1)} - (\sigma + 2 - s(s+1))]\, \psi(x) = 0,$$

  where $-2 < \sigma < \frac{1}{4}$ for $s = 0$, and $0 < \sigma < \frac{1}{4}$ for $s = 1, 2, \cdots$.

Concrete calculus in terms of coordinates

- In ambient space notations, a tensor field $\mathcal{K}_{\alpha_1\alpha_2\cdots}(x)$ can be viewed as a homogeneous function in the $\mathbb{R}^5$-variables x^α with some arbitrarily chosen degree σ which therefore satisfies:
$$x^\alpha \frac{\partial}{\partial x^\alpha}\mathcal{K}(x) = x\cdot\partial\mathcal{K}(x) = \sigma\mathcal{K}(x).$$
- The choice for σ will be dictated by simplicity reasons when one has to deal with field equations. For instance, set $\sigma = 0$ so that the d'Alembertian operator $\Box_{\mathrm{dS}} \equiv \nabla_\mu\nabla^\mu$ on dS space (∇_μ being the covariant derivative) coincides with the d'Alembertian operator $\Box_5 \equiv \partial^2$ on $\mathbb{R}^5$.
- Of course, not every homogeneous vector field of $\mathbb{R}^5$ represents a physical dS entity! In order to ensure that, e.g. vector field, $\mathcal{K}_\alpha(x)$ lies in the de Sitter tangent space-time it also must satisfy the transversality condition
$$x\cdot\mathcal{K}(x) = 0.$$
- The symmetric, transverse projector $\theta_{\alpha\beta} = \eta_{\alpha\beta} + H^2 x_\alpha x_\beta$ (remind $x_\alpha x^\alpha = -H^{-2}$) which satisfies $\theta_{\alpha\beta}\, x^\alpha = \theta_{\alpha\beta}\, x^\beta = 0$ is the transverse form of the dS metric in ambient space notation and it is used in the construction of transverse entities like the transverse derivative $\bar\partial_\alpha = \theta_{\alpha\beta}\partial^\beta = \partial_\alpha + H^2 x_\alpha x.\partial$.

Concrete calculus in terms of coordinates (continued)

Since in most of the works devoted to dS field theory the tensor fields are written using local coordinates, it is very important to provide the link between intrinsic and ambient approaches.

- "Intrinsic" For instance, vector field $A_\mu(X)$ is locally determined by the field $\mathcal{K}_\alpha(x)$ through

$$A_\mu(X) = \frac{\partial x^\alpha}{\partial X^\mu}\mathcal{K}_\alpha(x(X)) \qquad \mathcal{K}_\alpha(x) = \frac{\partial X^\mu}{\partial x^\alpha}A_\mu(X(x)).$$

- The transverse projector θ is the only symmetric and transverse tensor which is linked to the dS metric $g_{\mu\nu}$: $\dfrac{\partial x^\alpha}{\partial X^\mu}\dfrac{\partial x^\beta}{\partial X^\nu}\,\theta_{\alpha\beta} = g_{\mu\nu}$.

- Covariant derivatives ∇ are related to the transverse derivative denoted by $\bar\partial$.

- Acting on a l-rank tensor they are transformed according to

$$\nabla_\mu\nabla_\nu..\nabla_\rho h_{\lambda_1..\lambda_l} = \frac{\partial x^\alpha}{\partial X^\mu}\frac{\partial x^\beta}{\partial X^\nu}..\frac{\partial x^\gamma}{\partial X^\rho}\frac{\partial x^{\eta_1}}{\partial X^{\lambda_1}}..\frac{\partial x^{\eta_l}}{\partial X^{\lambda_l}}\text{Trpr}\bar\partial_\alpha\text{Trpr}\bar\partial_\beta..\text{Trpr}\bar\partial_\gamma\mathcal{K}_{\eta_1..\eta_l}\,,$$

- Here the transverse projection defined by

$$(\text{Trpr}\,\mathcal{K})_{\lambda_1..\lambda_l} \equiv \theta^{\eta_1}_{\lambda_1}..\theta^{\eta_l}_{\lambda_l}\mathcal{K}_{\eta_1..\eta_l}\,,$$

guarantees the transversality in each index.

