Poisson gauge theory

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Outline of the talk:

1. The statement of the problem: NC gauge theories with a non-constant NC parameter $\Theta^{ab}(x)$
2. Semi-classical limit and Poisson gauge algebra
3. Symplectic embeddings
4. Gauge covariant derivative
5. Poisson field strength and Bianchi identity
6. Explicit examples with linear Poisson structures
   - $su(2)$-like structure
   - κ-Minkowski
7. Maxwell-Poisson equations
8. Arbitrariness in construction and field redefinitions
9. Conclusions and outlook

Gauge theories

The $U(1)$ gauge transformations of the gauge and matter fields are,

$$A_a \to A'_a = A_a + \partial_a f\,, \qquad \psi(x) \to \psi'(x) = e^{if(x)}\psi(x)\,,$$

then the gauge covariant derivative, $\mathcal{D}_a = \partial_a - iA_a$, transforms as,

$$\begin{aligned}\mathcal{D}'_a\psi' &= (\partial_a - iA_a - i\partial_a f)\left[e^{if(x)}\psi(x)\right] \\ &= e^{if(x)}(\partial_a - iA_a)\psi(x) + i\partial_a f e^{if(x)}\psi(x) - i\partial_a f e^{if(x)}\psi(x) \\ &= e^{if}\mathcal{D}_a\psi\,.\end{aligned}$$

The field strength is defined by,

$$[\mathcal{D}_a, \mathcal{D}_b] = -i\,(\partial_a A_b - \partial_b A_a) := -iF_{ab}\,.$$

Jacoby identity, $[\mathcal{D}_a, [\mathcal{D}_b, \mathcal{D}_c]] + \text{cycl.} = 0$, implies Bianchi identity for the field strength, $\partial_a F_{ab} + \text{cycl.} = 0$. The field equations are,

$$\partial_a F^{ab} = 0\,.$$

Non-commutativity of space-time

How precise can be the measurement of the position in the space, how small can be the error Δr? [Bronstein '1931]

QM: test particle, $\Delta r \sim \lambda = hc/E$. The higher the energy the higher the precision of the measurement.

GR: Each mass or energy creates the curvature of the space. The highest precision cannot exceed the Schwarzschild radius, $r_s \sim E$.

The space becomes non-local at the Planck scale,

$$\Delta r \Delta r_s \geq \ell_p^2 = \frac{\hbar G}{c^3} \,.$$

This can be introduced in the theory supposing that the coordinates are non-commutative,

$$[\hat{x}^a, \hat{x}^b] = i\Theta^{ab}(\hat{x}) \,.$$

The most simple case, the flat NC space, $\Theta^{ab} = const$, is not very plausible from the point of view of the theory of gravity.

Star product

Instead of working with NC operators $f(\hat{x})$, work with commutative functions $f(x)$ substituting the standard multiplication $f \cdot g$ with a star product, $\hat{f}(\hat{x}) \circ \hat{g}(\hat{x}) = \widehat{f \star g}(\hat{x})$,

$$f \star g = f \cdot g + \frac{i\hbar}{2}\{f, g\} + \mathcal{O}(\hbar^2)\,, \qquad \{f, g\} = \Theta^{ij}(x)\,\partial_i f\,\partial_j g\,,$$

$$f \star (g \star h) = (f \star g) \star h\,,$$

$$[f, g]_\star = f \star g - g \star f = i\hbar\{f, g\} + \mathcal{O}(\hbar^2)\,.$$

The existence of $\star$ for Poisson bi-vector $\Theta^{ij}(x)$ follows from Formality Th. [Kontsevich '1997]. For constant Θ^{ij}, Moyal star product,

$$f \star_{can} g = \exp\left(\frac{i\hbar}{2}\Theta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}\right) f(x)\, g(y)\Big|_{y=x}\,.$$

Non-commutative gauge theory

Standard covariant derivative, $\mathcal{D}_a = \partial_a - iA_a(x)$, is based on the Leibniz rule, which is violated for the star product for $\Theta^{ij}(x)$,

$$\partial_a(f \star g) \neq \partial_a f \star g + f \star \partial_a g \,.$$

So, if to substitute in the action all point-wise products with a star products, the theory will not be gauge invariant.

