Universidade Federal do ABC

Introduction to black hole perturbation theory

Scattering, quasinormal modes and quasibound states

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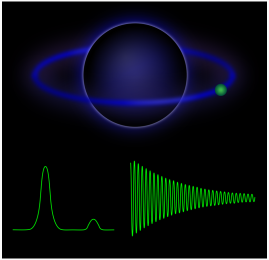


Ref: APS/Alan Stonebraker

These slides and the associated Mathematica notebooks are available at:

– Slides:
http://professor.ufabc.edu.br/~mauricio.richartz/bh_pert.pdf

– Notebook:
http://professor.ufabc.edu.br/~mauricio.richartz/bh_pert.nb

Objectives

The main goal is to introduce the basic ideas of black hole perturbation theory and to learn some of the numerical methods associated with it. I hope that, at the end of the lectures, the student will understand superradiance, quasinormal modes, and quasibound states of a black hole. The student will know how to calculate reflection and transmission coefficients associated with black hole scattering, and will be able to determine the frequency of the quasinormal modes (QNMs) and of the quasibound states of a black hole (QBSs).

For full understanding, the recommended background is:

- General Relativity (in particulas, the Schwarzschild and Kerr metrics), at the level of Hartle or Carroll;
- Non-relativistic Quantum Mechanics (in particular, 1D scattering and the hydrogen atom);
- Mathematical Physics (in particular, separation of variables, series solutions, and Frobenius method).

Outline

This mini-course is divided into 2 parts:

- **Part I: Test fields in Schwarzschild and Kerr.**
  – Basic notions of scattering, quasinormal modes, quasibound states, superradiance and instabilities.
- **Part II: Numerical methods.**
  – Scattering in the frequency domain. The continued fraction (CFM) method for QNMs and QBSs. Scattering in the time domain.

References

These lectures are based in the following main references:

- Black hole perturbation theory (Lecture notes by E. Berti): *https://www.icts.res.in/event/page/3071*
- Quasinormal modes of black holes and black branes (E. Berti, V. Cardoso, A. Starinets) - CQG 26 163001 (2009) - arXiv:0905.2975
- Quasinormal modes of black holes: from astrophysics to string theory (R. Konoplya, A. Zhidenko) - Rev. Mod. Phys. 83, 793 (2011) - arXiv:1102.4014
- Superradiance – the 2020 Edition (R. Brito, V. Cardoso, P. Pani) - Lec. Notes in Physics 971 (2020) - arXiv:1501.06570

Part 0:

Notation and basic definitions

Notation and basic definitions

We use units in which $G = c = 1$ and we adopt the Einstein notation for sums. Given a metric $g_{\mu\nu}$, we denote its inverse by $g^{\mu\nu}$ and its determinant by g. We work with 4-dimensional spacetimes and we adopt the $(-,+,+,+)$ signature.

$$\text{Christoffel symbols: } \Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta} \right) \tag{1}$$

$$\text{Riemann tensor: } R^{\alpha}{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^{\alpha}_{\beta\delta} - \partial_\delta \Gamma^{\alpha}_{\beta\gamma} + \Gamma^{\alpha}_{\mu\gamma}\Gamma^{\mu}_{\beta\delta} - \Gamma^{\alpha}_{\mu\delta}\Gamma^{\mu}_{\beta\delta} \tag{2}$$

$$\text{Ricci tensor: } R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta}, \quad \text{Ricci scalar: } R = g^{\alpha\beta} R_{\alpha\beta} \tag{3}$$

$$\text{Einstein tensor: } G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \tag{4}$$

Covariant derivative:

$$\nabla_\beta A^\mu = \partial_\beta A^\mu + \Gamma^\mu_{\alpha\beta} A^\alpha, \quad \nabla_\beta A^{\mu\nu} = \partial_\beta A^{\mu\nu} + \Gamma^\mu_{\alpha\beta} A^{\alpha\nu} + \Gamma^\nu_{\alpha\beta} A^{\mu\alpha} \tag{5}$$

$$\nabla_\beta A_\mu = \partial_\beta A_\mu - \Gamma^\alpha_{\mu\beta} A_\alpha, \quad \nabla_\beta A_{\mu\nu} = \partial_\beta A_{\mu\nu} - \Gamma^\alpha_{\mu\beta} A^{\alpha\nu} + \Gamma^\alpha_{\nu\beta} A_{\mu\alpha} \tag{6}$$

Important identities:

$$\Gamma^\mu_{\mu\beta} = \frac{1}{\sqrt{-g}} \partial_\beta \sqrt{-g} \tag{7}$$

$$\nabla_\alpha A^\alpha = \frac{1}{\sqrt{-g}} \partial_\alpha \left(\sqrt{-g} A^\alpha \right) \tag{8}$$

$$\Box \Phi = g^{\alpha\beta} \nabla_\alpha \nabla_\beta \Phi = \nabla_\alpha \nabla^\alpha \Phi = \frac{1}{\sqrt{-g}} \partial_\alpha \left(\sqrt{-g} g^{\alpha\beta} \partial_\beta \Phi \right) \tag{9}$$

Notation and basic definitions

Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \quad (10)$$

The event horizon is located at $r = 2M$.

Kerr metric:

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mra\sin^2\theta}{\Sigma} dt d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\Sigma}\right)\sin^2\theta \, d\phi^2, \quad (11)$$

where $a = J/M$, $\quad \Delta = r^2 - 2Mr + a^2$, $\quad \Sigma = r^2 + a^2\cos^2\theta$.

The event horizon is located at $r = r_+ = M + \sqrt{M^2 - a^2}$ and the angular velocity of stationary observers at r_+ is $\Omega = a/2Mr_+$.