Concrete calculus in terms of coordinates (continued)

Relations between intrinsic and ambient formulations of wave equations :

- For scalar fields, the d'Alembertian reads as

$$\Box_{\mathrm{dS}}\phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = g^{\mu\nu}\frac{\partial x^\alpha}{\partial X^\mu}\frac{\partial x^\beta}{\partial X^\nu}\left(\bar\partial_\alpha\bar\partial_\beta\phi - H^2 x_\beta\bar\partial_\alpha\phi\right)$$
$$= \theta^{\alpha\beta}\left(\bar\partial_\alpha\bar\partial_\beta\phi - H^2 x_\beta\bar\partial_\alpha\phi\right) = \bar\partial^2\phi\,.$$

- and so its expression in terms of the first dS Casimir for scalar fields :

$$Q_0^{(1)} = -\frac{1}{2}M_{\alpha\beta}M^{\alpha\beta} = -H^{-2}(\bar\partial)^2 = -H^{-2}\Box_{\mathrm{dS}},$$

where $M_{\alpha\beta} = -i(x_\alpha\partial_\beta - x_\beta\partial_\alpha)$

- For vector fields,

$$\Box_{\mathrm{dS}}A_\mu = \nabla^\lambda\nabla_\lambda A_\mu = \frac{\partial x^\alpha}{\partial X^\mu}\left[\bar\partial^2\mathcal{K}_\alpha - H^2\mathcal{K}_\alpha - 2H^2 x_\alpha\bar\partial\cdot\mathcal{K}\right]\ .$$

- and so its expression in terms of the first dS Casimir for vector fields

$$Q_1^{(1)}\mathcal{K}(x) = -\frac{1}{2}L_{\alpha\beta}L^{\alpha\beta}\mathcal{K}(x)$$
$$= \left(Q_0^{(1)} - 2\right)\mathcal{K}(x) + 2x\,\bar\partial\cdot\mathcal{K}(x) - 2\partial\,x\cdot\mathcal{K}(x),$$

where $L_{\alpha\beta} = M_{\alpha\beta} + S_{\alpha\beta}$.

De Sitter plane waves I

The solution of the dS Klein -Gordon equation is given by dS “plane waves”

$$x \mapsto (Hx \cdot \xi)^{\mu = -\frac{3}{2} - i\nu}, \; \nu \approx mH^{-1}$$

is, as a function of ξ , homogeneous with degree μ on the null upper cone $C^{+} = \{\xi \in \mathbb{R}^5 : \; \xi \cdot \xi = 0, \; \text{sgn}(\xi^0) = +\}$

De Sitter plane waves II

As boundary values (bv) of analytic functions, de Sitter plane waves have the general form

$$\psi(x) = \mathrm{bv}_{z=x} \left[\mathcal{D}(\xi, z)(z \cdot \xi)^{\mu} \right] ,$$

where

- → $\mathcal{D}(\xi, z)$ is a vector-valued differential operator such that $\psi(x)$ is a relevant tensor-spinor solution of the wave equation.
- → The vector $\xi = (\xi^0, \vec{\xi}, \xi^4) \in C^{\pm} = \{\xi \in \mathbb{R}^5 : \ \xi \cdot \xi = 0, \ \mathrm{sgn}(\xi^0) = \pm\}$, "future" null cone in ambient space $\mathbb{R}^5$ ($\sim$ a four-momentum).
- → The complex five-vector $z \in$ tubular domains $\mathcal{T}^{\pm}$: $\mathcal{T}^{\pm} = (\mathbb{R}^5 \pm iV^+) \cap M_H^{(c)}$, where

  $$M_H^{(c)} = \{z = x + iy \in \mathbb{R}^5 + i\mathbb{R}^5 \ | \ \eta_{\alpha\beta} z^{\alpha} z^{\beta} = -H^{-2}\}$$

  is the complexification of the dS hyperboloid M_H and $\mathbb{R}^5 \pm iV^+$,
  $V^+ \equiv \{x \in \mathbb{R}^5; \ \ x^0 > \sqrt{\parallel \vec{x} \parallel^2 + (x^4)^2}\}$, are the forward and backward tubes in $\mathbb{C}^5$.
- → The complex power μ is such that ψ is solution to the wave equation.

