Hidden Symmetries

Lecture 1

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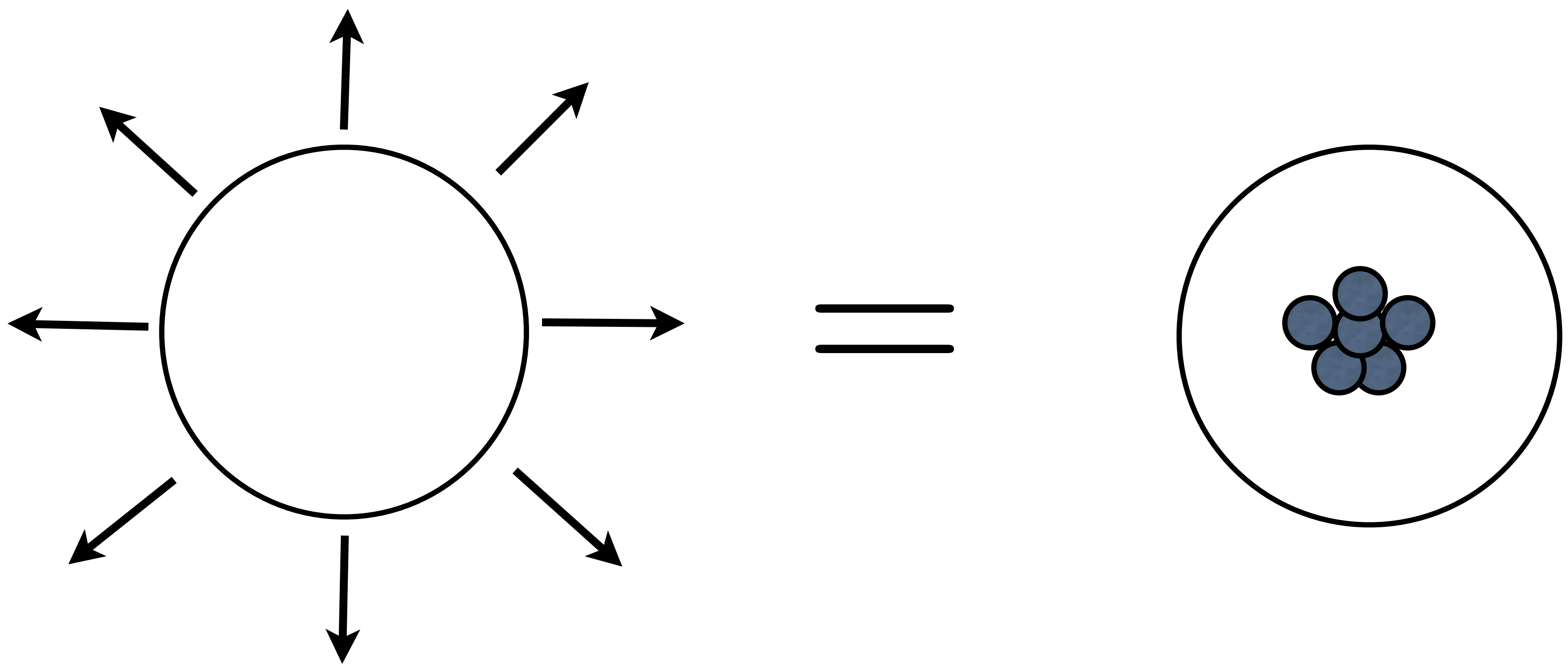
The Main Idea



Flux Charge

- That is a conservation law that leads to an isospectral evolution

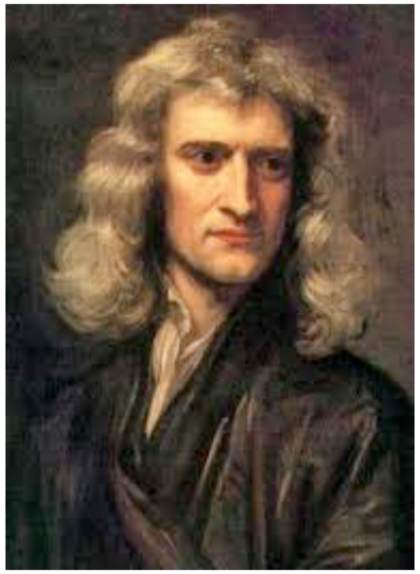
$$V(t) = U\, V(0)\, U^{-1} \qquad \text{(non-Noether)}$$

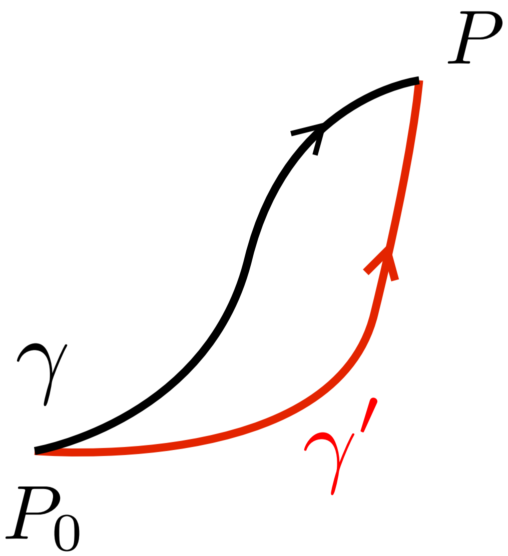
- It underlies gauge theories

- It underlies integrable theories

- Use it to construct the integral eqs. for Yang-Mills, its conserved charges and hidden symmetries in loop space

Hidden Symmetries in a nutshell: Conservative Forces





$$\text{Work}(P_0 \to P) = \int_\gamma \vec{F} \cdot d\vec{l} = \int_{\gamma'} \vec{F} \cdot d\vec{l}$$

Potential $$U(P) = -\int_{P_0}^{P} \vec{F} \cdot d\vec{l}$$ on any curve from P_0 to P

(Stokes Theorem) $\vec{\nabla} \wedge \vec{F} = 0 \longrightarrow \partial_i F_j - \partial_j F_i = 0$ → flat connection

Theorem: Work done = variation of kinetic energy

Conservation of mechanical energy $$E = \frac{p^2}{2\,m} + U$$

Energy as an eigenvalue

$$A = \frac{1}{2}\begin{pmatrix} p & K(q) \\ K(q) & -p \end{pmatrix} \xrightarrow{\det(A - \lambda 1) = 0} \lambda^2 = \frac{1}{2}\left(\frac{p^2}{2} + \frac{K^2}{2}\right)$$

$U = \frac{1}{2}K^2$

Look for iso-spectral evolution

$$A(t) = U(t)\,A(0)\,U^{-1}(t) \longrightarrow \frac{dA}{dt} = \left[\frac{dU}{dt}U^{-1},\, A\right]$$

Take $$B = -\frac{dU}{dt}U^{-1} = \frac{1}{2}\begin{pmatrix} 0 & \frac{dK}{dq} \\ -\frac{dK}{dq} & 0 \end{pmatrix}$$

$$\frac{dA}{dt} - [A,\, B] = \frac{1}{2}\left[\frac{dp}{dt} + \frac{dU}{dq}\right]\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 0$$

Newton's equation

Hidden symmetry

$$A \to g\,A\,g^{-1}$$

$$B \to g\,B\,g^{-1} - \frac{dg}{dt}g^{-1}$$

$g \in SU(2)$

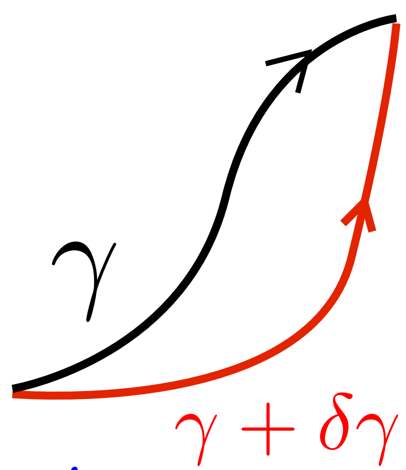
Path Independency

Introduce "fake" variable x and denote $\quad A_x \equiv A \qquad A_t \equiv B$

$$F_{tx} \equiv \partial_t A_x - \partial_x A_t + [A_t\,,\, A_x] = \partial_t A - [A\,,\, B] = 0$$

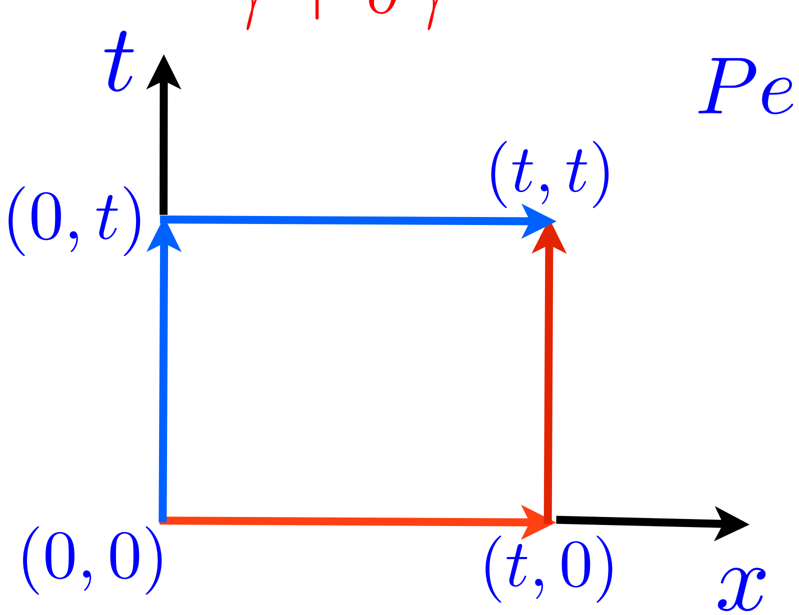
Wilson line $\quad \dfrac{dW}{dt} + A_\mu \dfrac{d\,x^\mu}{d\,\sigma}\, W = 0 \longrightarrow W = P\, e^{-\int_\gamma d\sigma A_\mu \frac{d\,x^\mu}{d\,\sigma}}$

$$W = 1 - \int_{\sigma_0}^{\sigma} d\sigma'\, A_\mu \frac{dx^\mu}{d\sigma'} + \int_{\sigma_0}^{\sigma} d\sigma'\, A_\mu(\sigma')\frac{dx^\mu}{d\sigma'} \int_{\sigma_0}^{\sigma'} d\sigma''\, A_\nu(\sigma'')\frac{dx^\nu}{d\sigma''} - \dots$$