The manipulation of trigonometric functions is faster in Mathematica if we introduce the change of variables $\chi = \cos\theta$, with $\chi \in [-1, 1]$, such that:

$$\sin\theta = \sqrt{1-\chi^2}, \quad d\chi = -\sin\theta\, d\theta, \quad d\theta = -\frac{d\chi}{\sqrt{1-\chi^2}}. \tag{12}$$

Therefore, all metric components of Schwarzschild and Kerr are rational functions. In Kerr, for example, we have:

$$\begin{aligned} ds^2 = &-\left(1-\frac{2Mr}{r^2+a^2\chi^2}\right)dt^2 - \frac{4Mra(1-\chi^2)}{r^2+a^2\chi^2}dtd\phi \\ &+\frac{r^2+a^2\chi^2}{r^2-2Mr+a^2}dr^2+\frac{r^2+a^2\chi^2}{1-\chi^2}d\chi^2 \\ &+(1-\chi^2)\left[r^2+a^2+\frac{2Mra^2\left(1-\chi^2\right)}{r^2+a^2\chi^2}\right]d\phi^2, \end{aligned} \tag{13}$$

Part I:

Test fields in Schwarzschild and in Kerr - Basic notions on scattering, quasinormal modes, quasibound states, super-radiance, and instabilities.

When quantizing the energy-momentum relation of Special Relativity, we get (remember that $c = 1$)

$$E^2 = p^2 + m^2 \longrightarrow \hat{E}^2 = \hat{p}^2 + m^2 \mathbb{1}, \tag{14}$$

where $\hat{E} = i\hbar\,\partial_t$, $\hat{p} = -i\hbar\nabla$ and $\mathbb{1}$ is the identity operator. Applying the identity to a scalar field Φ, we find the Klein–Gordon (KG) equation in the Minkowski spacetime:

$$\left(-\partial_t^2 + \nabla^2 - \mu^2\right)\Phi = \left(\partial_\alpha \partial^\alpha - \mu^2\right)\Phi = 0, \text{ com } \mu = m/\hbar. \tag{15}$$

In a generic spacetime, after replacing the ordinary derivatives by covariant derivatives and using the identity (9), we find:

$$\left(\Box - \mu^2\right)\Phi = \frac{1}{\sqrt{-g}}\partial_\alpha\left(\sqrt{-g}g^{\alpha\beta}\partial_\beta \Phi\right) - \mu^2\Phi = 0. \tag{16}$$

Since the Schwarzschild spacetime is stationary and axisymmetric, we separate the coordinates t and ϕ in the KG equation using the *ansatz*

$$\Phi(t,r,\chi,\phi) = \exp\left(-i\omega t + im\phi\right)\tilde{\Phi}(r,\chi) = \exp\left(-i\omega t + im\phi\right)R(r)S(\chi),$$

where $R(r)$ and $S(\chi)$ are functions that will be determined if the separation of the coordinates r and χ succeeds.

Proceeding with the separation (see notebook), we find:

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{r-M}{r-2M}\frac{dR}{dr} + \left[\frac{\omega^2 r^2}{(r-2M)^2} + \frac{\mu^2 r^2 - \lambda}{r(r-2M)}\right]R = 0, \tag{17}$$

$$(1-\chi^2)\frac{d^2S}{d\chi^2} - 2\chi\frac{dS}{d\chi} + \left(\lambda - \frac{m^2}{1-\chi^2}\right)S = 0, \tag{18}$$

where λ is a separation constant. Eq. (18) is the associated Legendre equation. $S(\chi)$ is well behaved (non-singular) at the poles (i.e. $\chi = -1$ and $\chi = 1$) only if $\lambda = \ell(\ell+1)$, $\ell \in \mathbb{Z}$. In such a case, $S = P_{\ell m}(\chi)$ are the associated Legendre polynomials.

The solutions defined in the previous slide are modes of frequency ω, orbital number $\ell \in \mathbb{N}$ and azimuthal $m \in \mathbb{Z}$, with $-\ell \leq m \leq \ell$:

$$\Phi_{\omega\ell m}(t, r, \chi, \phi) = \exp\left(-i\omega t + im\phi\right) R_{\omega\ell}(r) S_{\ell m}(\chi), \tag{19}$$

Since the KG is linear, any linear combination of modes is also a solution:

$$\Phi(t, r, \chi, \phi) = \int_{-\infty}^{\infty} d\omega \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m\omega} \exp\left(-i\omega t + im\phi\right) R_{\omega\ell}(r) P_{\ell m}(\chi), \tag{20}$$

where $a_{\ell m\omega}$ are arbitrary functions of ω for any given pair (ℓ, m). The functions $a_{\ell m\omega}$ can be determined from the initial conditions associated with the problem under investigation.

It is convenient to look for changes of variables that transform the radial equation into a Schrödinger-like equation and that take extend the domain by moving the event horizon to infinity. This allows a simple physical interpretation of the results, as we will see shortly.

We define a new dependent variable ψ and a new independent variable r_* by:

$$\psi_{\omega\ell}(r) = rR_{\omega\ell}(r), \quad \frac{dr_*}{dr} = \frac{r^2}{r^2 - 2Mr} \Rightarrow r_* = r + 2M \ln\left(\frac{r}{2M} - 1\right). \quad (21)$$

Consequently, the radial equation (17) becomes

$$\frac{d^2\psi_{\omega\ell}}{dr_*^2} + \left[\omega^2 - V(r)\right]\psi_{\omega\ell} = 0, \quad (22)$$

with $r = r(r_*)$ and $V(r) = \left(1 - \frac{2M}{r}\right)\left(\mu^2 + \frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3}\right)$.

We want to find solutions of the radial equation near and far away from the black hole. Recall that:

$$\frac{d^2\psi_{\omega\ell}}{dr_*^2} + \left[\omega^2 - V(r)\right]\psi_{\omega\ell} = 0, \quad r_* = r + 2M\ln\left(\frac{r}{2M} - 1\right),$$

$$V(r) = \left(1 - \frac{2M}{r}\right)\left(\mu^2 + \frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3}\right).$$

- Near the event horizon, i.e. when $r \to 2M$, we have:

$$r_* \approx 2M\ln\left(\frac{r}{2M} - 1\right) \to -\infty \;\text{ and }\; V(r) \to 0 \Rightarrow \frac{d^2\psi_{\omega\ell}}{dr_*^2} + \omega^2\psi_{\omega\ell} = 0$$

Therefore, $\psi_{\omega\ell} \to \exp(\pm i\,\omega\, r_*)$ when $r_* \to -\infty$.