The occurrence of complex variables in these expressions is not fortuitous. It is actually at the heart of the analyticity requirements (iv), as will appear through the following explicit examples.

(Complex) DS plane waves for the scalar case $s = 0$

$$\psi(z) = (Hz\cdot\xi)^{\mu}\,, \quad H \equiv \sqrt{\frac{\Lambda}{3}},$$

where $\mu = -\frac{3}{2} + i\nu,\ \nu\in\mathbb{R}$ for the principal series $\Upsilon_{0,\sigma=\nu^2+\frac{1}{4}}$ and $-3 < \mu = -\frac{3}{2} \pm \sqrt{1-2\sigma} < 0$ for the complementary series $\Upsilon_{0,\sigma}$.

The term *plane wave* in the case of the principal series is consistent with the null curvature limit :

At the $R\to\infty$ limit, $x\to(\tau, \rho\,\vec{n}, \infty)$, that we consider as the point $X := (X^0 = \tau, \vec{X} = \rho\vec{n}) \in \mathbb{R}^{1,3}$. With $m = H\nu$,

$$\lim_{H\to 0}(Hx(X)\cdot\xi)^{-\frac{3}{2}+imH^{-1}} = \exp ik\cdot X,$$

where, in Minkowskian-like coordinates,

$$\xi = \left(\frac{k^0}{m}, \frac{\vec{k}}{m}, -1\right) \in C^{+}$$

$$x(X) = (x^0 = H^{-1}\sinh HX^0, \vec{x} = H^{-1}\frac{\vec{X}}{\|\vec{X}\|}\cosh HX^0 \sin H\|\vec{X}\|, x^4 = H^{-1}\cosh HX^0\cos H\|\vec{X}\|)$$

.

Precisions on orbital basis for the cone

Let us make more precise the notion of orbital basis γ for the future null cone

$$\mathcal{C}^+ = \{\xi \in \mathcal{C};\ \xi^0 > 0\}$$

Let us choose a unit vector e in $\mathbb{R}^5$ and let H_e be its stabilizer subgroup in $\mathrm{SO}_0(1,4)$. Then two types of orbits are interesting in the present context :

(i) the spherical type γ_0 corresponds to $e \in V^+ \equiv \{x \in \mathbb{R}^5;\ \ x^0 > \sqrt{\| \vec{x} \|^2 + (x^4)^2}\}$, and is an orbit of $H_e \simeq \mathrm{SO}(4)$.

$$\gamma_0 = \{\xi\ ;\ e \cdot \xi = a > 0\} \cap \mathcal{C}^+\ .$$

(ii) the hyperbolic type γ_4 corresponds to $e^2 = -1$. It is divided into two hyperboloid sheets, both being orbits of $H_e \simeq \mathrm{SO}_0(1,3)$.

The most suitable parametrization when one has in view the link with massive Poincaré UIR's is to work with the orbital basis of the second type

$$\gamma_4 = \{\xi\ \in C^+, \xi^{(4)} = 1\} \cup \{\xi\ \in C^+, \xi^{(4)} = -1\},$$

with the null vector ξ given in terms of the four-momentum $(k^0, \vec{k})$ of a Minkowskian particle of mass m

$$\xi_\pm = \left(\frac{k^0}{mc} = \sqrt{\frac{\vec{k}^2}{m^2c^2} + 1}, \frac{\vec{k}}{mc}, \pm 1\right)\ .$$

(Analytic) Wightman two-point function for the scalar case $s = 0$

The two-point function is analytic in the tuboid $\mathcal{T}^- \times \mathcal{T}^+$ and reads (for the principal series)

$$W_\nu(z_1, z_2) = c_\nu \int_\gamma (z_1 \cdot \xi)^{-\frac{3}{2}+i\nu} (\xi \cdot z_2)^{-\frac{3}{2}-i\nu} \, \mathrm{d}\mu_\gamma(\xi)$$
$$= \frac{H^2 \Gamma(\frac{3}{2}+i\nu)\Gamma(\frac{3}{2}-i\nu)}{2^4\pi^2} P^5_{-\frac{3}{2}+i\nu}(H^2 z_1 \cdot z_2).$$