Newton's equation

$$W^{-1}\,\delta W = \int_\gamma d\sigma\, W^{-1} F_{\mu\nu}\, W \frac{d\,x^\mu}{d\,\sigma}\,\delta x^\nu = 0$$



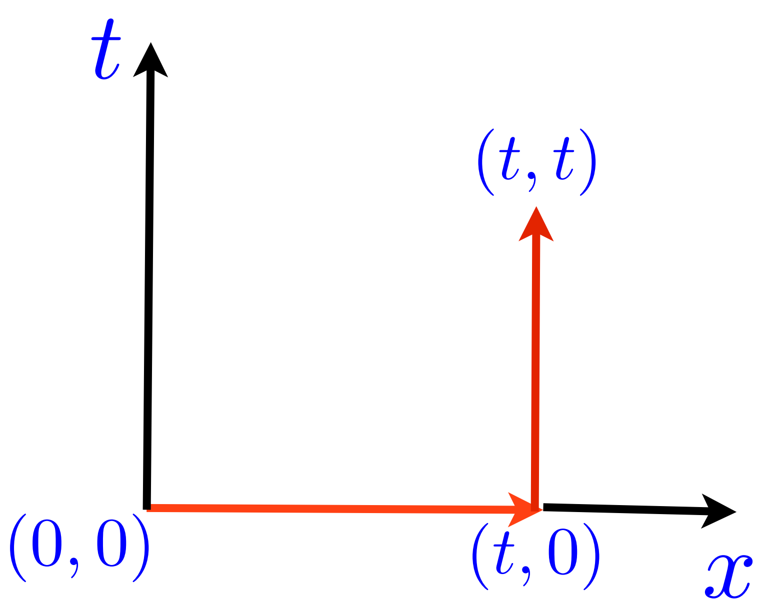
$$Pe^{-\int_0^t dx\, A(t)}\; Pe^{-\int_0^t dt'\, B(t')} = Pe^{-\int_0^t dt'\, B(t')}\; Pe^{-\int_0^t dx\, A(0)}$$

$$e^{-t\,A(t)} = U(t)\, e^{-t\,A(0)}\, U^{-1}(t)$$

$$U(t) = Pe^{-\int_0^t dt'\, B(t')}$$

$$A(t) = U(t)\, A(0)\, U^{-1}(t)$$

The general solution



$$W(t) = Pe^{-\int_0^t dt'\, B(t')}\, Pe^{-\int_0^t dx\, A(0)}\, W(0)$$
$$= U\,(t)\; e^{-t\, A(0)}\, W\,(0)$$

But A is hermitian and B anti-hermitian   (U is unitary)

$$A = \frac{1}{2}\begin{pmatrix} p & K\,(q) \\ K\,(q) & -p \end{pmatrix} \qquad B = \frac{1}{2}\begin{pmatrix} 0 & \frac{d\,K}{d\,q} \\ -\frac{d\,K}{d\,q} & 0 \end{pmatrix}$$

initial data

$$X \equiv W^\dagger\, W = W(0)^\dagger\, e^{-2\,t\, A(0)}\, W(0)$$

Another way of looking at it

$$A = \frac{1}{2}\begin{pmatrix} p & K(q) \\ K(q) & -p \end{pmatrix} = p\,T_3 + K\,T_1 \qquad B = \frac{1}{2}\begin{pmatrix} 0 & \frac{dK}{dq} \\ -\frac{dK}{dq} & 0 \end{pmatrix} = i\,\frac{dK}{dq}\,T_2$$

Automorphism: $\sigma(T_3) = -T_3 \qquad \sigma(T_1) = -T_1 \qquad \sigma(T_2) = T_2$

$[T_a\,,\,T_b] = i\,\varepsilon_{abc}\,T_c \qquad \sigma(A) = -A \qquad \sigma(B) = B \qquad \sigma(U) = U$

But $\qquad W = U(t)\,e^{-t\,A(0)}\,W(0)$

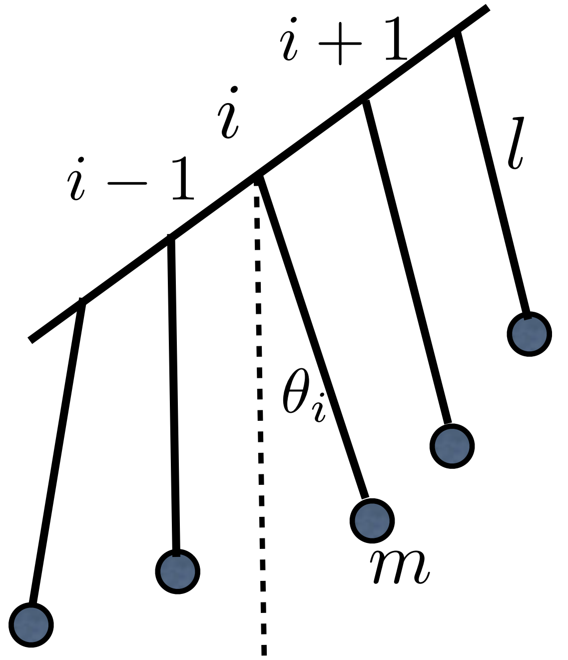
Then $\qquad X(W) \equiv \sigma(W)^{-1}\,W = \sigma(W(0))^{-1}\,e^{-2\,t\,A(0)}\,W(0)$

Note $\qquad X(h\,W) = X(W) \qquad h \in U(1)$

Parametrizes the symmetric space $SU(2)/U(1)$

Phase space of a particle in 1-d

Hidden Symmetry in a coconut shell: Pendula on a clothes line



Newton's equation $(s_i \equiv l\,\theta_i)$

$$m\frac{d^2 s_i}{dt^2} = F_i^g + F_i^\tau$$

$$F_i^g \to -m\,g\,\sin\theta_i \qquad F_i^\tau \to -\frac{\alpha}{l}\,(\theta_i - \theta_{i-1}) + \frac{\alpha}{l}\,(\theta_{i+1} - \theta_i)$$

Define $\theta_i \equiv \theta(x_i)$ $\quad x_i = i\,\Delta$ $\quad \Delta \equiv$ spacing between pendula

$$\frac{\theta_{i+1} - 2\,\theta_i + \theta_{i-1}}{\Delta^2} \to \frac{\partial^2\theta}{\partial x^2} \qquad \Delta \to 0$$

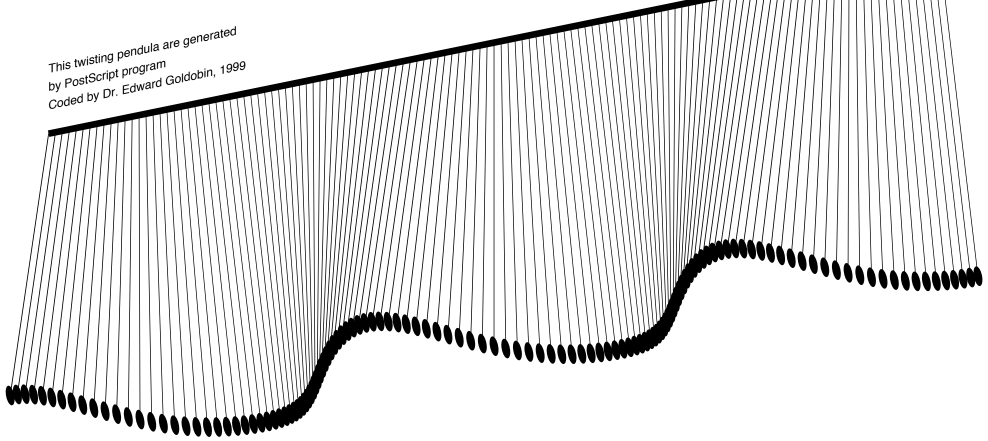
In the limit $\Delta \to 0$, and $\alpha \to \infty$, with $\Delta^2\,\alpha \equiv$ finite, we get

sine-Gordon eq. $$\frac{1}{c^2}\,\frac{\partial^2\theta}{\partial t^2} - \frac{\partial^2\theta}{\partial x^2} = -\frac{\omega^2}{c^2}\,\sin\theta \qquad \left(c^2 = \frac{\alpha\,\Delta^2}{m\,l^2}\;;\;\omega^2 = \frac{g}{l}\right)$$