The statement of the problem: We are looking for the non-commutative theory satisfying the following two properties:

1. Gauge invariance
2. Correct commutative limit ($U(1)$ gauge theory)

We would like to understand how the effects of quantum gravity (taken into account through the non-commutativity of space-time) may affect the fundamental interactions.

Previous approaches

Seiberg-Witten map [Seiberg & Witten, JHEP '99] Relates degrees of freedom of NC gauge theory to their commutative counterpart. For $\Theta^{ij}(x)$ there is a perturbative and quite complicated expression.

Covariant coordinates [Wess et al, EPJC '00] Instead of partial derivative ∂_a consider, $D_a = i[x_a, \cdot]_\star$. Solves the problem with Leibniz, $D_a[f, g]_\star = [D_a f, g]_\star + [f, D_a g]_\star$, instead problems with commutative limit. Works with the Universal enveloping algebra of gauge fields, additional dof without clear physical meaning.

Twist [Vassilevich, MPLA '06] Based on Hopf algebras. Twist is known for a very few examples of $\Theta^{ij}(x)$.

L_∞ algebra [Blumenhagen et al, JHEP '18] is a powerful tool for the construction of order by order deformations of gauge theories. For explicit all-orders expressions need something more.

Poisson gauge algebra

NC $U(1)$ gauge transformations $\delta_f^{NC} A_a$ are deformations of the abelian gauge transformations, $\delta_f^0 A_a = \partial_a f$, closing the algebra,

$$[\delta_f^{NC}, \delta_g^{NC}] A_a = \delta_{-i[f,g]_\star}^{NC} A_a \,.$$

For simplicity in this talk we discuss the semi-classical limit,

$$\{f, g\} = \lim_{\hbar \to 0} \frac{[f, g]_\star}{i\hbar} \,,$$

which defines the Poisson gauge algebra,

$$[\delta_f, \delta_g] A_a = \delta_{\{f,g\}} A_a \,.$$

$\delta_f A$ are Poisson gauge transformations,

$$\lim_{\Theta \to 0} \delta_f A_a = \partial_a f \,.$$

Poisson gauge transformations

If Θ^{ij} is constant, one may just set,

$$\delta_f^{can} A_a = \partial_a f + \{A_a, f\}_{can} \, .$$

The direct calculation using Leibniz and Jacobi gives,

$$\begin{aligned}
[\delta_f^{can}, \delta_g^{can}] A_a &= \delta_f^{can}(\partial_a g + \{A_a, g\}_{can}) - \delta_g^{can}(\partial_a f + \{A_a, f\}_{can}) \\
&= \{\partial_a f + \{A_a, f\}_{can}, g\}_{can} - \{\partial_a g + \{A_a, g\}_{can}, f\}_{can} \\
&= \partial_a \{f, g\}_{can} + \{A_a, \{f, g\}_{can}\}_{can} = \delta_{\{f,g\}_{can}} A_a
\end{aligned}$$

For non-constant $\Theta^{ij}(x)$, the standard Leibniz rule is violated,

$$\partial_a \{f, g\} \neq \{\partial_a f, g\} + \{f, \partial_a g\} \, , \qquad \{f, g\} = \Theta^{ij}(x) \, \partial_i f \, \partial_j g \, .$$

To fix this problem one has to introduce corrections with derivatives $\partial_a \Theta^{ij}(x)$ which would compensate the violation of the Leibniz rule.

Symplectic embeddings, [Weinstein; Karasev, Maslov '80s]

The problem with the violation of the Leibniz rule can be solved in the extended space. Start with given set of Poisson brackets,

$$\{x^i, x^j\} = \Theta^{ij}(x)\,, \qquad i,j = 1,\dots d\,,$$
$$\{x^i,\{x^j,x^k\}\} + \{x^k,\{x^i,x^j\}\} + \{x^j,\{x^k,x^i\}\} = 0 \quad \Leftrightarrow$$
$$\Theta^{im}\partial_m\Theta^{jk} + \Theta^{km}\partial_m\Theta^{ij} + \Theta^{jm}\partial_m\Theta^{ki} = 0\,.$$