- The integration is performed on an "orbital basis" $\gamma \subset C^+$. The symbol P_α^λ stems for a generalized Legendre function, and the coefficient factor is fixed by Hadamard condition.
- The corresponding Wightman function $\mathcal{W}_\nu(x_1, x_2) = \langle \Omega, \phi(x_1)\phi(x_2)\Omega \rangle$, where Ω is the Fock vacuum and ϕ is the field operator seen as an operator-valued distribution on M_H, is the boundary value $\mathrm{bv}_{\mathcal{T}^\mp \ni z_{\substack{1\\2}} \to x_{\substack{1\\2}}} W_\nu(z_1, z_2)$.
- Its integral representation is given by:

$$\mathcal{W}_\nu(x_1, x_2) = c_\nu \int_\gamma ((x_1 \cdot \xi)_+^{-\frac{3}{2}+i\nu} + e^{-i\pi(-\frac{3}{2}+i\nu)} (x_1 \cdot \xi)_-^{-\frac{3}{2}+i\nu})((\xi \cdot x_2)_+^{-\frac{3}{2}-i\nu}$$
$$+ e^{-i\pi(-\frac{3}{2}-i\nu)} (\xi \cdot x_2)_-^{-\frac{3}{2}-i\nu}) \, \mathrm{d}\mu_\gamma(\xi).$$

About the Hadamard condition

The Hadamard condition imposes that short-distance behavior of the two-point function of the field should be the same for Klein-Gordon fields on curved space-time as for corresponding Minkowskian free field.

(i) In case of dS (and many other curved space-times) it selects a unique vacuum state

(ii) In case of dS, this selected vacuum coincides with the *euclidean* or *Bunch-Davies* vacuum state structure.

More about the contraction dS UIR $\rightarrow$ Poincaré UIR

- It has been shown[24] that the representations of the de Sitter group associated with the massive scalar field, i.e. the principal series of $SO_0(1,4)$, contract (in the zero curvature limit) toward the direct sum of two UIR's of the group Poincaré associated with positive and negative frequencies massive scalar fields respectively, namely:

$$U_\nu \underset{H\to 0,\nu\to\infty}{\longrightarrow} \underbrace{\mathcal{P}^{>}(m)}_{\text{positive energies}} \oplus \underbrace{\mathcal{P}^{<}(m)}_{\text{negative energies}} ,$$

- This result could appear as somewhat confusing since it suggests that the curvature is in some sense responsible for the emergence of negative frequency modes in QFT.

- Actually, we will see that the negative energy plane waves do not appear when the curvature vanishes as soon as the Euclidean vacuum has been chosen. The latter choice is ruled by our physical reality.

[24] J. Mickelsson and J. Niederle, Commun. Math. Phys. **27**, 167 (1972)

More about dS wave planes and the contraction dS UIR → Poincaré UIR (continued)

- Remind that the plane-wave solution of the dS Klein -Gordon equation defined by

$$x \mapsto (Hx \cdot \xi)^{\mu=-\frac{3}{2}-\mathrm{i}\nu}, \ \nu \approx mH^{-1}$$

is, as a function of ξ , homogeneous with degree μ on the null upper cone $C^+ = \{\xi \in \mathbb{R}^5 : \ \xi \cdot \xi = 0, \ \mathrm{sgn}(\xi^0) = +\}$ and thus is entirely determined by specifying its values on a well chosen three-dimensional submanifold (an orbital basis) γ of C^+.

- These dS waves, as functions on de Sitter spacetime, are only locally defined because they are singular on specific lower dimensional subsets of M_H and multivalued since $Hx \cdot \xi$ can be negative.