- Far away from the black hole, i.e. when $r \to \infty$:

$$r_* \approx r \to +\infty \;\text{ and }\; V(r) \to \mu^2 \Rightarrow \frac{d^2\psi_{\omega\ell}}{dr_*^2} + \left[\omega^2 - \mu^2\right]\psi_{\omega\ell} = 0.$$

Therefore, $\psi_{\omega\ell} \to \exp\left(\pm i\sqrt{\omega^2 - \mu^2}\, r_*\right)$ when $r_* \to +\infty$.

To sum up:

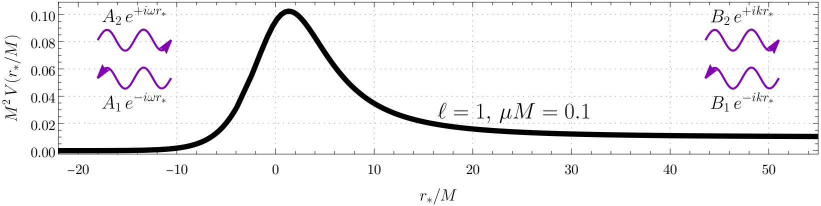
$$\psi_{\omega\ell}(r_*) \approx \begin{cases} A_1 \exp\left(-i\,\omega\,r_*\right) + A_2 \exp\left(+i\,\omega\,r_*\right), & r_* \to -\infty, \\ B_1 \exp\left(-i\,k\,r_*\right) + B_2 \exp\left(+i\,k\,r_*\right), & r_* \to +\infty, \end{cases} \tag{23}$$

where $k = \sqrt{\omega^2 - \mu^2}$. What is the physical interpretation of the constants A_1, A_2, B_1, B_2? Recalling the temporal dependence $\exp\left(-i\,\omega\,t\right)$, we have that:

– A_1 represents the amplitude of a wave that enters the black hole and A_2 represents the amplitude of a wave that exits the black hole.

– If $\omega > \mu$: B_1 represents the amplitude of an incoming wave at infinity and B_2 represents the amplitude of an outgoing wave at infinity.

– If $\omega < \mu$: B_1 represents an exponential growing mode at infinity and B_2 represents an exponential decaying mode at infinity.



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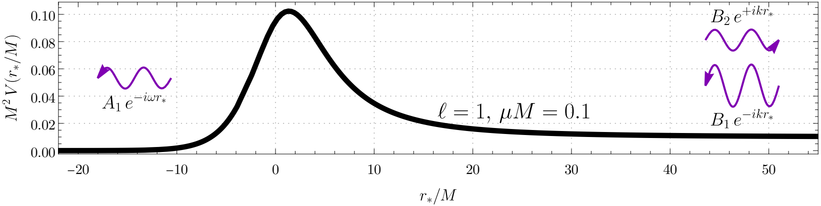
- If $\omega < \mu$: B_1 represents an exponential growing mode at infinity and B_2 represents an exponential decaying mode at infinity.

We want to study three types of problem: scattering, quasinormal modes, and quasibound states. Recall that:

$$\psi_{\omega\ell}(r_*) \approx \begin{cases} A_1 \exp(-i\omega r_*) + A_2 \exp(+i\omega r_*), & r_* \to -\infty, \\ B_1 \exp(-ik r_*) + B_2 \exp(+ik r_*), & r_* \to +\infty. \end{cases} \tag{24}$$

- At the event horizon: classically, nothing can escape the black hole $\Rightarrow \boxed{A_2 = 0.}$
- **Scattering:** a frequency $\omega \in \mathbb{R}$, with $\omega > \mu$, is given. We choose the amplitude B_1 of the incoming wave from infinity and determine the amplitudes A_1 and B_2.
- **Quasinormal modes (QNMs) and quasibound states (QBSs):** we impose a boundary condition at infinity - only outgoing waves (QNM) or only exponentially decaying solution (QBS). Due to the extra boundary condition, the frequency $\omega \in \mathbb{C}$ is unknown (it is an eigenvalue problem).

We can understand black hole scattering by making an analogy with 1-D problems from QM. Assuming that $\omega \in \mathbb{R}$, with $\omega > \mu$, we have the following representation of $\boxed{\psi''_{\omega\ell} + \left[\omega^2 - V(r)\right]\psi_{\omega\ell} = 0 :}$



$$\psi_{\omega\ell}(r_*) \approx \begin{cases} A_1 \exp\left(-i\,\omega\, r_*\right), & r_* \to -\infty, \\ B_1 \exp\left(-i\,k\,r_*\right) + B_2 \exp\left(+i\,k\,r_*\right), & r_* \to +\infty. \end{cases} \quad (25)$$

Part I: Understanding scattering modes

Note that if $\psi_{\omega\ell}$, with $\omega \in \mathbb{R}$, is a solution of the differential equation

$$\frac{d^2\psi_{\omega\ell}}{dr_*^2} + \left[\omega^2 - V(r)\right]\psi_{\omega\ell} = 0, \tag{26}$$

then $\psi^*_{\omega\ell}$ is also a solution of the same equation. Since the equation has no first order term, we find that the wronskian W between two solutions is contant. Therefore,

$$W\left(\psi_{\omega\ell}, \psi^*_{\omega\ell}\right) = \psi_{\omega\ell}\frac{d\psi^*_{\omega\ell}}{dr_*} - \psi^*_{\omega\ell}\frac{d\psi_{\omega\ell}}{dr_*} = \text{constant} \tag{27}$$

Equating the wronskian at the horizon and at infinity, we find:

$$W|_{r_*\to-\infty} = W|_{r_*\to\infty} \Rightarrow 2i\omega|A_1|^2 = 2ik\left(|B_1|^2 - |B_2|^2\right). \tag{28}$$

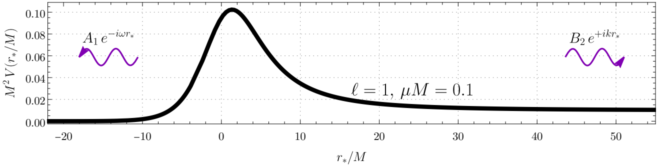
Hence,

$$R_\omega + T_\omega = 1, \quad \text{onde} \quad R_\omega = \left|\frac{B_2}{B_1}\right|^2 \quad \text{e} \quad T_\omega = \frac{\omega}{k}\left|\frac{A_1}{B_1}\right|^2 \tag{29}$$

are, respectively, the reflection and transmission coefficients.