Small Oscillations

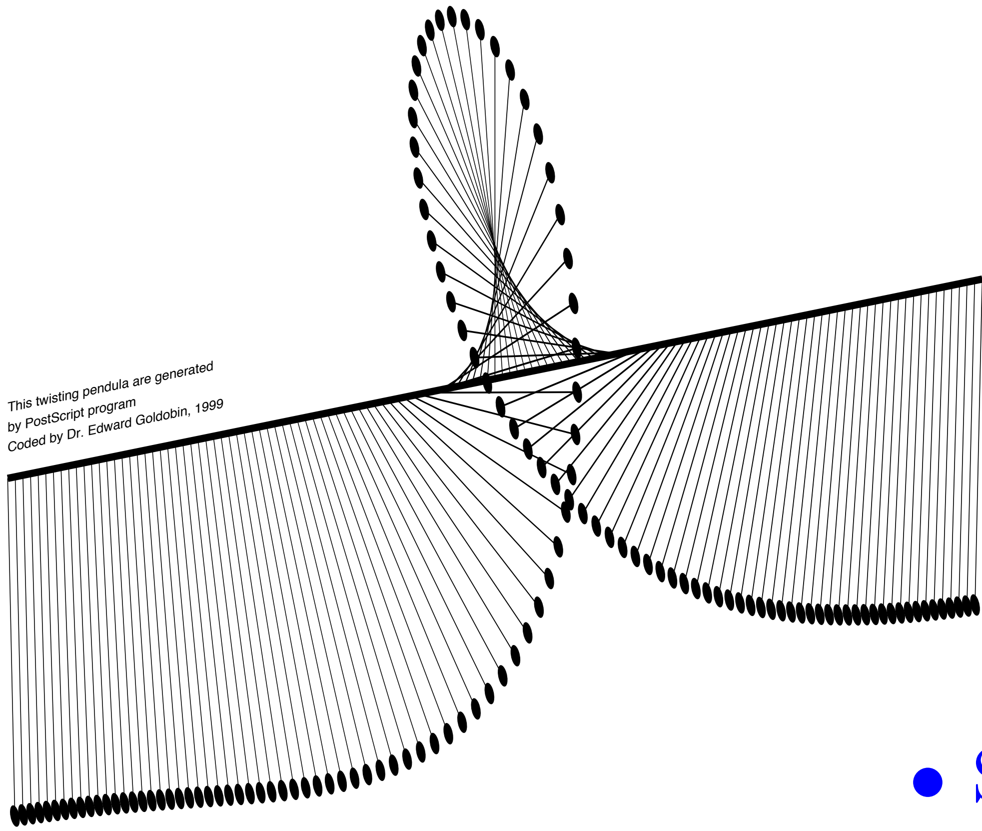
$$\frac{1}{c^2}\,\frac{\partial^2\theta}{\partial t^2} - \frac{\partial^2\theta}{\partial x^2} = -\frac{\omega^2}{c^2}\,\theta \qquad\qquad (\sin\theta \sim \theta)$$

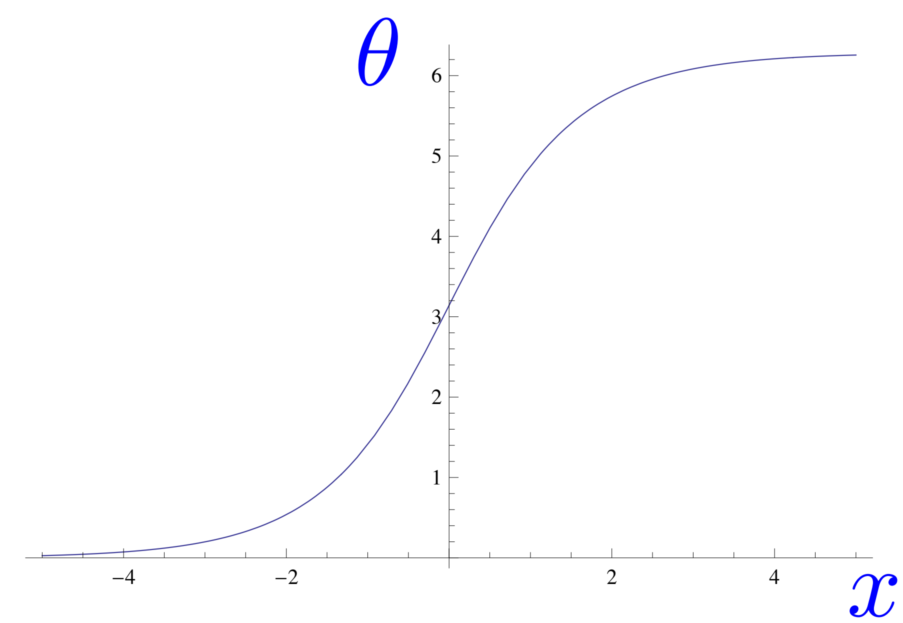
$$\theta = \theta_0\,\cos(k\,x - \Omega\,t + \delta) \qquad\qquad k^2 = \frac{\Omega^2 - \omega^2}{c^2}$$



Solitons Solutions

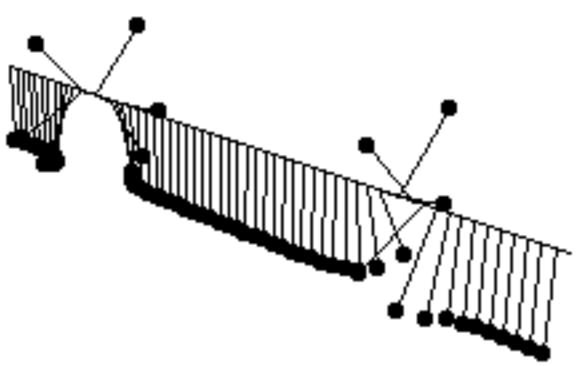
$$\theta = 4\,\mathrm{ArcTan}\left[e^{\gamma(x - v\,t)/\sqrt{1 - v^2/c^2}}\right] \qquad \gamma = \pm\frac{\omega}{c}$$





- Solitons propagate without dissipating energy

- Under collision they are not destroyed. Suffer a time delay only



- They have a "topology"

- They become particles in the quantum theory (Thirring model)

The magic of it ...

$$A_0 = \frac{i}{4} \begin{pmatrix} \frac{\partial\theta}{\partial x^1} & \frac{\omega}{c}\left(e^{i\,\theta/2} + \frac{1}{\lambda}\,e^{-i\,\theta/2}\right) \\ \frac{\omega}{c}\left(e^{i\,\theta/2} + \lambda\,e^{-i\,\theta/2}\right) & -\frac{\partial\theta}{\partial x^1} \end{pmatrix}$$

$$A_1 = \frac{i}{4} \begin{pmatrix} \frac{\partial\theta}{\partial x^0} & \frac{\omega}{c}\left(e^{i\,\theta/2} - \frac{1}{\lambda}\,e^{-i\,\theta/2}\right) \\ -\frac{\omega}{c}\left(e^{i\,\theta/2} - \lambda\,e^{-i\,\theta/2}\right) & -\frac{\partial\theta}{\partial x^0} \end{pmatrix}$$

$$x^0 = c\,t$$

$$x^1 = x$$

Note that λ is arbitrary

$$F_{01} \equiv \partial_0 A_1 - \partial_1 A_0 + [A_0\,,\,A_1] = \frac{i}{4}\left(\partial_0^2\theta - \partial_1^2\theta + \frac{\omega^2}{c^2}\,\sin\theta\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A_\mu \to g\,A_\mu\,g^{-1} + \partial_\mu g\,g^{-1} \qquad\qquad g = e^{i\,\zeta_a^m\,T_a^m}$$

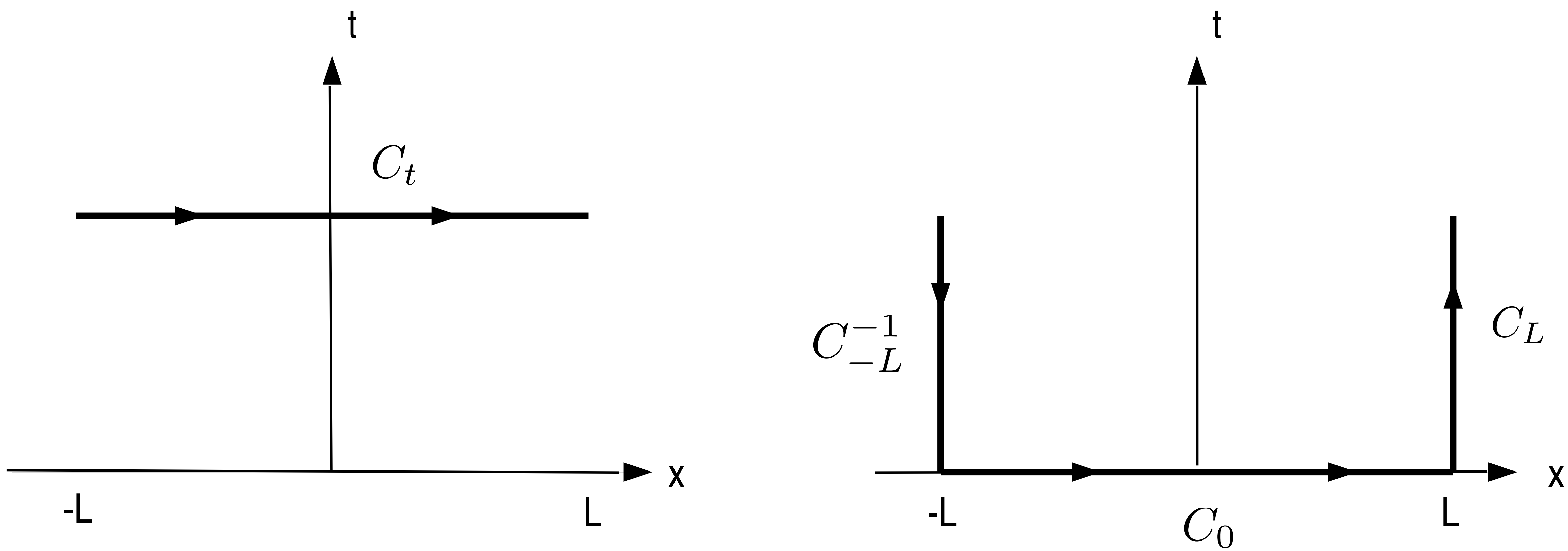
Loop Algebra $\quad \left[T_i^m\,,\,T_j^n\right] = i\,\varepsilon_{ijk}\,T_k^{m+n} \qquad (T_i^n = \lambda^n\,T_i)$

hidden symmetries

Kac-Moody Alg. $\left[T_i^m\,,\,T_j^n\right] = i\,\varepsilon_{ijk}\,T_k^{m+n} + C\,m\,\delta_{m+n,0}\,\delta_{i,j}$

Path independency

$F_{\mu\nu} = 0$ means that $W = P\, e^{-\int_\gamma d\sigma A_\mu \frac{d\,x^\mu}{d\,\sigma}}$ is path independent



Boundary Condition $\quad A_t(-L,t) = A_t(L,t)$

iso-spectral evolution $\quad W(C_t) = U\, W(C_0)\, U^{-1}$

$$U = Pe^{-\int_0^t d\sigma\ A_t(L\,,\,t)\,\frac{d\,t}{d\,\sigma}}$$

power series in λ: infinite number of conserved quantities

Soliton theory in $2d$: quite well established

Flat connections in Kac-Moody algebras

(path independency)