- In order to get a single-valued global definition, they have to be viewed as distributions which are boundary values of analytic continuations to suitable domains in the complexified de Sitter space $M_H^{(c)}$:

$$M_H^{(c)} = \{z = x + \mathrm{i}y \in \mathbb{R}^5 + \mathrm{i}\mathbb{R}^5 \mid \eta_{\alpha\beta} z^\alpha z^\beta = -H^{-2}\}$$

More about the contraction dS UIR → Poincaré UIR (continued)

- The minimal domains of analyticity which yield single-valued functions on de Sitter spacetime are the forward and backward tubes of $M_H^{(c)}$: $\mathcal{T}^{\pm} = (\mathbb{R}^5 \pm \mathrm{i}V^+) \cap M_H^{(c)}$
- When z varies in $\mathcal{T}^+$ and ξ lies in the positive cone C^+, the dS KG solutions $z \mapsto (Hz \cdot \xi)^{\mu}$ are globally well defined since the imaginary part of $(z \cdot \xi)$ is nonpositive.
- The de Sitter waves $\phi_{\mu,\xi}(x)$ are then defined as boundary value of the analytic continuation to the future tube $\mathcal{T}^+$ of $x \mapsto (Hx \cdot \xi)^{\mu}$:

$$\begin{aligned}\phi_{\mu,\xi}(x) &:= c_{\nu}\, \mathrm{bv}\,(Hz \cdot \xi)^{\mu} \\ &= c_{\nu}\left[\Theta(Hx \cdot \xi) + \Theta(-Hx \cdot \xi)e^{-\mathrm{i}\pi\mu}\right] |Hx \cdot \xi|^{\mu},\end{aligned}$$

  where Θ is the Heaviside function.
- The real valued constant c_{ν} is determined by imposing the Hadamard condition on the two-point function. This choice of modes corresponds to the Euclidean vacuum.

More about the contraction dS UIR $\to$ Poincaré UIR (continued)

- In terms of de Sitter waves, the analytic two-point function reads

$$W_\nu(z_1, z_2) = c_\nu \int_\gamma (z_1 \cdot \xi)^{-\frac{3}{2}+\mathrm{i}\nu} (\xi \cdot z_2)^{-\frac{3}{2}-\mathrm{i}\nu} \, \mathrm{d}\mu_\gamma(\xi) \;\; z_1 \in \mathcal{T}^-, \;\; z_2 \in \mathcal{T}^+.$$

- The measure $\mathrm{d}\mu_\gamma(\xi)$ on the orbital basis γ is chosen to be m^2 times the natural one induced from the R^5 Lebesgue measure.

- Calculation yields :

$$c_\nu = \sqrt{\frac{H^2(\nu^2 + 1/4)}{2(2\pi)^3(1 + e^{-2\pi\nu})m^2}}$$

More about the contraction dS UIR → Poincaré UIR (continued)

For the flat limit of dS waves, consider a region around any point x_A in which all the distances are small compared to H^{-1}. Then the considered modes do not generate negative frequency modes in the flat limit :

$$\lim_{H\to 0} \phi_{\mu,\xi}(x) = \frac{1}{2(2\pi)^3} e^{-\mathrm{i}kx} \quad \text{for } x_A \cdot \xi > 0,$$

$$\lim_{H\to 0} \phi_{\mu,\xi}(x) = 0 \quad \text{for } x_A \cdot \xi < 0.$$

More about the contraction dS UIR $\to$ Poincaré UIR (continued)

To prove it,

1. choose coordinates such that $x_A^4 = H^{-1}$ and $x_A^\mu = 0$,
2. in the neighborhood of this point, dS space time meets its tangent plane $\simeq 4d$ Minkowski spacetime, and coordinates x^α in this neighborhood read:

$$x^\mu = X^\mu + o(H),\; x^4 = H^{-1} + o(1),$$

3. for $\mu \approx -\frac{3}{2} - \mathrm{i}mH^{-1}$, $\exp(-\mathrm{i}\pi\mu) \to 0$ and so :

$$\phi_{\mu,\xi}(x) \approx \frac{|\xi_4|^\mu}{\sqrt{2(2\pi)^3}} \left(1 + \frac{H\xi_\mu X^\mu}{|\xi_4|}\right)^{-3/2-\mathrm{i}mH^{-1}} \Theta(-\xi^4),$$

4. this limit exists only for $|\xi_4| = 1$. As a consequence, we use the orbital basis $\gamma = \mathcal{C}_1 \cup \mathcal{C}_2$ with

$$\xi = \left(\frac{\omega_k}{m}, \frac{\vec{k}}{m}, -1\right) \in \mathcal{C}_1,\; \xi = \left(\frac{\omega_k}{m}, \frac{\vec{k}}{m}, 1\right) \in \mathcal{C}_2,$$

and $\omega_k = \sqrt{\vec{k}^2 + m^2}$.