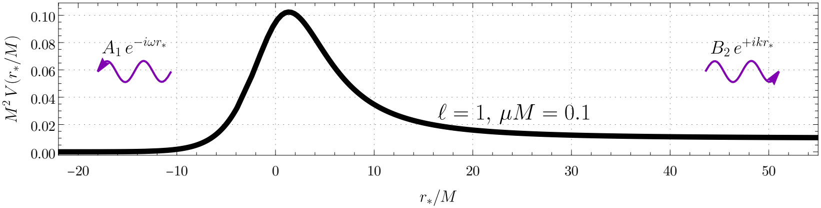
Part I: Understanding QNMs

Normal modes are the characteristic modes of oscillation of a system (as in the cases of a mass-spring system and a string with fixed ends). Due to the boundary conditions, normal modes do not exist for black holes. For black holes, the characteristic oscillations are always associated with dissipation, originating QNMs. The QNMs are the natural modes of relaxation of a black hole. They are characterized by the absence incoming waves from infinity.



In terms of the scattering problem, we can think of the QNMs as poles of the scattering coefficients, i.e.

$$\omega \in \mathbb{C} \text{ such that } 1/T_\omega \to 0 \text{ and } 1/R_\omega \to 0. \tag{30}$$



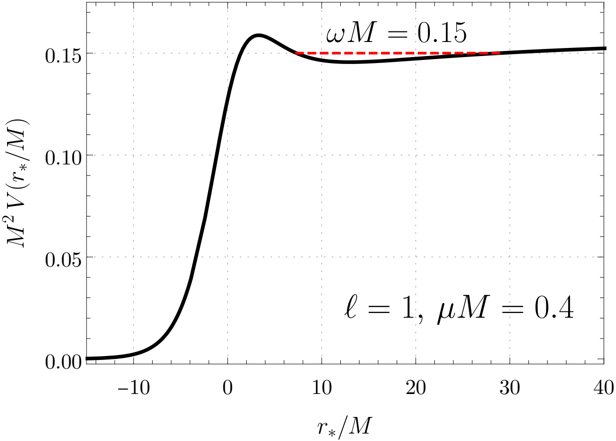
Writing $\boxed{\omega = \omega_R + i\omega_I}$ and $\boxed{k = k_R + ik_I}$ (without loss of generality, we assume $\omega_R > 0$ and $k_R > 0$), we have (at $r \to \infty$):

$$\phi_{\omega\ell m} \propto \exp\left(-i\,\omega\,t + i\,k\,x\right) = \exp\left[-i\,\omega_R\left(t - \frac{k_R}{\omega_R}\,r\right)\right]\exp\left(\omega_I\,t\right)\exp\left(-k_I\,r\right)$$

– QNMs are characterized by $\boxed{\omega_I < 0,}$ leading to energy dissipation. If $\boxed{\omega_I > 0,}$ the corresponding mode is an unstable mode.

– QNMs are not normalizable - since $k_I < 0$, they are not localized in space, diverging in the limit $r \to \infty$.

Unlike QNMs, the QBSs are localized in space (far away from the black hole, they decay exponentially). The QBSs remain localized inside the potential well generated by the mass of the field:



- As in the case of QNMs, due to the boundary conditions the frequency of the QBSs are complex numbers.

- Writing $\boxed{\omega = \omega_R + i\omega_I}$ and $\boxed{k = k_R + ik_I}$ (without loss of generality, we assume $\omega_R > 0$ and $k_I > 0$), we find that (at $r \to \infty$):

$$\phi_{\omega \ell m} \propto \exp\left(-i\,\omega\,t + i\,k\,r\right) = \exp\left[-i\,\omega_R\,\left(t - \frac{k_R}{\omega_R}\,r\right)\right] \exp\left(\omega_I\,t\right) \exp\left(-k_I\,r\right)$$

- QBSs have $\boxed{\omega_I < 0.}$ If $\boxed{\omega_I > 0,}$ the corresponding mode is unstable.

Summary: scattering, QNMs, QBSs, instabilities

In all cases, we impose the boundary condition that requires only ingoing waves at the event horizon. What differentiate the modes is the boundary condition at infinity:

- **Scattering modes:** real frequency ($\omega_I = 0$) is given; boundary condition not required at infinity (both ingoing and outgoing waves).
- **QNMs:** complex frequency ($\omega_I < 0$) to be determined; boundary condition at infinity required (only outgoing wave).
- **EQLs:** complex frequency ($\omega_I < 0$) to be determined; boundary condition at infinity required (only exponentially decaying wave). Only exists when $\mu \neq 0$.

– In general, there is a discrete set (infinite, but countable) of QNMs and QBSs.

– If $\omega_I > 0$ is found, instead of a QNM or a QBS, we have an unstable mode. In Schwarzschild, unstable modes are not allowed.

The study of scalar fields in Kerr is similar to Schwarzschild, but there are important differences that we will highlight in the next few slides.

Since the Kerr spacetime is stationary and axisymmetric, we separate the coordinates in the KG equation using the *ansatz*

$$\Phi(t,r,\chi,\phi) = \exp\left(-i\omega t + im\phi\right)\tilde{\Phi}(r,\chi) = \exp\left(-i\omega t + im\phi\right)R(r)S(\chi).$$

The functions $R(r)$ e $S(\chi)$ satisfy (see notebook):

$$\Delta^2\frac{d^2R}{dr^2} + \Delta\frac{d\Delta}{dr}\frac{dR}{dr} + \left[K^2(r) - \left(\lambda + \mu^2 r^2\right)\Delta\right]R = 0, \tag{31}$$

$$(1-\chi^2)\frac{d^2S}{d\chi^2} - 2\chi\frac{dS}{d\chi} + \left(a^2k^2\chi^2 + A - \frac{m^2}{1-\chi^2}\right)S = 0, \tag{32}$$

where $k^2 = \omega^2 - \mu^2$, $\Delta = (r-r_+)(r-r_-)$, $K = \omega\left(r^2+a^2\right) - am$, $\lambda = A + a^2\omega^2 - 2am\omega$, and A is a separation constant. Recall that $\Delta = r^2 - 2Mr + a^2 = (r-r_+)(r-r_-)$.