To each x^i we introduce p_i, in such a way that,

$$\{x^i, p_j\} = \gamma^i_j(x,p) = \delta^i_j + \mathcal{O}(\Theta)\,, \qquad \{p_i, p_j\} = 0\,,$$

should satisfy the Jacobi identity,

$$\{x^i,\{x^j,p_k\}\} + \{p_k,\{x^i,x^j\}\} + \{x^j,\{p_k,x^i\}\} = 0\,,$$
$$\{x^i,\{p_j,p_k\}\} + \{p_k,\{x^i,p_j\}\} + \{p_j,\{p_k,x^i\}\} = 0\,.$$

Symplectic embeddings

The JI implies an equation on $\gamma(x,p)$,

$$\gamma^l_b\,\partial^b_p\gamma^k_a - \gamma^k_b\,\partial^b_p\gamma^l_a + \Theta^{lm}\,\partial_m\gamma^k_a - \Theta^{km}\,\partial_m\gamma^l_a - \gamma^m_a\,\partial_m\Theta^{lk} = 0\,,$$

where, $\partial^b_p = \frac{\partial}{\partial p_b}$, and, $\partial_m = \frac{\partial}{\partial x^m}$. Perturbative solution is given by,

$$\begin{aligned}\gamma^k_a(x,p) &= \delta^k_a - \frac{1}{2}\,\partial_a\Theta^{kb}p_b \\ &\quad -\frac{1}{12}\left(2\,\Theta^{cm}\partial_a\partial_m\Theta^{bk} + \partial_a\Theta^{bm}\partial_m\Theta^{kc}\right)p_b p_c + \mathcal{O}(\Theta^3)\,.\end{aligned}$$

The recursive formula can be found in [VK, J.Phys.A '19].

In particular, for constant Θ^{ij} one finds, $\gamma^i_j(x,p) = \delta^i_j$,

Explicit constructions

For linear Poisson structures (f_k^{ij} are structure constants of a Lie algebra),

$$\{x^i, x^j\} = f_k^{ij}\, x^k \,,$$
$$f_i^{kl} f_l^{ja} + f_i^{jl} f_l^{ak} + f_i^{al} f_l^{kj} = 0.$$

One finds [Gutt, LMP '83],

$$\gamma_j^i(p) = \delta_j^i - \frac{1}{2}\, f_j^{ij_1}\, p_{j_1} + \mathcal{X}_j^i\,(-M/2)\ ,$$

where

$$M_l^i = f_k^{ij_1} f_l^{kj_2} p_{j_1} p_{j_2}\ ,$$

and $\mathcal{X}_l^i\,(-M/2)$ is a matrix valued function with,

$$\mathcal{X}(t) = \sqrt{\frac{t}{2}}\cot\sqrt{\frac{t}{2}} - 1 = \sum_{n=1}^{\infty} \frac{(-2)^n\, B_{2n}\, t^n}{(2n)!}\ .$$

Examples

- $su(2)$ structure, $\{x^i, x^j\} = 2\theta\,\varepsilon^{ijk} x_k$, [VK & Vitale, JHEP '15]

  $$[\gamma_\varepsilon]^k_a(p) = \left[1 + \theta^2 p^2 \chi\left(\theta^2 p^2\right)\right]\delta^k_a - \theta^2 \chi\left(\theta^2 p^2\right) p_a p^k - \theta\,\varepsilon_a{}^{kl} p_l\,,$$

  where,

  $$\chi(t) = \frac{1}{t}\left(\sqrt{t}\,\cot\sqrt{t} - 1\right)\,, \qquad \chi(0) = -\frac{1}{3}\,.$$

- κ-Minkowski, $\{x^k, x^l\} = 2(a^i\,x^j - a^j\,x^i)$, [VK, Kurkov, Vitale, JHEP '21]

  $$[\gamma_\kappa]^k_a(p) = \left[\sqrt{1 + (a\cdot p)^2} + (a\cdot p)\right]\delta^k_a - a^k\,p_a\,.$$

- Nonlinear structure, $\{x^i, x^j\} = 2\theta\,\varepsilon^{ijk} x_k\, e^{-\frac{\theta^2 x^2}{2}}$, [VK, JHEP '21]

  $$\gamma^i_j(x,p) = e^{-\frac{\theta^2 x^2}{2}}\left(\delta^i_k - \frac{\theta^2\,x_k\,x^i}{1 + \theta^2\,x^2}\right)[\gamma_\varepsilon]^k_j(p)\,.$$

Poisson gauge transformations

To overcome, $\partial_a\{f,g\} \neq \{\partial_a f, g\} + \{f, \partial_a g\}$, instead of standard partial derivative ∂_a we introduce the 'twisted' one:

$$\partial^\tau_a f(x) := \{f(x), p_a\} = \gamma^i_a(x,p)\,\partial_i f(x)\,.$$

For constant Θ^{ij}, $\gamma^i_j(x,p) = \delta^i_j$, so, $\{f(x), p_a\} = \partial_a f(x)$.