More about the contraction dS UIR → Poincaré UIR (continued)

- Thus, due to the analyticity condition at the origin of the $\exp(-2i\pi\mu)$ term, the negative energy modes are exponentially suppressed whereas the positive energy modes give the Minkowskian on-shell modes corresponding to a particle of mass m.
- Thus, any vacuum different from the Euclidean vacuum would lead to physically unacceptable Minkowskian QFT. The Euclidean vacuum has therefore to be preferred with respect to the flat limit criterion.
- Now, one can realize the dS one-particle sector $\mathcal{H}_H$ as distributions on spacetime through the following de Sitter Fourier-like inverse transform:

$$\mathcal{H}_H \ni \psi(x) = \int_{\xi\in\gamma} \widetilde{\psi}(\xi)\phi_{\mu,\xi}(x)\mathrm{d}\mu_\gamma(\xi),\ \widetilde{\psi} \in L^2(\gamma, \mathrm{d}\mu_\gamma(\xi)).$$

- Then, using the decomposition $\gamma = \mathcal{C}_1 \cup \mathcal{C}_2$ of the orbital basis $\mathcal{H}_H$ can be decomposed into $\mathcal{H}_H = \mathcal{H}_H^1 \oplus \mathcal{H}_H^2$, and, at the limit of null curvature, the second part vanishes and only the positive frequency remains.
- As a consequence, the ordinary Fourier transform is the flat limit of the de Sitter Fourier transform.

Properties of the Wightman two-point function for $s = 0$

It satisfies all QFT requirements:

(i) Covariance: $\mathcal{W}_\nu(\Lambda^{-1}x_1, \Lambda^{-1}x_2) = \mathcal{W}_\nu(x_1, x_2)$, for all $\Lambda \in SO_0(1,4)$.

(ii) Local commutativity:
$$\mathcal{W}_\nu(x_1, x_2) = \mathcal{W}_\nu(x_2, x_1)$$
for every space-like separated pair (x_1, x_2).

(iii) Positive definiteness:
$$0 \leq \int_{M_H \times M_H} \bar{f}(x_1)\mathcal{W}_\nu(x_1, x_2) f(x_2)\, d\sigma(x_1)\, d\sigma(x_2)$$
for any test function f, and where $d\sigma(x)$ is the $O(1,4)$ invariant measure on M_H.

(iv) Maximal analyticity: $W_\nu(z_1, z_2)$ can be analytically continued in the cut-domain $\Delta = (M_H^{(c)} \times M_H^{(c)}) \setminus \Sigma^{(c)}$ where the cut is defined by $\Sigma^{(c)} = \{(z_1, z_2) \in M_H^{(c)} \times M_H^{(c)};\ (z_1 - z_2)^2 = \rho,\ \rho \geq 0\}$.

J. Bros, J.-P. Gazeau and U. Moschella, *Quantum Field Theory in the de Sitter Universe*, Phys. Rev. Lett., Vol.73 (1994)1746-1749.

Construction of the field operators from the Wightman function

Given the two-point function, one realizes the Hilbert space as functions on M_H as follows.

- For any test function $f \in \mathcal{D}(M_H)$ (C^∞ + compact support), we define the (vector-valued in general) distribution taking values in the space generated by the modes $\mathcal{K}(x,\xi) \equiv$ bv $\mathcal{K}(z,\xi)$ by :

$$x \to p(f)(x) = \int_{M_H} \mathcal{W}(x,x')f(x')\mathrm{d}\sigma(x') = \int_\gamma \mathrm{d}\sigma_\gamma(\xi)\mathcal{K}_\xi(f)\,\mathcal{K}(x,\xi)\,,$$

- Here $\mathcal{K}_\xi(f)$ is the smeared form of the modes:

$$\mathcal{K}_\xi(f) = \int_{M_H} \mathcal{K}^*(x,\xi)f(x)\,\mathrm{d}\sigma(x)\,.$$

- The space generated by the $p(f)$'s is equipped with the positive invariant inner product

$$\langle p(f), p(g)\rangle = \int_{M_H\times M_H} f^*(x)\mathcal{W}(x,x')g(x')\mathrm{d}\sigma(x')\,\mathrm{d}\sigma(x)\,.$$