Integrable Field Theories in $(1+1)$-dimensions

Linear Problem $$(\partial_\mu + A_\mu)\Psi = 0$$

Lax-Zakharov-Shabat Equation (zero curvature condition)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu\, ,\, A_\nu] = 0 \qquad\qquad \mu\, ,\, \nu = 0, 1$$

A_μ lives on a Kac-Moody algebra (infinite dimensional)

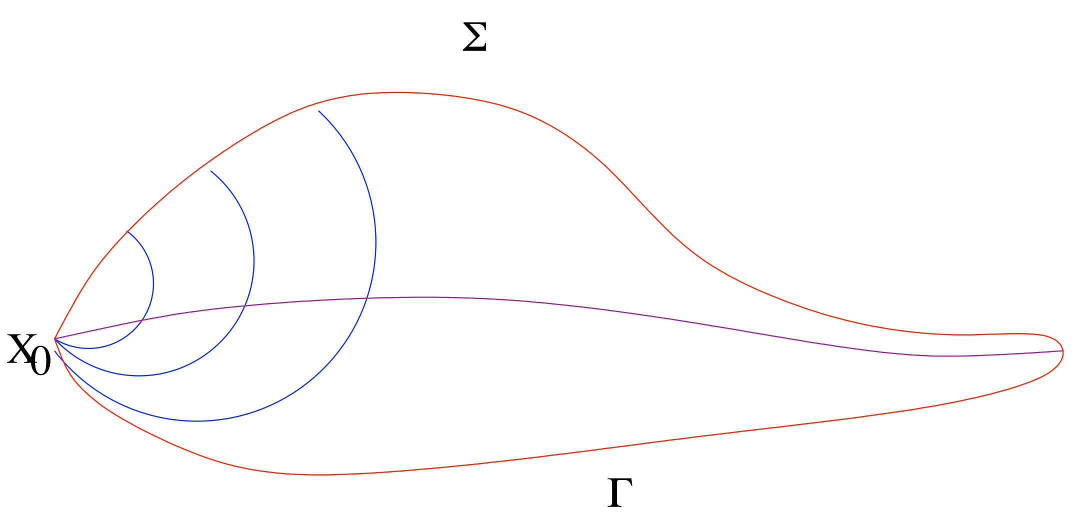
- Infinite number of conservation laws
- Inverse scattering method
- Dressing method
- Hirota method
- Classical r-matrix
- Quantum R-matrix
- Spin chains and $N = 4$ Super Yang-Mills
- etc

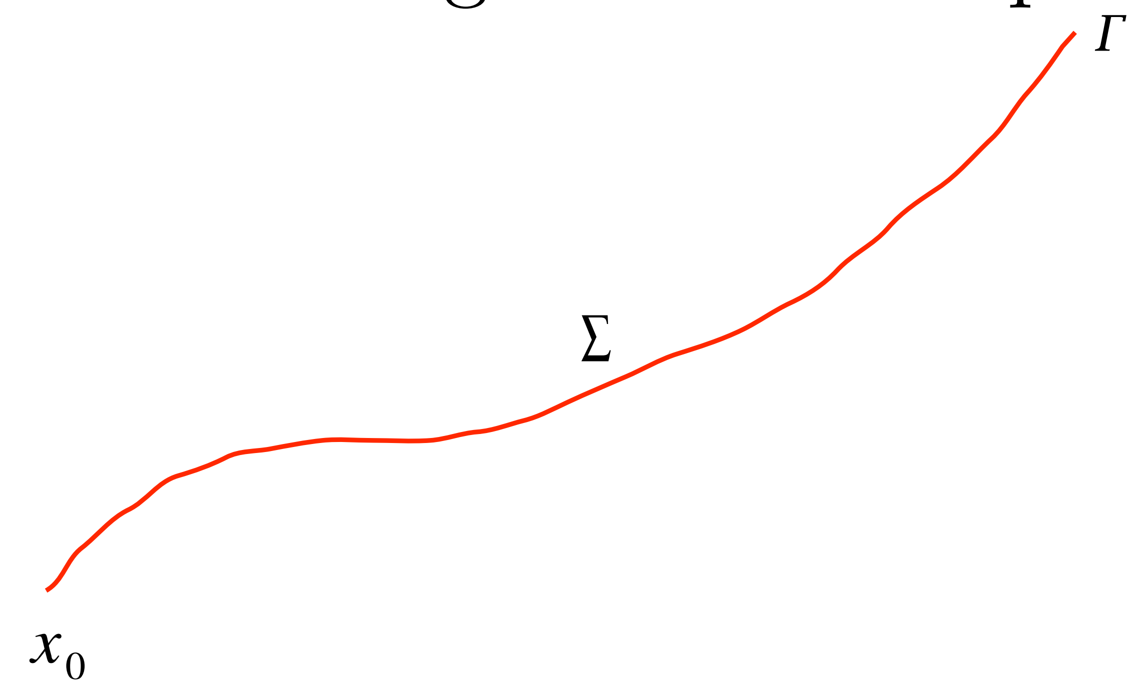
Going to higher dimensions...

How to find the charges Mr. Holmes?

Quite elementary Mr. Watson!

Charges in $(2+1)$-dimensions should be integrals over 2d space





space-time surface ⟷ path in loop space

Loop Space: $\Omega^{(1)} = \{ f : S^1 \to M \mid \text{north pole } \to x_0 \}$

Introduce a flat connection $\mathcal{A}$ in loop space

$$\mathcal{F} = \delta \mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0$$

Construct the charges using path independency!

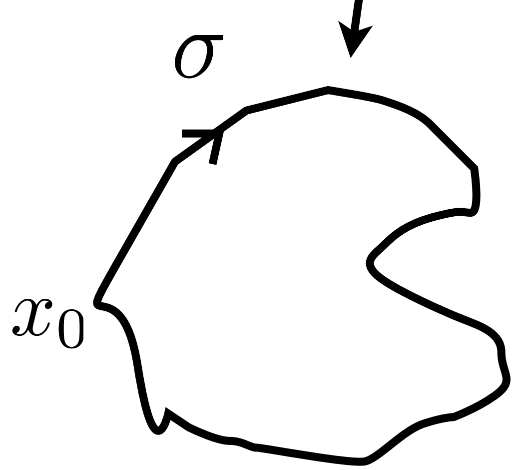
The one-form connection on loop space

$$\mathcal{A}[x(\sigma)] = \int_0^{2\pi} d\sigma \; W(\sigma)^{-1} B_{\mu\nu}(x(\sigma)) W(\sigma) \frac{dx^\mu}{d\sigma} \delta x^\nu(\sigma)$$

antisymmetric tensor

integrated over a loop

$$\frac{dW}{d\sigma} + A_\mu \frac{dx^\mu}{d\sigma} W = 0$$



variations perpendicular to the loop

$$A_\mu \to g\, A_\mu\, g^{-1} + \frac{i}{e} \partial_\mu g\, g^{-1} \qquad \text{then} \qquad W \to g(x)\, W\, g^{-1}(x_0)$$

$$B_{\mu\nu} \to g\, B_{\mu\nu}\, g^{-1} \qquad B^W_{\mu\nu} \to g(x_0)\, B^W_{\mu\nu}\, g^{-1}(x_0) \qquad (B^W_{\mu\nu} \equiv W^{-1} B_{\mu\nu} W)$$

The curvature on loop space

$$\mathcal{F} = \delta\mathcal{A} + \mathcal{A}\wedge\mathcal{A}$$

$$\begin{aligned}\mathcal{F} = & -\frac{1}{2}\int_0^{2\pi} d\sigma\; W^{-1}(\sigma)\left[D_\lambda B_{\mu\nu} + D_\mu B_{\nu\lambda} + D_\nu B_{\lambda\mu}\right](x(\sigma))W(\sigma)\frac{dx^\lambda}{d\sigma}\;\delta x^\mu(\sigma)\wedge\delta x^\nu(\sigma)\\ + & \int_0^{2\pi} d\sigma\int_0^\sigma d\sigma'\left[B^W_{\kappa\mu}(x(\sigma')) - F^W_{\kappa\mu}(x(\sigma')), B^W_{\lambda\nu}(x(\sigma))\right]\frac{dx^\kappa}{d\sigma'}\frac{dx^\lambda}{d\sigma}\;\delta x^\mu(\sigma')\wedge\delta x^\nu(\sigma)\end{aligned}$$

$$D_\mu * \equiv \partial_\mu * + i\,e\,[A_\mu\,,\,*]$$

Problems to have $\mathcal{F} = 0$:

- Non-locality
- Dependency upon reparameterization
- Hard to reconcile with local field theories

Connects to:

- Gerbes
- Two-form connections
- Higher spin gauge theories
- etc

Local conditions:

$$[T_a\,,\,T_b] = i\,f_{abc}\,T_c$$

$$[T_a\,,\,P_i] = P_j\,R_{ji}(T_a)$$

$$[P_i\,,\,P_j] = 0$$

$$A_\mu = A^a_\mu\,T_a \qquad F_{\mu\nu} = 0$$

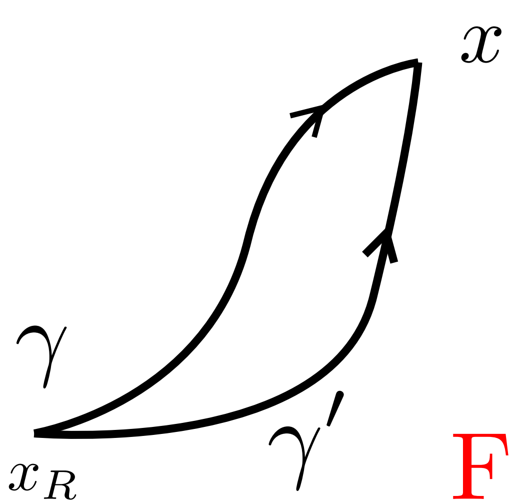
$$B_{\mu\nu} = B^i_{\mu\nu}\,P_i \qquad D\wedge B = 0$$

CP^1-model
Skyrme model
Skyrme-Faddeev model
Self-dual YM, etc

Orlando Alvarez, LAF and J. Sánchez Guillén
hep-th/9710147, *Nucl. Phys.* **B529** (1998) 689-736
IJMPA, **24** (2009) 1825 - 1888; arXiv:0901.1654 [hep-th].