The equation for $S(\chi)$ is called the spheroidal equation:

$$(1-\chi^2)\frac{d^2S}{d\chi^2} - 2\chi\frac{dS}{d\chi} + \left(a^2k^2\chi^2 + A - \frac{m^2}{1-\chi^2}\right)S = 0.$$

- The solutions $S(\chi)$ are well behaved (non-singular) at the poles (i.e. $\chi = -1$ and $\chi = 1$) only for a discrete set of constants A_ℓ, $\ell \in \mathbb{Z}$.
- The associated functions $S_{\ell m}(\chi)\exp(im\phi)$ are called spheroidal harmonics.
- In general, there is no analytic expression for the eigenvalues A_ℓ. But one can show that:

$$A_\ell = \ell(\ell+1) + \mathcal{O}\left(a^2k^2\right)$$

  In other words, when $a = 0$ we have $A_\ell = \ell(\ell+1)$ and the spheroidal harmonics reduce to the spherical harmonics.

The radial equation can be put into a Schrödinger like form if we define ψ and r_* through [recall that $\Delta = (r - r_+)(r - r_-)$]:

$$\psi_{\omega\ell}(r) = \sqrt{r^2 + a^2} R_{\omega\ell}(r), \qquad \frac{dr_*}{dr} = \frac{r^2 + a^2}{\Delta} \tag{33}$$

At the horizon ($r \to r_+$), we have $r_* \to -\infty$. At infinity ($r \to +\infty$), we have $r_* \to +\infty$. The radial equation (31), in terms of ψ and r_*, is

$$\frac{d^2\psi_{\omega\ell}}{dr_*^2} + \left[\omega^2 - V(\omega, r)\right] \psi_{\omega\ell} = 0. \tag{34}$$

Unlike Schwarzschild, in Kerr the potential V depends on ω. To save space, we do not write it explicitly (see notebook). For now, it is enough to know that (recall that $\Omega = a/2Mr_+$):

$$\text{Em } r_* \to -\infty : \quad V \to (\omega - m\Omega)^2 = \widetilde{\omega}^2 \;\Rightarrow\; \psi_{\omega\ell} \propto \exp\left(\pm i\widetilde{\omega} r_*\right) \tag{35}$$

$$\text{Em } r_* \to \infty : \quad V \to \omega^2 - \mu^2 = k^2 \;\Rightarrow\; \psi_{\omega\ell} \propto \exp\left(\pm i k r_*\right) \tag{36}$$

The most general solution of the radial equation is, therefore:

$$\psi_{\omega\ell}(r_*) \approx \begin{cases} A_1 \exp\left(-i\,\widetilde{\omega}\,r_*\right) + A_2 \exp\left(+i\,\widetilde{\omega}\,r_*\right), & r_* \to -\infty, \\ B_1 \exp\left(-i\,k\,r_*\right) + B_2 \exp\left(+i\,k\,r_*\right), & r_* \to +\infty, \end{cases} \tag{37}$$

where $\widetilde{\omega} = \omega - m\Omega$ and $k^2 = \omega^2 - \mu^2$.

- The boundary condition at the horizon implies $A_2 = 0$. Imposing a bounding condition at infinity, we can study QNMs and QBSs.
- Note that $\partial\omega/\partial\widetilde{\omega}$ is always positive, but the signal of $\omega/\widetilde{\omega}$ depends on the relation between ω and $m\Omega$. It means that, although the ingoing wave at the event horizon always has a negative group velocity, it may have a positive phase velocity if $\omega < m\Omega$.
- Repeating the wronskian analysis, we find:

$$R_\omega + T_\omega = 1, \quad \text{where} \quad R_\omega = \left|\frac{B_2}{B_1}\right|^2 \quad \text{e} \quad T_\omega = \frac{\tilde{\omega}}{k}\left|\frac{A_1}{B_1}\right|^2 \tag{38}$$

When $0 < \omega < m\Omega$, **superradiance** happens: $T_\omega < 0$ and $R_\omega > 1$.

Summary: differences between Schwarzschild and Kerr

The main differences when studying scalar fields around Schwarzschild and Kerr are:

- In Scharzschild, the angular part is a spherical harmonic, with eigenvalue $\ell(\ell+1)$. In Kerr, we have a **spheroidal harmonic** instead, and there is no analytic expression for the eigenvalue.
- Scattering by a Schwarzschild black hole always produces $0 \leq R_\omega \leq 1$. In Kerr, however, if the frequency is sufficiently small ($0 < \omega < m\Omega$), we have a **superradiant scattering**, which is characterized by $\boxed{R_\omega > 1.}$
- In Schwarzschild, unstable solutions do not exist: when looking for QNMs and QBSs, we always find $\mathrm{Im}(\omega) < 0$. In Kerr, on the other hand, unstable modes $[\mathrm{Im}(\omega) < 0]$ exist if the field is massive. These unstable modes are associated with superradiance: **superradiant instabilities**.

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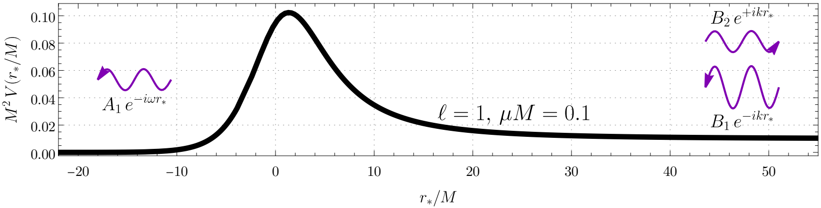
Part II:

Numerical Methods - Scattering in the frequency domain and in the time domain. The continued fraction method (CFM) for QNMs and for QBSs.