The fact that, $\{p_i, p_j\} = 0$, and Jacobi imply, $[\partial^\tau_a, \partial^\tau_b] = 0$.

Moreover, the Jacobi identity,

$$\{\{f(x), g(x)\}, p_i\} = \{\{f(x), p_i\}, g(x)\} + \{f(x), \{g(x), p_i\}\}\,.$$

implies the Leibniz rule for ∂^τ_a,

$$\partial^\tau_a\{f,g\} = \{\partial^\tau_a f, g\} + \{f, \partial^\tau_a g\},$$

The price to pay is an explicit dependance of $\{f(x), p_a\}$ on p.

Poisson gauge transformations; e-Print: 2101.12618

Theorem: [VK, Szabo '21] *Poisson gauge transformations, defined by,*

$$\begin{aligned} \delta_f A_a : &= \{f(x), p_a\}|_{p_a = A_a(x)} + \{A_a(x), f(x)\} \\ &= \gamma_a^l(A)\, \partial_l f + \{A_a, f\}\,, \end{aligned}$$

where,

$$\gamma_a^l(A) := \gamma_a^l(x,p)|_{p_a = A_a(x)}\,,$$

close the Poisson gauge algebra,

$$[\delta_f, \delta_g] A_a = \delta_{\{f,g\}} A_a$$

and reproduce the standard $U(1)$ gauge transformations in the commutative limit,

$$\lim_{\Theta \to 0} \delta_f A_a = \partial_a f\,. \qquad \square$$

Covariant derivative

To be consistent with the semi-classical limit we define the gauge variation of the mater field ψ as,

$$\begin{aligned}
&\delta_f \psi = \{\psi, f\} \qquad \Rightarrow \\
&[\delta_f, \delta_g]\psi = \delta_f\left(\{\psi, g\}\right) - \delta_g\left(\{\psi, f\}\right) \\
&\qquad\qquad = \{\{\psi, f\}, g\} - \{\{\psi, g\}, f\} = \{\psi, \{f, g\}\} = \delta_{\{f,g\}}\psi\,.
\end{aligned}$$

Gauge covariant derivative of matter field should transform covariantly under the gauge transformation,

$$\delta_f \mathcal{D}_a(\psi) = \{\mathcal{D}_a(\psi), f\}\,,$$

and reproduce the standard derivative in the commutative limit,

$$\lim_{\Theta\to 0} \mathcal{D}_a(\psi) = \partial_a \psi\,.$$

Covariant derivative, [VK, JHEP '21]

Theorem: [VK '21] *The operator,*

$$\mathcal{D}_a(\psi) = \rho^i_a(A)\left(\gamma^l_i(A)\,\partial_l\psi + \{A_a,\psi\}\right)\,,$$

satisfies the above conditions if, $\rho^i_a(A) := \rho^i_a(x,p)_{p_a=A_a(x)}$, *and,*

$$\{f(x),\rho^i_a(x,p)\} + \rho^b_a(x,p)\,\partial^i_p\{f(x),p_b\} = 0\,, \qquad \forall f(x)\,.\,\square$$

In local coordinates,

$$\gamma^j_b\,\partial^b_p\,\rho^i_a + \rho^b_a\,\partial^i_p\gamma^j_b + \alpha\,\Theta^{jb}\,\partial_b\rho^i_a = 0\,.$$

Perturbative solution reads,

$$\begin{aligned}\rho^i_a(x,p) &= \delta^i_a - \frac{1}{2}\,\partial_a\Theta^{ib}p_b + \\ &\quad \frac{1}{6}\left(2\,\Theta^{cm}\partial_a\partial_m\Theta^{ib} - \partial_a\Theta^{bm}\partial_m\Theta^{ic}\right)p_bp_c + \mathcal{O}(\Theta^3)\,.\end{aligned}$$

For constant Θ^{ij} one finds just, $\rho^i_a(x,p) = \delta^i_a$.