Not quite Mr. Holmes!!!

Revisit integrable field theories in $1+1$ dimensions



Let $g(x)$ and A_μ be functionals of the fields of the theory and impose on any curve γ the integral equation

Flux

$$g\,(x)\;g^{-1}\,(x_R) = P_1\,e^{-\int_\gamma d\sigma\,A_\mu\,\frac{d\,x^\mu}{d\sigma}}$$

$$= P_1 e^{-\int_{\gamma'} d\sigma\,A_\mu\,\frac{dx^\mu}{d\sigma}}$$

charge

path independent → conservation laws

Take γ infinitesimal → $A_\mu = -\partial_\mu g\,g^{-1}$

So, A_μ is flat and we have Lax-Zakharov-Shabat equation

$$\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu\,,\,A_\nu] = 0$$

Look for integral equations!! $P_{d-1}e^{\int_{\partial\Omega}\mathcal{A}} = P_d e^{\int_\Omega \mathcal{F}}$

Basic property of gauge theories: Flux=Charge

Hidden Symmetries

Lecture 2

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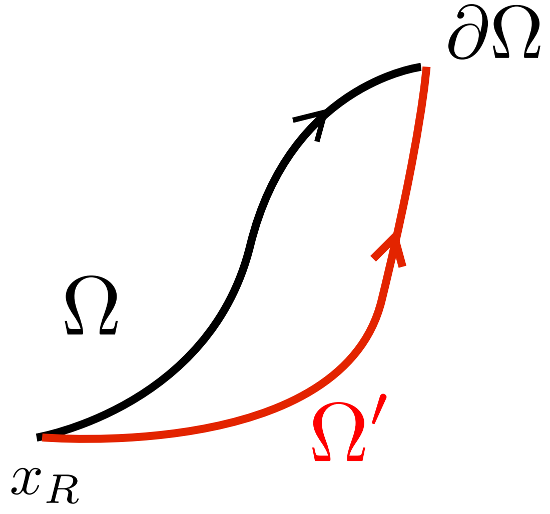
What have we learnt in Lecture 1?

Look for integral equations!! $P_{d-1} e^{\int_{\partial\Omega} \mathcal{A}} = P_d e^{\int_\Omega \mathcal{F}}$

Basic property of gauge theories: Flux=Charge

$$P_{d-1}\, e^{\int_{\partial\Omega} \mathcal{A}} = P_d\, e^{\int_\Omega \mathcal{F}}$$
$$= P_d\, e^{\int_{\Omega'} \mathcal{F}}$$

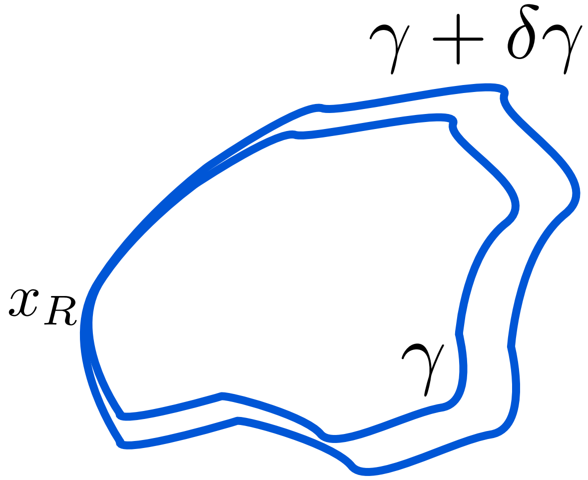


Ω and Ω': two 3-volumes with the same border

Path independency $\rightarrow$ conservation laws

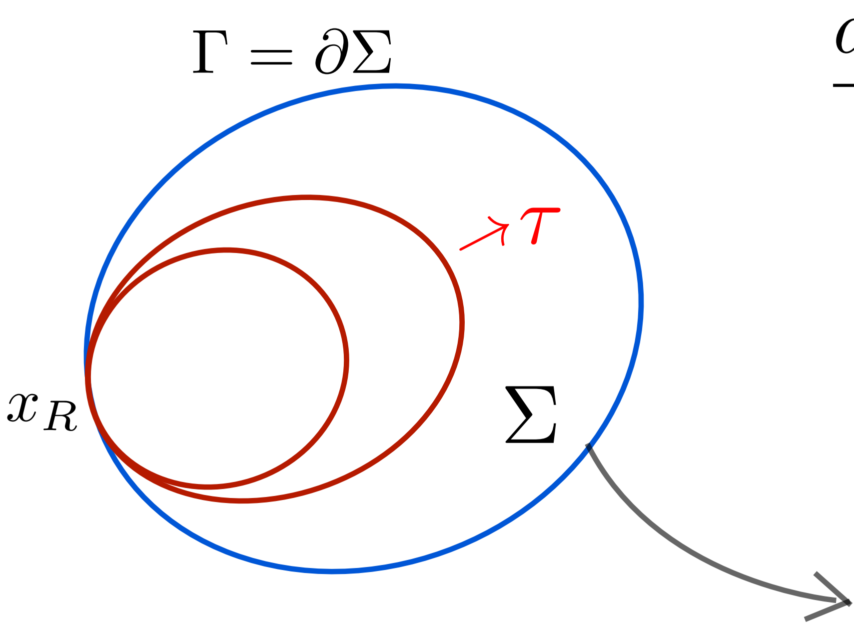
Non-Abelian Stokes Theorem

Wilson line
$$\frac{d\,W}{d\,\sigma} + A_\mu \frac{d\,x^\mu}{d\,\sigma} W = 0 \qquad W = Pe^{-\int_\Gamma d\sigma\; A_\mu \frac{d\,x^\mu}{d\,\sigma}}$$



$$W^{-1}(\gamma)\delta W(\gamma) = \int_0^{2\pi} d\sigma\, W^{-1} F_{\mu\nu} W \frac{dx^\mu}{d\sigma}\delta x^\nu$$

$$\downarrow$$



$$\frac{dW}{d\tau} = W \int_0^{2\pi} d\sigma W^{-1}(\sigma)\, F_{\mu\nu}\, W(\sigma)\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}$$

$$W = P_2\, e^{\int_\Sigma d\sigma d\tau W^{-1}\, F_{\mu\nu}\, W \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}}$$

$$W = P_1\, e^{-\int_\Gamma d\sigma A_\mu \frac{dx^\mu}{d\sigma}}$$

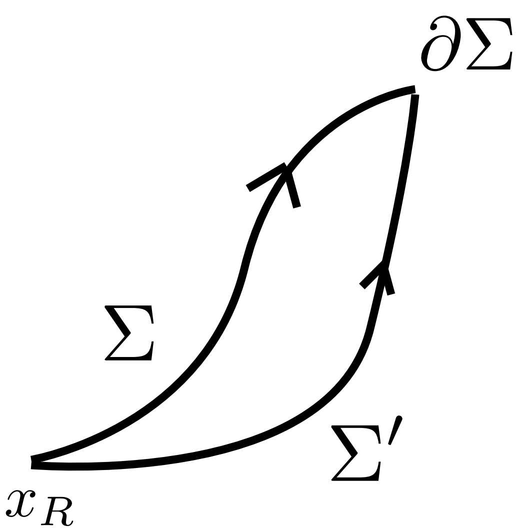
$$P_1\, e^{-\int_\Gamma d\sigma A_\mu \frac{dx^\mu}{d\sigma}} = P_2\, e^{\int_\Sigma d\sigma d\tau W^{-1}\, F_{\mu\nu}\, W \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}}$$

$$\int_{\partial\Sigma} A = \int_\Sigma d \wedge A$$

The Flatness Condition

$$P_1\, e^{-\int_{\partial\Sigma} d\sigma A_\mu \frac{dx^\mu}{d\sigma}} = P_2\, e^{\int_\Sigma d\sigma d\tau W^{-1}\, F_{\mu\nu}\, W \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau}}$$

$$= P_2\, e^{\int_{\Sigma'} d\sigma d\tau W^{-1}\, F_{\mu\nu}\, W \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau}}$$



$$\mathcal{A}[x(\sigma)] = \int_0^{2\pi} d\sigma\; W(\sigma)^{-1} F_{\mu\nu}(x(\sigma)) W(\sigma) \frac{dx^\mu}{d\sigma} \delta x^\nu(\sigma)$$

Flat connection on loop space

Path independency on loop space $\rightarrow$ conservation laws in $2+1$

Examples:

Chern-Simons in $2+1$

Yang-Mills in $2+1$

LAF and G. Luchini, NPB 858 (2012), 336; arXiv 1109.2606

An example: Chern-Simons theory

$A_\mu \in$ Lie algebra $\mathcal{G}$

Eq. of motion $\rightarrow$ $$F_{\mu\nu} = \frac{1}{\kappa} \varepsilon_{\mu\nu\rho} J^\rho \equiv \frac{1}{\kappa} \tilde{J}_{\mu\nu}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu \, , \, A_\nu]$

For any surface Σ impose the integral equation:

$$P_1 \, e^{-\int_{\partial\Sigma} d\sigma A_\mu \frac{dx^\mu}{d\sigma}} = P_2 \, e^{\frac{1}{\kappa} \int_\Sigma d\sigma d\tau W^{-1} \, \tilde{J}_{\mu\nu} \, W \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau}}$$