We assume that the scattering process is a stationary process: there is a constant flux of waves, and the Kerr background is fixed.



$$\Delta^2 \frac{d^2R}{dr^2} + \Delta \frac{d\Delta}{dr}\frac{dR}{dr} + \left[K^2(r) - \left(\lambda + \mu^2 r^2\right)\Delta\right] R = 0, \tag{39}$$

$$(1-\chi^2)\frac{d^2S}{d\chi^2} - 2\chi\frac{dS}{d\chi} + \left(a^2k^2\chi^2 + A - \frac{m^2}{1-\chi^2}\right) S = 0, \tag{40}$$

where $k^2 = \omega^2 - \mu^2$, $\Delta = (r - r_+)(r - r_-)$, $K = \omega\left(r^2 + a^2\right) - am$, $\lambda = A + a^2\omega^2 - 2am\omega$, and A is a separation constant.

To solve the radial equation, we need to first determine the eigenvalues A of the spheroidal equation.

– Mathematica has it implemented:

$$A = A_{\ell m} = -k^2 a^2 + \mathrm{SpheroidalEigenvalue}[l, m, i\, k\, a] \tag{41}$$

– For electromagnetic or gravitational perturbations, the angular equation is slightly different (i.e. the spin-weighted spheroidal equation) and there is no native implementation (see, however, the Black hole Perturbation Toolkit - `http://bhptoolkit.org`)

– Here, instead of using the native Mathematica function, I will explain how to determine the eigenvalues of the spheroidal equation using the continued fraction method (CFM).

– The continued fraction method is also known as Leaver's method in the context of black holes. It was introduced by Leaver for the calculation of QNMs and is based on a 1934 paper by Jaffé about the spectrum of a hydrogen ion.

– The basic idea is to write a series solution for the differential equations that automatically satisfies the boundary conditions. The requirement that the series satisfies the differential equation translates into a recursion relation between the series coefficients. One can then show that the series is everywhere convergent if and only if a certain equation involving a continued fraction is satisfied. By solving this algebraic equation numerically, one determines the eigenvalues of the differential equation.

We want to find the eigenvalues A of

$$(1-\chi^2)\frac{d^2S}{d\chi^2} - 2\chi\frac{dS}{d\chi} + \left(a^2k^2\chi^2 + A - \frac{m^2}{1-\chi^2}\right)S = 0$$

which are compatible with the requirement that $S(\chi)$ be regular at $\chi = -1$ and at $\chi = 1$.

– We note that $\chi = \pm 1$ are regular singular points of the ODE while $\chi = \infty$ is an irregular singular point.

– Using the Frobenius method, one finds that the asymptotic solutions near $\chi = -1$ and $\chi = 1$ have, repectively, the following form:

$$S(\chi) \sim (1+\chi)^{\pm\frac{m}{2}} \text{ and } S(\chi) \sim (1-\chi)^{\pm\frac{m}{2}} \tag{42}$$

– Consequently, we want to find the eigenvalues A that imply a regular solution S, meaning that

$$S(\chi) \to (1\pm\chi)^{\frac{|m|}{2}} \quad \text{when} \quad \chi \to \mp 1. \tag{43}$$

Considering the asymptotic solutions, we introduce the *ansatz*

$$S(\chi) = \exp\left(a\,k\,\chi\right) \cdot (1-\chi)^{\frac{|m|}{2}} \cdot (1+\chi)^{\frac{|m|}{2}} \cdot \sum_{n=0}^{\infty} b_n\,(1+\chi)^n\,. \qquad (44)$$

Plugging the expression above into the differential equation, one finds that the b_n must satisfy the following 3-term recurrence relation:

$$\begin{cases} \alpha_0 b_1 + \beta_0 b_0 = 0, \\ \alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} = 0, \quad n \geq 1, \end{cases} \qquad (45)$$

where the coefficients α_n, β_n, and γ_n are:

$$\begin{cases} \alpha_n = -2(n+1)(|m|+n+1), \\ \beta_n = -A + |m|\,(-2ak+2n+1) - ak(ak+2) + m^2 + n^2 - 4akn + n, \\ \gamma_n = 2a\,k(|m|+n). \end{cases}$$

If we define $R_n = b_n/b_{n-1}$, we can rewrite the recurrence relation as

$$\frac{-\gamma_n}{\beta_n + \alpha_n R_{n+1}} = R_n \tag{46}$$

Therefore:

$$R_1 = \frac{-\gamma_1}{\beta_1 + \alpha_1 R_2} \text{ and } R_2 = \frac{-\gamma_2}{\beta_2 + \alpha_2 R_3} \Rightarrow R_1 = \frac{-\gamma_1}{\beta_1 - \frac{\alpha_1\gamma_2}{\beta_2 + \alpha_2 R_3}} \tag{47}$$

Repeating the idea for $R_3, R_4, \ldots$, we find:

$$R_1 = \cfrac{-\gamma_1}{\beta_1 - \cfrac{\alpha_1\gamma_2}{\beta_2 - \cfrac{\alpha_2\gamma_3}{\beta_3 - \ldots}}} \tag{48}$$

From Eq. (45), we also have $R_1 = b_1/b_0 = -\beta_0/\alpha_0$. Hence, we obtain the main equation of the CFM:

$$0 = \beta_0 - \cfrac{\alpha_0\gamma_1}{\beta_1 - \cfrac{\alpha_1\gamma_2}{\beta_2 - \cfrac{\alpha_2\gamma_3}{\beta_3 - \dots}}} \tag{49}$$

Given a, k and m, the coefficients α_n, β_n, γ_n only depend on the eigenvalue A. The infinite continued fraction, therefore, is an equation for A. In order to solve the equation numerically, we need to truncate the continued fraction at some order $n = N$. In other words, we stop the recursion process at R_N for a sufficiently large N. According to Leaver's original procedure, we choose $R_N = 0$. Nevertheless, we can improve convergence by using the so-called Nollert's improvement, which gives a prescription for estimating R_N.

– See the notebook for more details on the numerical implementation.

Now that we know how to find the angular eigenvalues, we will learn to calculate the scattering coefficients.