Explicit constructions

For linear Poisson structures, $\{x^i, x^j\} = f^{ij}_k \, x^k$, the expression for $\gamma^i_j(p)$ reads,

$$\gamma^i_j(p) = \delta^i_j - \frac{1}{2} \, f^{ij_1}_j \, p_{j_1} + \mathcal{X}^i_j \, (-M/2) \ ,$$

where

$$M^i_l = f^{ij_1}_k f^{kj_2}_l p_{j_1} p_{j_2} \, , \qquad \mathcal{X}(t) = \sqrt{\frac{t}{2}} \cot \sqrt{\frac{t}{2}} - 1 = \sum_{n=1}^{\infty} \frac{(-2)^n \, B_{2n} \, t^n}{(2n)!} \ .$$

For this $\gamma^i_j(p)$ the matrix $\rho^i_a(p)$ which solves the equation,

$$\gamma^j_b \, \partial^b_p \, \rho^i_a + \rho^b_a \, \partial^i_p \gamma^j_b = 0 \, ,$$

can be constructed according to [Abla,VK, Kurkov, Vitale, work in progress]

$$[\rho^{-1}]^i_j = \delta^i_j + \frac{1}{2} \, f^{ij_1}_j \, p_{j_1} + \mathcal{X}^i_j \, (-M/2) \ .$$

Examples, $su(2)$-like structure

Let us take, $\{x^i, x^j\} = 2\theta\, \varepsilon^{ijk} x_k$, corresponding to the rotationally invariant NC space. One finds [VK, Fortsch.Phys '19],

$$\delta_f A_a = \partial_a f + \{A_a, f\} + \theta \varepsilon_a{}^{bc} A_b \partial_c f + \theta^2 \left(\partial_a f\ A^2 - \partial_b f\ A^b A_a\right) \chi(\theta^2 A^2)\,,$$

where,

$$\chi(t) = \frac{1}{t}\left(\sqrt{t}\cot\sqrt{t} - 1\right)\,, \qquad \chi(0) = -\frac{1}{3}\,.$$

The matrix ρ which enters the definition of $\mathcal{D}(\psi)$ is [VK, JHEP '21],

$$\rho^i_a(A) = \delta^i_a - \theta\, \varepsilon_a{}^{ik} A_k\, \lambda\left(\theta^2 A^2\right) - \theta^2 \left(\delta^i_a A^2 - A^i A_a\right) \tau(\theta^2 A^2)\,.$$

where,

$$\lambda(t) = \frac{\sin^2\sqrt{t}}{t}\,, \qquad \tau(t) = -\frac{1}{t}\left(\frac{\sin 2\sqrt{t}}{2\sqrt{t}} - 1\right)\,.$$

Examples: κ-Minkowski

κ-Minkowski non-commutativity corresponds to the linear Poisson structure,

$$\{x^i, x^j\} = 2\left(a^i\, x^j - a^j\, x^i\right)\ .$$

In this case one finds for the gauge transformation [VK, Kurkov, Vitale, JHEP '21],

$$\delta_f A_b = \left[\sqrt{1+(a\cdot A)^2} + (a\cdot A)\right]\partial_b f - A_b\,(a\cdot\partial) f + \{A_b, f\}\,.$$

While the matrix $\rho(A)$ becomes,

$$\rho^i_b(A) = \left[\sqrt{1+(a\cdot A)^2} + (a\cdot A)\right]\delta^i_b - \frac{\sqrt{1+(a\cdot A)^2} + (a\cdot A)}{\sqrt{1+(a\cdot A)^2}}\, a^i\, A_b\,.$$

Poisson field strength & Bianchi; [VK, JHEP '21]

The commutator relation for the covariant derivatives,

$$[\mathcal{D}_a, \mathcal{D}_b] = \{\mathcal{F}_{ab}, \cdot\} + \left(\mathcal{F}_{ad}\,\Lambda_b{}^{de} - \mathcal{F}_{bd}\,\Lambda_a{}^{de}\right)\mathcal{D}_e \,,$$

defines the Poisson field strength:

$$\mathcal{F}_{ab} := \frac{1}{2}\left(\rho_a^c\rho_b^d - \rho_b^c\rho_a^d\right)\left(\gamma_c^m\,\partial_m A_d + \{A_c, A_d\}\right) \,,$$