Flux $\quad$ Charge

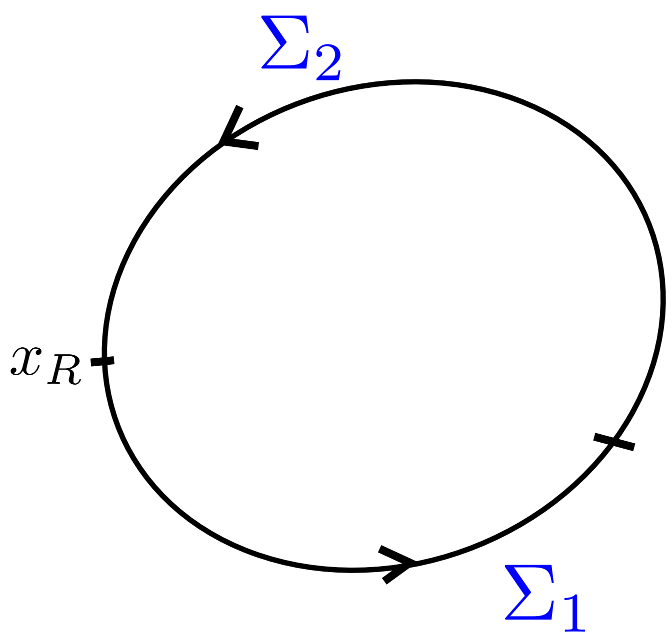
For an infinitesimal Σ one gets the differential equation $F_{\mu\nu} = \tilde{J}_{\mu\nu}$

The flatness condition

For a closed surface Σ_c the integral equation implies

$$P_2\, e^{\frac{1}{\kappa}\int_{\Sigma_c} d\sigma d\tau W^{-1}\,\tilde{J}_{\mu\nu}\,W\,\frac{dx^\mu}{d\sigma}\,\frac{dx^\nu}{d\tau}} = \mathbb{1}$$

On the loop space $\Sigma_c \equiv$ closed path



$\Sigma_c = \Sigma_1 + \Sigma_2$

$$P_2\, e^{\int_{\Sigma_c}\mathcal{A}} = \mathbb{1}$$

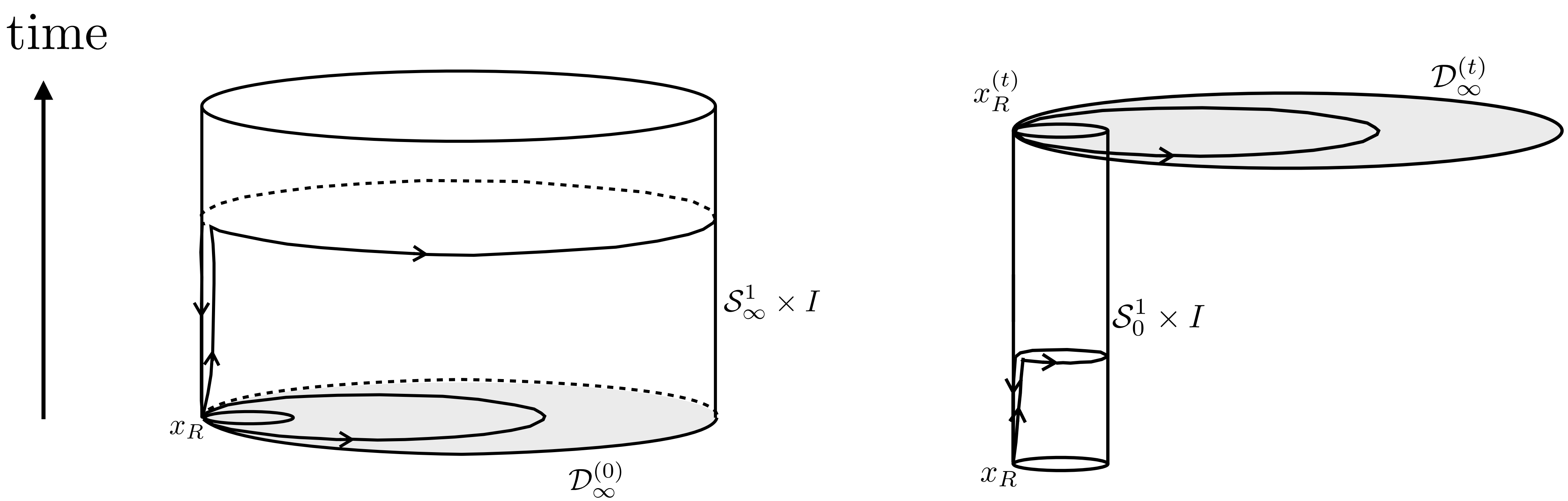
$$\mathcal{A} = \frac{1}{\kappa}\int_0^{2\pi} d\sigma W^{-1}\,\tilde{J}_{\mu\nu}\,W\,\frac{dx^\mu}{d\sigma}\delta x^\nu$$

$$P_2\, e^{\int_{\Sigma_1}\mathcal{A}}\, P_2\, e^{\int_{\Sigma_2}\mathcal{A}} = \mathbb{1}$$

$$\longrightarrow \quad P_2\, e^{\int_{\Sigma_1}\mathcal{A}} = P_2\, e^{\int_{\Sigma_2^{-1}}\mathcal{A}}$$

Path (surface) independency

Construction of conserved charges



Surface $\Sigma_1 = \mathcal{D}_\infty^{(0)} \cup \left(S_\infty^1 \times I\right)$ Surface $\Sigma_2 = \left(S_0^1 \times I\right) \cup \mathcal{D}_\infty^{(t)}$

Path independency: $P_2\, e^{\int_{\mathcal{D}_\infty^{(0)}} \mathcal{A}}\, P_2\, e^{\int_{S_\infty^1 \times I} \mathcal{A}} = P_2\, e^{\int_{S_0^1 \times I} \mathcal{A}}\, P_2\, e^{\int_{\mathcal{D}_\infty^{(t)}} \mathcal{A}}$

Boundary conditions: $\tilde{J}_{12} = J_0 \sim \frac{1}{r^{2+\delta}}\, T\left(\hat{r}\right) \qquad \text{for} \quad r \to \infty$

Change of base point: $P_2\, e^{\int_{\mathcal{D}_\infty^{(t)}} \mathcal{A}}\, |_{x_R^{(t)}} = W\left(x_R^{(t)}, x_R\right)\, P_2\, e^{\int_{\mathcal{D}_\infty^{(t)}} \mathcal{A}}\, |_{x_R}\, W^{-1}\left(x_R^{(t)}, x_R\right)$

Conserved charges → eigenvalues of the operator:

$$V_{x_R^{(t)}}\left(\mathcal{D}_\infty^{(t)}\right) = P_2\, e^{\frac{ie}{\kappa} \int_{\mathcal{D}_\infty^{(t)}} d\tau\, d\sigma\, W^{-1} \widetilde{J}_{\mu\nu} W \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau}} = P_1 e^{-ie \oint_{S_\infty^1} d\sigma\, A_\mu \frac{dx^\mu}{d\sigma}}$$

Integral Equations for Yang-Mills in $(2+1)$-Dimensions

Non-Abelian Stokes theorem $$P_1\, e^{\int_{\partial\Sigma}\, \mathcal{A}_\mu \frac{d\,x^\mu}{d\,\sigma}} = P_2\, e^{\int_\Sigma \mathcal{W}^{-1}\mathcal{F}_{\mu\nu}\mathcal{W} \frac{d\,x^\mu}{d\,\sigma}\frac{d\,x^\nu}{d\,\tau}}$$

Take $\mathcal{A}_\mu = A_\mu + \beta\, \widetilde{F}_\mu$ $\qquad \widetilde{F}_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho}F^{\nu\rho}$ $\qquad \beta$ is a free parameter

$$P_1 e^{-ie\oint_{\partial\Sigma} d\sigma\, \left(A_\mu+\beta\,\widetilde{F}_\mu\right)\frac{dx^\mu}{d\sigma}} = P_2\; e^{ie\,\int_\Sigma d\tau\; d\sigma\; W^{-1}\left(F_{\mu\nu}-\beta\,\widetilde{J}_{\mu\nu}+ie\,\beta^2\left[\widetilde{F}_\mu\,,\,\widetilde{F}_\nu\right]\right)W\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}}$$

$$\widetilde{J}_{\mu\nu} \equiv \varepsilon_{\mu\nu\rho}J^\rho$$

For $\Sigma \rightarrow 0$, gets the Yang-Mills equations

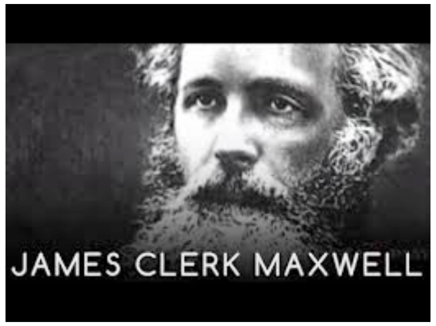
$$D_\mu\widetilde{F}_\nu - D_\nu\widetilde{F}_\mu = -\widetilde{J}_{\mu\nu} \longrightarrow D_\nu F^{\nu\mu} = J^\mu$$

Conserved charges are obtained the same way as for Chern-Simons

How to do it in $3 + 1$ dimensions, Mr. Holmes?

Quite elementary, Mr. Watson?

Ask Prof. Maxwell !!!