Given a, M, m, μ, and $\omega > \mu$, we want to solve the radial equation

$$\Delta^2 \frac{d^2 R}{dr^2} + \Delta \frac{d\Delta}{dr}\frac{dR}{dr} + \left[K^2(r) - \left(\lambda + \mu^2 r^2\right)\Delta\right] R = 0$$

assuming that

$$\psi(r_*) \approx \begin{cases} A_1 \exp\left(-i\,\widetilde{\omega}\,r_*\right), & r_* \to -\infty, \\ B_1 \exp\left(-i\,k\,r_*\right) + B_2 \exp\left(+i\,k\,r_*\right), & r_* \to +\infty, \end{cases} \tag{50}$$

where $\widetilde{\omega} = \omega - m\Omega$, $\Omega = am/(2Mr_+)$, and $k = \sqrt{\omega^2 - \mu^2}$. Recall that

$$\psi(r) = \sqrt{r^2 + a^2} R_{\omega\ell}(r) \quad \text{and} \quad \frac{dr_*}{dr} = \frac{r^2 + a^2}{\Delta}. \tag{51}$$

To avoid having to invert the relation $r = r(r_*)$, we will solve the equation for $R(r)$ instead of the equation for $\psi(r_*)$.

The first step is to rewrite the asymptotic solutions in terms of R and r instead of ψ and r_*. Without loss of generality, we assume $M = 1$. To save some time and space, we assume $\mu = 0$ (but it is straightforward to repeat for $\mu \neq 0$). We find:

$$R(r) \approx \begin{cases} Y_t\,(r - r_+)^{-i\sigma}\,, & r \to r_+, \\ Y_i r^{-1-2iM\omega} \exp\left(-i\,\omega\, r\right) + Y_o r^{-1+2iM\omega} \exp\left(+i\,\omega\, r\right), & r \to +\infty, \end{cases}$$

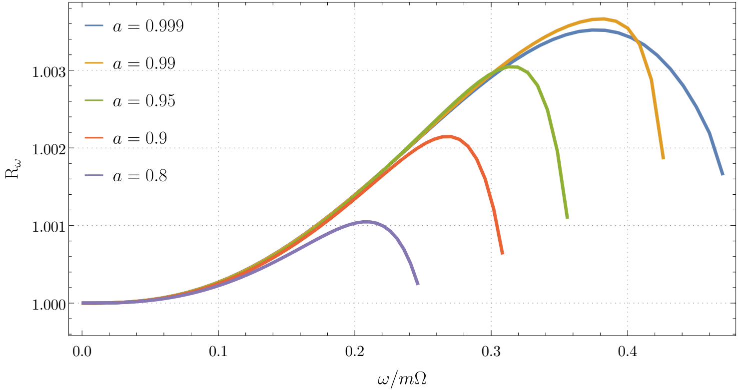
where $\sigma = \frac{2M\widetilde{\omega} r_+}{r_+ - r_-}$, with $\widetilde{\omega} = \omega - m\Omega$.

– We set $Y_t = 1$ and impose initial conditions for R and R' at $r = r_+ + \epsilon$ using the near-horizon behaviour given above.

– We solve the ODE in the domain $[r_+ + \epsilon, r_1]$, where $r_1 >> r_+$. In the end, we find R and R' at $r = r_1$.

– Using the far-away behaviour given above, we determine Y_i and Y_o from R and R'. The reflection coefficient is then calculated.

Repeating the procedure outlined previously for different ω, we find (for $\ell = m = 1$ and selected spin values a):



– Note the occurrence of superradiance. See the notebook for more details on the numerical implementation.

The exact same idea used to determine the eigenvalues of the angular equation can be used to determine the eigenvalues of the radial equation:

$$\Delta^2 \frac{d^2R}{dr^2} + \Delta \frac{d\Delta}{dr}\frac{dR}{dr} + \left[K^2(r) - \left(\lambda + \mu^2 r^2\right)\Delta\right] R = 0,$$

where $k^2 = \omega^2 - \mu^2$, $\Delta = (r - r_+)(r - r_-)$, $K = \omega\left(r^2 + a^2\right) - am$, $\lambda = A + a^2\omega^2 - 2am\omega$, and A is a separation constant.

By inserting the appropriate *ansatz* into the ODE,

$$R(r) = \exp\left(-i\,k\,r\right)\left(r - r_-\right)^{-1-2iMk-iM\frac{\mu^2}{k}} \sum_{n=0}^{\infty} \left(\frac{r - r_+}{r - r_-}\right)^{-i\sigma+n} \tag{52}$$

we find a 3-term recurrence relation as before

$$\begin{cases} \alpha_0 b_1 + \beta_0 b_0 = 0, \\ \alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} = 0, \quad n \geq 1. \end{cases} \tag{53}$$

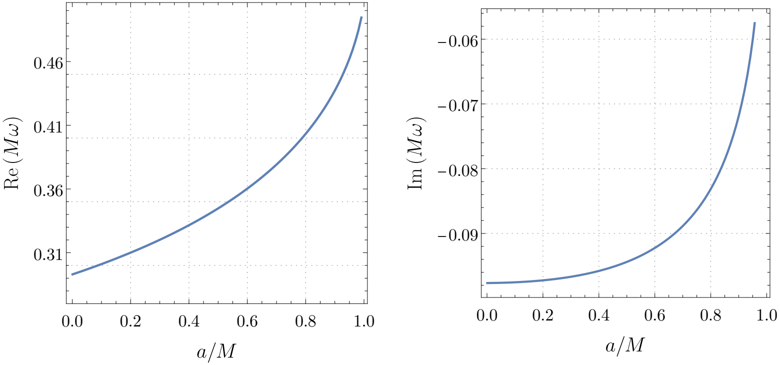
Part II: Quasinormal modes with the CFM

To find the eigenvalues ω and A we have to solve two continued-fraction equations of the form

$$0 = \beta_0 - \cfrac{\alpha_0\gamma_1}{\beta_1 - \cfrac{\alpha_1\gamma_2}{\beta_2 - \cfrac{\alpha_2\gamma_3}{\beta_3 - \ldots}}}, \tag{54}$$

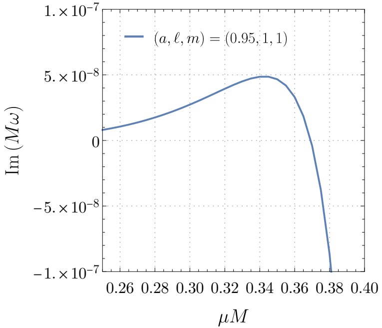
one associated with the radial ODE and another associated with the angular ODE. We need to solve them simultaneously, since in both cases the coefficients α_n, β_n, and γ_n depend on ω and A. We use a root-finding algorithm for that (initial guesses for ω and A are needed).