which transforms covariantly, $\delta_f \mathcal{F}_{ab} = \{f, \mathcal{F}_{ab}\}$, and

$$\lim_{\Theta\to 0}\, \mathcal{F}_{ab} = \partial_a A_b - \partial_b A_a \,.$$

In addition, JI for $\mathcal{D}_a$ implies deformed Bianchi identity for $\mathcal{F}_{ab}$:

$$\mathcal{D}_a\left(\mathcal{F}_{bc}\right) - \mathcal{F}_{ad}\,\Lambda_b{}^{de}\,\mathcal{F}_{ec} + \mathrm{cycl.}(abc) = 0 \,,$$

where, $\Lambda_b{}^{de} = \left(\rho^{-1}\right)_j^d\left(\partial_A^j\rho_b^m - \partial_A^m\rho_b^j\right)\left(\rho^{-1}\right)_m^e$.

Gauge invariant action

The gauge covariance of the field strength,

$$\mathcal{F}_{ab} = \frac{1}{2}\left(\rho_a^c \rho_b^d - \rho_b^c \rho_a^d\right)\left(\gamma_i^m\, \partial_m A_j + \{A_i, A_j\}\right) ,$$

implies that the following Lagrangian, $\mathcal{L} = -\frac{1}{4}\mathcal{F}_{ab}\mathcal{F}^{ab}$, also transforms covariantly,

$$\begin{aligned} \delta_f \mathcal{L} &= -\frac{1}{4}\delta_f\left(\mathcal{F}_{ab}\right)\mathcal{F}^{ab} - \frac{1}{4}\mathcal{F}_{ab}\delta_f\left(\mathcal{F}^{ab}\right) \\ &= -\frac{1}{4}\{f, \mathcal{F}_{ab}\}\mathcal{F}^{ab} - \frac{1}{4}\mathcal{F}_{ab}\{f, \mathcal{F}^{ab}\} = \{f, \mathcal{L}\} . \end{aligned}$$

Define the integration measure such that, $\int \mu(x)\{f, g\} = 0$, for Schwartz functions. It requires, $\partial_i\left(\mu(x)\Theta^{ij}(x)\right) = 0$. E.g., for, $\Theta^{ij}(x) = \varepsilon^{ijk}x^k$, the measure $\mu(x) = 1$.

Then the action, $S = \int \mu(x)\mathcal{L}$, is gauge invariant, $\delta_f S \equiv 0$.

Maxwell-Poisson equations

For, $\{x^i, x^j\} = 2\theta\, \varepsilon^{ijk} x_k$, one writes [VK & Vitale, JHEP '20],

$$\begin{aligned} \mathcal{F}^{\varepsilon}_{ab} &= \partial_c\, \pi_{ab}{}^{c} + 2\,\{\rho_{ab}{}^{c}, A_c\}\,, \\ \pi_{ab}{}^{c} &= (\delta^c_a A_b - \delta^c_b A_a)\ \phi\left(\theta^2 A^2\right) - \theta\, \varepsilon_{abm} A^m A^c\, \lambda\left(\theta^2 A^2\right) + \frac{1}{\theta}\varepsilon_{ab}{}^{c}\, \Phi\left(\theta^2 A^2\right)\ , \\ \rho_{ab}{}^{c} &= \frac{1}{4}\,(\delta^c_b A_a - \delta^c_a A_b)\ \lambda\left(\theta^2 A^2\right) - \frac{1}{4\theta}\varepsilon_{ab}{}^{c}\, \Lambda\left(\theta^2 A^2\right)\ , \end{aligned}$$

with, $\lambda(t) = \frac{\sin^2\sqrt{t}}{t}$, $\phi(t) = \frac{\sin 2\sqrt{t}}{2\sqrt{t}}$, $\Phi'(t) = \phi(t)$, and $\Lambda'(t) = \lambda(t)$.