- The field is defined by the operator valued distribution

$$\mathcal{K}(f) = a\left(p(f)\right) + a^\dagger\left(p(f)\right)$$

Construction of the field operators from the Wightman function (continued)

- One gets:

$$\mathcal{K}(f) = \int_\gamma \mathrm{d}\sigma_\gamma(\xi)\left[\mathcal{K}^*_\xi(f)\, a(\xi) + \mathcal{K}_\xi(f)\, a^\dagger(\xi)\right] ,$$

where the operators $a(\mathcal{K}(\xi)) \equiv a(\xi)$ and $a^\dagger(\mathcal{K}(\xi)) \equiv a^\dagger(\mathcal{K}(\xi))$ are respectively antilinear and linear in their arguments.

- The unsmeared operator reads

$$\mathcal{K}(x) = \int_\gamma \mathrm{d}\sigma_\gamma(\xi)\left[\, \mathcal{K}(x,\xi)\, a(\xi) + \mathcal{K}^*(x,\xi)\, a^\dagger(\xi)\right] ,$$

- The operator $a(\xi)$ satisfies the canonical commutation relations (ccr) and is defined by

$$a(\xi)|\Omega >= 0.$$

- The integral representation of $\mathcal{K}(x)$ is independent of the orbital basis γ

- For the hyperbolic type submanifold γ_4 the measure is $\mathrm{d}\sigma_{\gamma_4}(\xi) = \mathrm{d}^3\vec{\xi}/\xi_0$ and the ccr are represented by

$$[a(\xi), a^\dagger(\xi')] = \xi^0\delta^3(\vec{\xi} - \vec{\xi}').$$

- The field commutation relations are

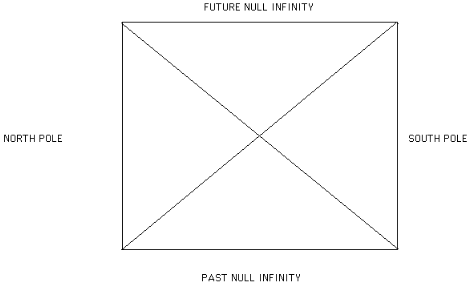
$$\left[\mathcal{K}(x), \mathcal{K}(x')\right] = 2\mathrm{iIm}\langle p(x), p(x')\rangle = 2\mathrm{iIm}\mathcal{W}(x,x') \,.$$

Maximal analyticity and KMS condition

- In the maximal analytic framework here described, the Wightman 2-point function determine free fields in a preferred representation.

- The latter is characterized by a well-defined Kubo-Martin-Schwinger condition formulated in appropriate regions of de Sitter universe associated with geodesic observers

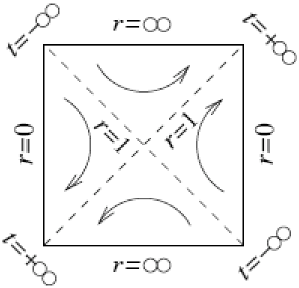
Penrose diagram for de Sitter



PENROSE DIAGRAM FOR DS

The north and south poles are timelike lines, while every point in the interior represents an S^{d-2}. A horizontal slice is an S^{d-1}. The diagonals are the past and future horizons of an observer at the south pole. (The conformal time coordinate runs from $-\frac{\pi}{2}$ at the past null infinity $\mathbf{T}_{-\infty}$ to $\frac{\pi}{2}$ at the future null infinity $\mathbf{T}_{+\infty}$). Note that a classical observer sitting on the south pole will never be able to observe anything past the diagonal line stretching from the north pole at $\mathbf{T}_{-\infty}$ to the south pole at $\mathbf{T}_{+\infty}$. This is qualitatively different from Minkowski space, for example, where a timelike observer will eventually have the entire history of the universe in his/her past light cone.