$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{\nabla} \times \vec{B} - \mu_0\, \varepsilon_0\, \frac{\partial\, \vec{E}}{\partial\, t} = \mu_0\, \vec{J}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{E} + \frac{\partial\, \vec{B}}{\partial\, t} = 0$$

$\mu_0\, \varepsilon_0 = 1/c^2$

$$E_i = F_{0i} \qquad B_i = -\frac{1}{2\, c}\, \varepsilon_{ijk} F_{jk} \qquad j^\mu \equiv \frac{1}{\varepsilon_0} \left(\rho\,,\, -\frac{1}{c}\, J^i \right)$$

$$\partial_\mu F^{\mu\nu} = j^\nu \qquad \partial_\mu \widetilde{F}^{\mu\nu} = 0$$

$$\widetilde{F}_{\mu\nu} = \frac{1}{2}\, \varepsilon_{\mu\nu\rho\sigma}\, F^{\rho\sigma}$$

$$\partial_\nu j^\nu = 0$$

$$Q = \int_V d^3x\, j^0 = \int_V d^3x\, \partial_i F^{i0} = \int_{\partial V} d\vec{\Sigma} \cdot \vec{E}$$

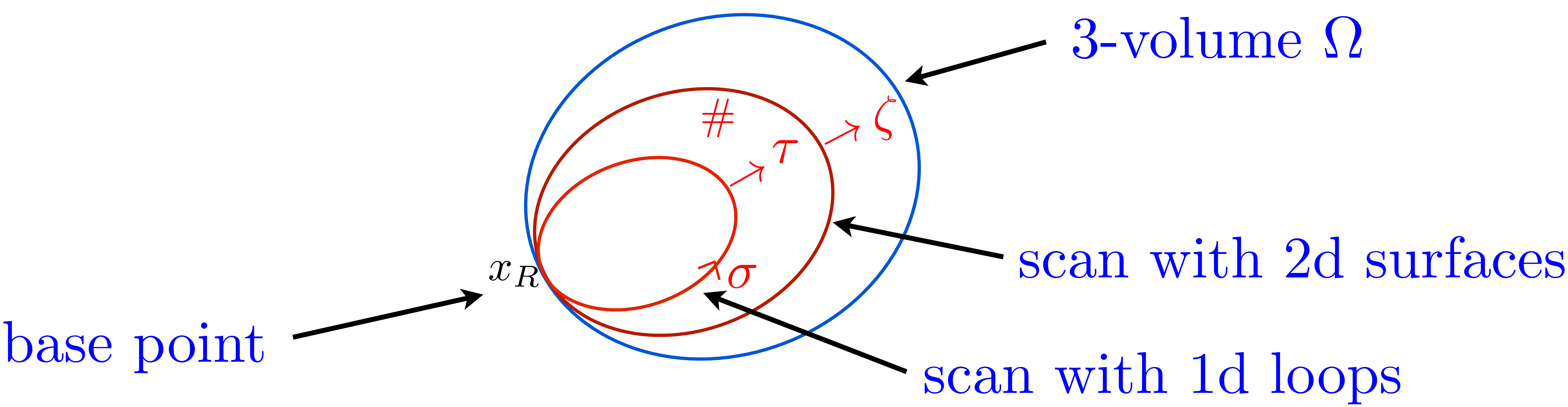
Q is conserved and gauge invariant

Abelian Stokes Theorem

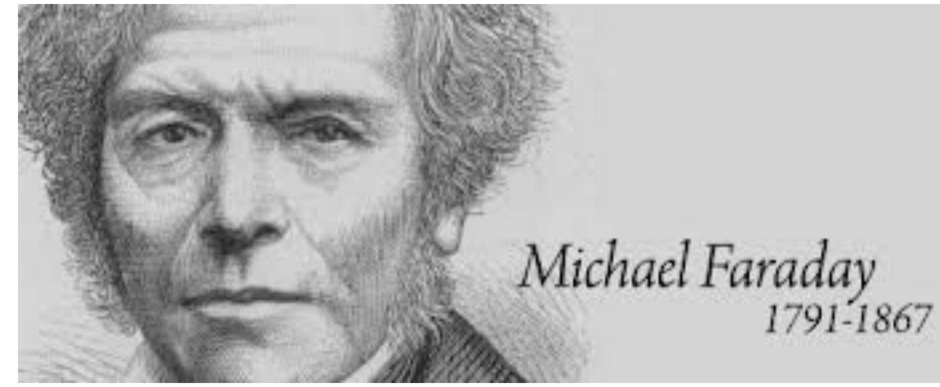
$$\int_{\partial\Omega} B = \int_{\Omega} d \wedge B$$

For an abelian two-form $B_{\mu\nu}$ and a 3-volume Ω

$$\int_{\partial\Omega} B_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau} = \int_{\Omega} \left[\partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu} + \partial_\mu B_{\nu\rho}\right] \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\zeta}$$



Back to Faraday: Integral Equations



$$\partial_\mu F^{\mu\nu} = j^\nu$$

$$\partial_\mu \widetilde{F}^{\mu\nu} = 0$$

$\longrightarrow$

$$\partial_\rho \tilde{F}_{\mu\nu} + \partial_\nu \tilde{F}_{\rho\mu} + \partial_\mu \tilde{F}_{\nu\rho} = \tilde{j}_{\rho\mu\nu}$$

$$\partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} + \partial_\mu F_{\nu\rho} = 0$$

$$j^\lambda = \frac{1}{3!}\varepsilon^{\lambda\rho\mu\nu}\,\tilde{j}_{\rho\mu\nu}$$

In Stokes theorem, take $B_{\mu\nu} \equiv \alpha\, F_{\mu\nu} + \beta\, \tilde{F}_{\mu\nu}$ to get

$$\int_{\partial\Omega} \left[\alpha\, F_{\mu\nu} + \beta\, \tilde{F}_{\mu\nu}\right] \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau} = \beta \int_\Omega \tilde{j}_{\mu\nu\rho}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\zeta}$$

For $\alpha = 0$, $\beta = 1$, and Ω purely spatial

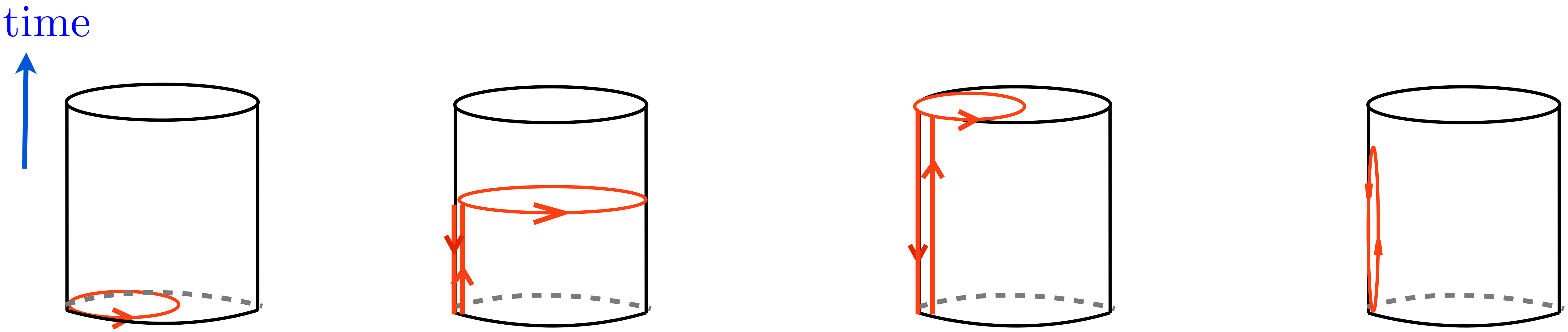
$$\int_{\partial\Omega} \widetilde{F}_{ij}\frac{\partial\, x^i}{\partial\,\sigma}\frac{\partial\, x^j}{\partial\,\tau}\,d\sigma\, d\tau = -\int_{\partial\Omega} \vec{E}\cdot d\vec{\Sigma} = -\int_\Omega \frac{\rho}{\varepsilon_0} = -\frac{Q}{\varepsilon_0}$$

Gauss law

Integral Equations and Conservation Laws

For a 3-volume Ω without border ($\partial\Omega = 0$)

$$0 = \int_{\partial\Omega} \left[\alpha\, F_{\mu\nu} + \beta\, \tilde{F}_{\mu\nu}\right] \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau} = \beta \int_{\Omega} \tilde{j}_{\mu\nu\rho} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\zeta}$$



$$\int_{\Omega_0} \tilde{j}_{\mu\nu\rho} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\zeta} \quad \int_{S^2_\infty \times I} \tilde{j}_{\mu\nu\rho} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\zeta} \quad - \int_{\Omega_t} \tilde{j}_{\mu\nu\rho} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\zeta} \quad \int_{S^2_0 \times I} \tilde{j}_{\mu\nu\rho} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\zeta}$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$Q(0) \qquad 0 \qquad -Q(t) \qquad 0$$

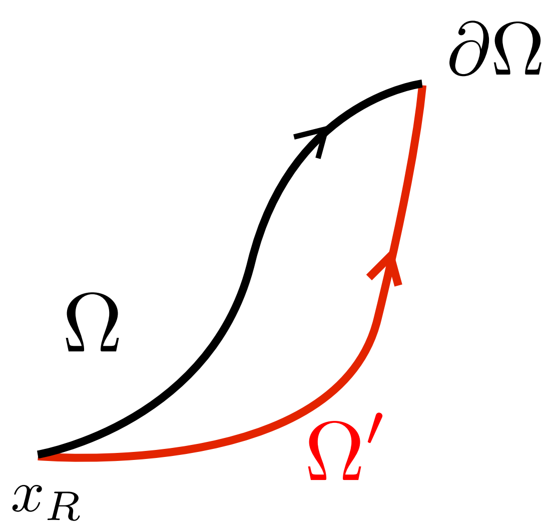
$$\int_{\Omega} \tilde{j}_{\mu\nu\rho} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\zeta} = 0 \longrightarrow Q(t) = Q(0)$$

Faraday's Path Independency

$$\int_{\partial\Omega} \left[\alpha\, F_{\mu\nu} + \beta\, \tilde{F}_{\mu\nu}\right] \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau} = \beta \int_{\Omega} \tilde{j}_{\mu\nu\rho} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\zeta}$$

$\Omega^{(2)} = \{f : S^2 \to M \mid \text{north pole} \to x_R\}$

$$= \beta \int_{\Omega'} \tilde{j}_{\mu\nu\rho} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\zeta}$$



Ω and Ω': two 3-volumes with the same border

connection in loop space:

$$\mathcal{A} \equiv \int_{\text{loop}} \tilde{j}_{\mu\nu\rho} \frac{d\,x^\mu}{d\,\sigma}\frac{d\,x^\nu}{d\,\tau}\,\delta x^\rho$$

$$\mathcal{F} = \delta\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0$$

$$\partial_\rho \tilde{F}_{\mu\nu} + \partial_\nu \tilde{F}_{\rho\mu} + \partial_\mu \tilde{F}_{\nu\rho} = \tilde{j}_{\rho\mu\nu}$$

$(d \wedge \tilde{j} = 0)$

abelian

The laws of Electromagnetism correspond to flat connections in loop space!!