The Maxwell-Poisson eom are:

$$\mathcal{D}_a\, \mathcal{F}^{ab}_{\varepsilon} = 0\,.$$

In addition we have deformed Bianchi identity [VK, JHEP '21]:

$$\mathcal{D}_a\left(\mathcal{F}^{\varepsilon}_{bc}\right) - \mathcal{F}^{\varepsilon}_{ad}\, \Lambda_b{}^{de}\, \mathcal{F}^{\varepsilon}_{ec} + \text{cycl.}(abc) = 0\,,$$

where, $\Lambda_b{}^{de} = \left(\rho^{-1}\right)^d_j \left(\partial^j_A \rho^m_b - \partial^m_A \rho^j_b\right)\left(\rho^{-1}\right)^e_m\ .$

Arbitrariness

Symplectic embedding is not unique. Suppose we have,

$$\begin{aligned} \{x^i, x^j\} &= \Theta^{ij}(x)\,, \qquad \{p_i, p\} = 0\,, \\ \{x^i, p_j\} &= \gamma^i_j(x,p)\,. \end{aligned}$$

Making an invertible change of variables in the extended space,

$$\phi:\; p \to \tilde{p}\,, \qquad \text{with} \qquad \tilde{p}_j(x,p) = p_j + \ldots\,,$$

one obtains another symplectic embedding of the same Poisson structure,

$$\begin{aligned} \{x^i, x^j\} &= \Theta^{ij}(x)\,, \qquad \{\tilde{p}_i, \tilde{p}\} = 0\,, \\ \{x^i, \tilde{p}_j\} &= \tilde{\gamma}^i_j(x,\tilde{p}) = \gamma^i_k(x,p)\,\partial^k_p \tilde{p}_j(p)|_{p=p(\tilde{p})}\,. \end{aligned}$$

Arbitrariness; [VK, JHEP '21]

Different embeddings define different representation of the same gauge algebra,

$$\tilde{\delta}_f \tilde{A}_a = \tilde{\gamma}_a^i(\tilde{A})\,\partial_i f + \{f, \tilde{A}_a\}\,,$$
$$\left[\tilde{\delta}_f, \tilde{\delta}_g\right] \tilde{A}_a = \tilde{\delta}_{\{f,g\}} \tilde{A}_a\,.$$

Two theories are related by invertible field redefinition,

$$\phi : A \to \tilde{A}, \qquad \text{and} \qquad \tilde{f} = f\,, \qquad \text{with,} \qquad \tilde{\delta}_f := \phi \circ \delta_f \circ \phi^{-1}\,,$$

such that the Seiberg-Witten condition holds true,

$$\tilde{A}\,(A + \delta_f A) = \tilde{A}(A) + \tilde{\delta}_f \tilde{A}(A)\,.$$

Moreover, the gauge algebra remains the same, as expected,

$$\left[\tilde{\delta}_f, \tilde{\delta}_g\right] = \tilde{\delta}_f \circ \tilde{\delta}_g - \tilde{\delta}_g \circ \tilde{\delta}_f = \phi \circ (\delta_f \circ \delta_g - \delta_g \circ \delta_f) \circ \phi^{-1} = \tilde{\delta}_{\{f,g\}}\,.$$

Discussion

- Poisson gauge theory is a self-consistent approximation of the full NC gauge theory; first order in $\hbar$, all orders in Θ. All formulas are exact, not approximated.
- Symplectic embeddings of Poisson manifold define Poisson GT closing, $[\delta_f, \delta_g]A_a = \delta_{\{f,g\}}A_a$.
- To define the dynamics one needs to solve an additional equation on $\rho^i_a(A)$.
- For linear Poisson structures, $\{x^i, x^j\} = f^{ij}_k\, x^k$, we have an explicit all-order constructions for everything.
- Symplectic embedding of $\Theta^{ij}(x)$ is not unique, as well as the constructed gauge theory. Different field theories constructed here are related by the invertible field redefinition [VK '21].

Open questions

- Is there any reasonable physics behind the constructed models? E.g. energy-momentum dispersion relations.
- Some simplest solutions of the field equations.
- Over explicit examples of linear NC structures, e.g. NC AdS_2, [Pinzul & Stern, PRD '17]
- Quadratic Poisson structures, $\Theta^{ij}(x) = R^{ij}{}_{kl}\, x^k x^l$, corresponding to the quantum groups and Yang-Baxter deformations. Are there some explicit formulas?
- Towards full non-commutative gauge theory, all orders in $\hbar$: $\{f,g\} \to -i[f,g]_\star$, and take, $\delta_f \psi = i\, f \star \psi$.
- Almost-Poisson gauge algebra [VK & Szabo '21] is a semiclassical limit of full non-associative gauge algebra. The dynamical part of almost-Poisson gauge theory needs to be constructed.