Penrose diagram for de Sitter



This Penrose diagram shows the direction of the flow generated by the Killing vector $\partial/\partial t$ in static coordinates. The latter are not global and read as:

$$x^0 = \sqrt{1-r^2}\,\sinh t,\ \vec{x} = r\hat{n},\ x^d = \sqrt{1-r^2}\,\cosh t$$

The horizons (dotted lines) are at $r^2 = 1$, and the southern causal diamond is the region with $0 \leq r \leq 1$ on the right hand side. Past and future null infinity $\mathbf{T}_{\pm\infty}$ are at $r = \infty$. The existence of a timelike Killing vector allows to define the Hamiltonian. But we see that at $r = 1$ that it becomes null. When extended to the various diamonds of the Penrose diagram, one sees that $\partial/\partial t$ is spacelike in the top and bottom diamonds while in the northern diamond it is pointing towards the past!
Thus $\partial/\partial t$ in static coordinates can only be used to define a sensible time evolution in the southern diamond of de Sitter space. This absence of a globally timelike Killing vector in de Sitter space has important implications for the quantum theory.

Unruh detector

- An observer moving along a timelike geodesic observes a thermal bath of particles when the scalar field ϕ is in the vacuum state $|\Omega\rangle$. Thus we will conclude that de Sitter space is naturally associated with a temperature.
- Since the notion of a particle is observer-dependent in a curved spacetime, let us attempt to give a coordinate invariant characterization of the temperature.
- A Unruh detector is a detector which has some internal energy states and can interact with the scalar field by exchanging energy, i.e. by emitting or absorbing scalar particles. It can be modeled by the coupling

$$g \int_{-\infty}^{\infty} \mathrm{d}\tau \, \delta(\tau)\phi(x(\tau))$$

  of the scalar field ϕ along the worldline $x(\tau)$ of the observer to some operator $\delta(\tau)$ acting on the internal detector states.

Detectors Bing Bing

- Let us compute the transition amplitude from a state $|\Omega\rangle|E_i\rangle \in \mathcal{H}_{\text{field}} \otimes \mathcal{H}_{\text{det}}$ to the state $|\beta\rangle|E_j\rangle$ where $|E_{i \atop j}\rangle$ are eigenstates of the detector hamiltonian H_{det} and $|\beta\rangle$ is any state of the scalar field.

- To first order in perturbation theory for small coupling g, the probability reads as

$$P(E_i \to E_j) = g^2 |\langle E_i|\delta(0)|E_j\rangle|^2 \times \\ \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\tau \mathrm{d}\tau' \, e^{-\mathrm{i}(E_j - E_i)(\tau' - \tau)} \mathcal{W}(x(\tau'), x(\tau)).$$

- Since $\mathcal{W}(x(\tau'), x(\tau)) = G(\mathcal{S}(x(\tau'), x(\tau)))$, and if we consider for simplicity an observer sitting on the south pole, then G is given in static coordinates by $G(\cosh(\tau - \tau'))$

Thermal Green function

- Dividing out the singular factor issued from partial integration, we get the transition probability per unit proper time along the detector worldline,

$$\dot{P}(E_i \to E_j) = g^2 |\langle E_i|\delta(0)|E_j\rangle|^2 \int_{-\infty}^{\infty} \mathrm{d}\tau \, e^{-\mathrm{i}(E_j - E_i)\tau} G(\cosh \tau).$$

- Now we observe that the function G is periodic in imaginary time under $\tau \to \tau + 2\pi i$, and Green functions which are periodic in imaginary time are *thermal* Green functions.

- Now we can prove from complex path integration arguments the following :

$$\dot{P}(E_i \to E_j) = \dot{P}(E_j \to E_i) e^{-\beta(E_j - E_i)}$$

- Suppose that the energy levels of the detector are thermally populated, so that $N_i = Ne^{-\beta E_i}$ (N is some normalization factor). Then it is clear that the total transition rate R from E_i to E_j is the same as from E_j to E_i :

$$R(E_i \to E_j) = Ne^{-\beta E_i} \dot{P}(E_i \to E_j) = R(E_j \to E_i)$$

Thermal bath

- Thus, there is no change in the probability distribution for the energy levels with time (principle of detailed balance in a thermal ensemble) : the system is in a thermal bath of particles at temperature $T = 1/\beta = H/2\pi$ (after reintroducing dS fundamental length).

- In conclusion, any geodesic observer in de Sitter space will feel that she/he is in a thermal bath of particles at a temperature $T = 1/\beta = H/2\pi$.

- A notion of dS entropy can be derived from this fact.