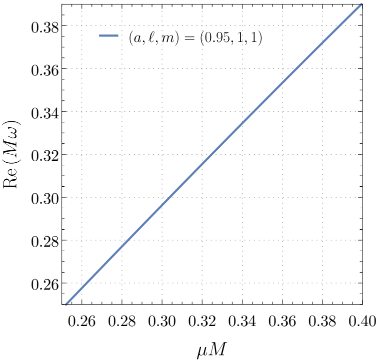
When $\mu = 0$ and $\ell = m = 1$, the fundamental (i.e. the most stable) QNM is given, as a function of the spin a, by (see notebook for details):



Note that these modes are stable, i.e. they have $\mathrm{Im}(\omega) < 0$.

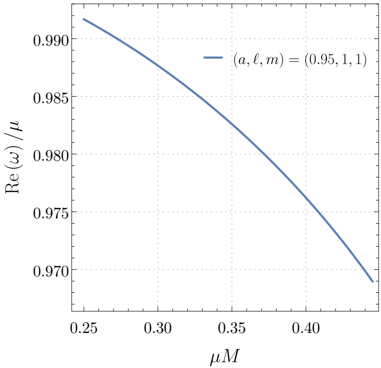
We can find QBSs using exactly the same procedure used to determine QNMs. When $a = 0.95$ and $\ell = m = 1$, the least energetic (i.e. lowest oscillation frequency) QBS is given, as a function of the mass μ, by (see notebook for details):

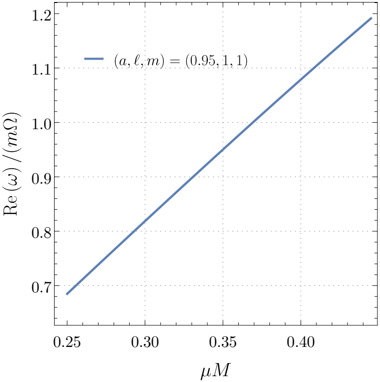




Note that these modes are unstable if μ is sufficiently small.

We can check that these modes are indeed QBSs by comparing the frequency ω with the mass μ of the scalar field. Recalling that QBSs can be understood as modes that are confined inside the potential well, we can verify if $\mathrm{Re}\,(\omega) < \mu$ or not. We can also check if they satisfy the superradiance condition ($\mathrm{Re}\,(\omega) < m\Omega$) or not:





Going back to the scattering problem, we can study differential equation without separating the time dependence. To illustrate the idea, we consider the scattering of a massless scalar field $\psi(t,r)Y_{\ell m})$ by a Schwarzschild black hole (*arXiv:gr-qc/9307009*):

$$\frac{\partial^2\psi}{\partial t^2} - \frac{\partial^2\psi}{\partial r_*^2} + V(r_*) = 0, \quad (55)$$

where $V(r) = \left(1 - \frac{2M}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3}\right)$ and $r_* = r + 2M\ln\left(\frac{r}{2M} - 1\right)$. Instead of (t, r_*), we use the light-cone coordinates (u, v), where $u = t - r_*$ is the retarded time and $v = t + r_*$ is the advanced time. In terms of u and v, the wave equation becomes:

$$4\frac{\partial^2\psi}{\partial u\partial v} + V(u,v)\psi = 0, \quad (56)$$

where $\psi = \psi(u, v)$ now. We discretize the equation and solve it subject to the initial data (which is typically assumed to be a Gaussian).

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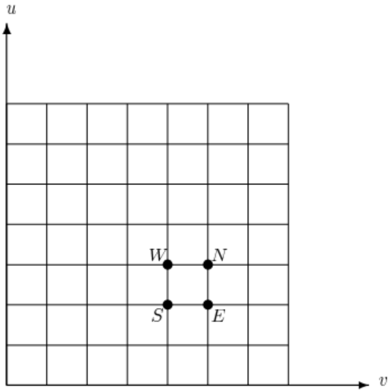
– Initial data is specified at $u = 0$ and at $v = 0$.

– Discretization of the wave equation yields (h is the cell length):

$$\psi(N) = \psi(W) + \psi(E) - \psi(S) - \frac{h^2}{8} V(S) \left[\psi(W) + \psi(E)\right] + \mathcal{O}\left(h^4\right)$$

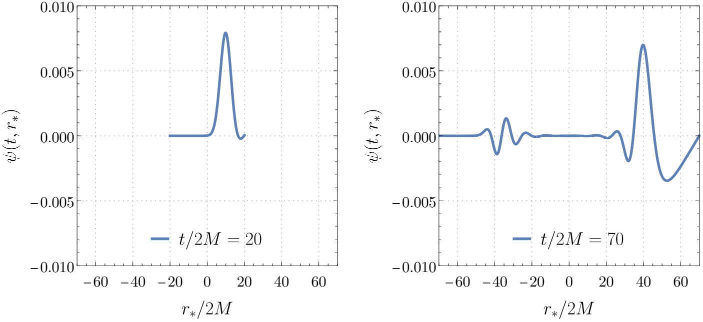
– Using the initial data and the equation above, we can determine ψ at every point of the grid.

– Converting back from (u, v) to (t, r_*), we find $\psi = \psi(t, r_*)$.



Ref: Konoplya, Zhidenko - Rev.Mod.Phys.83, 793 (2011)

After evolving a Gaussian initial data, we can observe the scattering process (see notebook). For $\ell = 1$, we compare the field ψ at the values $t = 40M$ and $t = 140M$:



On the left, we see the incident wave, going towards $r_* = -\infty$. On the right, we see the reflected wave going towards $r_* = \infty$ and the transmitted wave (smaller amplitude) going towards $r_* = -\infty$.

We can also plot $|\psi|$ as a function of t for a fixed r_*. We can identify three phases of the process: the passage of the initial perturbation, the quasinormal ringing, and the so-called late-time tail.