Generalizing Maxwell: Yang-Mills theory

Yang-Mills equations $D^\mu F_{\mu\nu} = J_\nu$ $\quad D^\mu \widetilde{F}_{\mu\nu} = 0$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\, e\, [\, A_\mu \,,\, A_\nu \,] \qquad D_\mu * = \partial_\mu * + i\, e\, [\, A_\mu \,,\, * \,]$$

$$J_\mu = \bar{\psi}\, \gamma_\mu\, R(T_a)\, \psi\, T_a$$

Invariant under a gauge group G

The (text book) conserved charges

$$j_\nu \equiv \partial^\mu F_{\mu\nu} = J_\nu - i\,e\,[\,A_\mu\,,\,F_{\mu\nu}\,] \qquad \rightarrow \qquad \partial^\mu j_\mu = 0$$

$$\widetilde{j}_\nu \equiv \partial^\mu \widetilde{F}_{\mu\nu} = -i\,e\left[\,A_\mu\,,\,\widetilde{F}_{\mu\nu}\,\right] \qquad \rightarrow \qquad \partial^\mu \widetilde{j}_\mu = 0$$

Charges

$$Q = \int d^3x\,\partial^i F_{i0} = \int d^3x\,\vec{\nabla}\cdot\vec{E} = \int d\vec{\Sigma}\cdot\vec{E}$$

$$\widetilde{Q} = \int d^3x\,\partial^i \widetilde{F}_{i0} = -\int d^3x\,\vec{\nabla}\cdot\vec{B} = -\int d\vec{\Sigma}\cdot\vec{B}$$

Under a gauge transformation

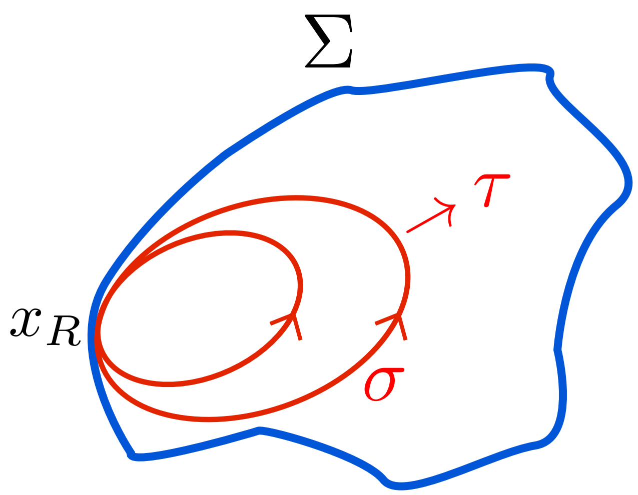
$$Q \rightarrow \int d\vec{\Sigma}\cdot g\,\vec{E}\,g^{-1} = g\,Q\,g^{-1} \longrightarrow$$

eigenvalues of Q are gauge invariant

$d\vec{\Sigma}\cdot g\,\vec{E}\,g^{-1}$: not gauge invariant

$= g\,Q\,g^{-1}$: if g is constant at infinity

Generalizing Faraday: Non-Abelian integrals



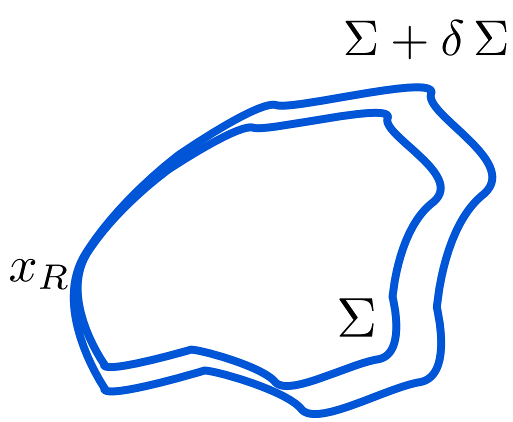
$$\frac{dV}{d\tau} - V\,T\,(A,B,\tau) = 0$$

$$T\,(B,A,\tau) \equiv \int_0^{2\pi} d\sigma\, W^{-1}\, B_{\mu\nu}\, W\, \frac{d\,x^\mu}{d\,\sigma}\,\frac{d\,x^\nu}{d\,\tau}$$

It is a surface ordered integral

$$V(\Sigma) = V_R\; P_2\, e^{\int_\Sigma d\sigma d\tau\; W^{-1}\, B_{\mu\nu}\, W\, \frac{d\,x^\mu}{d\,\sigma}\,\frac{d\,x^\nu}{d\,\tau}}$$

Vary Σ



$$\begin{aligned}\delta V\,V^{-1} \equiv& \int_0^{2\pi} d\tau \int_0^{2\pi} d\sigma\, V\,(\tau)\, \{ \\ & W^{-1}\, [D_\lambda B_{\mu\nu} + D_\mu B_{\nu\lambda} + D_\nu B_{\lambda\mu}]\, W \frac{d\,x^\mu}{d\,\sigma}\,\frac{d\,x^\nu}{d\,\tau}\,\delta x^\lambda \\ & - \int_0^\sigma d\sigma'\,\left[\, B^W_{\kappa\rho}\,(\sigma') - ieF^W_{\kappa\rho}\,(\sigma')\,,\, B^W_{\mu\nu}\,(\sigma)\,\right] \frac{dx^\kappa}{d\sigma'}\,\frac{dx^\mu}{d\sigma} \\ & \times \left(\frac{d\,x^\rho\,(\sigma')}{d\,\tau}\delta x^\nu\,(\sigma) - \delta x^\rho\,(\sigma')\,\frac{d\,x^\nu\,(\sigma)}{d\,\tau}\right)\Bigg\}\, V^{-1}\,(\tau)\end{aligned}$$

$(X^W \equiv W^{-1}\, X\, W)$

The generalized non-abelian Stokes Theorem

$$V_R\, P_2 e^{\int_{\partial\Omega} d\tau d\sigma W^{-1} B_{\mu\nu} W \frac{dx^\mu}{d\sigma}\frac{d\,x^\nu}{d\,\tau}} = P_3 e^{\int_\Omega d\zeta\,\mathcal{K}}\, V_R$$

$$\frac{d\,V}{d\,\tau} - V\,T\,(A,B,\tau) = 0$$

$$\frac{d\,V}{d\,\zeta} - \mathcal{K}\,V = 0$$

$$T\,(B,A,\tau) \equiv \int_0^{2\,\pi} d\sigma\; W^{-1}\, B_{\mu\nu}\, W\, \frac{d\,x^\mu}{d\,\sigma}\,\frac{d\,x^\nu}{d\,\tau}$$

$$\begin{aligned}\mathcal{K} \equiv & \int_0^{2\,\pi} d\tau \int_0^{2\,\pi} d\sigma\, V\,(\tau)\,\Big\{ \\ & W^{-1}\,\left[D_\lambda B_{\mu\nu} + D_\mu B_{\nu\lambda} + D_\nu B_{\lambda\mu}\right] W \frac{d\,x^\mu}{d\,\sigma}\,\frac{d\,x^\nu}{d\,\tau}\,\frac{d\,x^\lambda}{d\,\zeta} \\ & - \int_0^{\sigma} d\sigma'\,\left[\,B^W_{\kappa\rho}\,(\sigma') - ieF^W_{\kappa\rho}\,(\sigma')\,,\, B^W_{\mu\nu}\,(\sigma)\,\right] \frac{dx^\kappa}{d\sigma'}\,\frac{dx^\mu}{d\sigma} \\ & \times \left(\frac{d\,x^\rho\,(\sigma')}{d\,\tau}\,\frac{d\,x^\nu\,(\sigma)}{d\,\zeta} - \frac{d\,x^\rho\,(\sigma')}{d\,\zeta}\,\frac{d\,x^\nu\,(\sigma)}{d\,\tau}\right)\Big\}\, V^{-1}\,(\tau)\end{aligned}$$

$$(X^W \equiv W^{-1}\,X\,W)$$

Flat connection on the loop space:

$$\Omega^{(2)} = \{f : S^2 \rightarrow M \mid \text{north pole } \rightarrow x_0\}$$

The Integral Equations for Yang-Mills

$$P_2 e^{ie\int_{\partial\Omega} d\tau d\sigma\left[\alpha F^W_{\mu\nu}+\beta\widetilde{F}^W_{\mu\nu}\right]\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}} = P_3 e^{\int_\Omega d\zeta d\tau V\mathcal{J}V^{-1}}$$

$$\mathcal{J} \equiv \int_0^{2\pi} d\sigma \left\{ ie\beta\widetilde{J}^W_{\mu\nu\lambda}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\zeta} + e^2\int_0^\sigma d\sigma' \right.$$
$$\times \left[\left((\alpha-1)\,F^W_{\kappa\rho}+\beta\widetilde{F}^W_{\kappa\rho}\right)(\sigma'),\left(\alpha F^W_{\mu\nu}+\beta\widetilde{F}^W_{\mu\nu}\right)(\sigma)\right]$$
$$\left. \times \frac{dx^\kappa}{d\sigma'}\frac{dx^\mu}{d\sigma}\left(\frac{dx^\rho(\sigma')}{d\tau}\frac{dx^\nu(\sigma)}{d\zeta}-\frac{dx^\rho(\sigma')}{d\zeta}\frac{dx^\nu(\sigma)}{d\tau}\right)\right\}$$

$B_{\mu\nu} \to \alpha F_{\mu\nu} + \beta\widetilde{F}_{\mu\nu}$ $\qquad D^\mu F_{\mu\nu} = J_\nu$ $\qquad D^\mu \widetilde{F}_{\mu\nu} = 0$

Direct consequence of Stokes theorem and Yang-Mills eqs.

Implies Yang-Mills eqs. in the limit $\Omega \to 0$

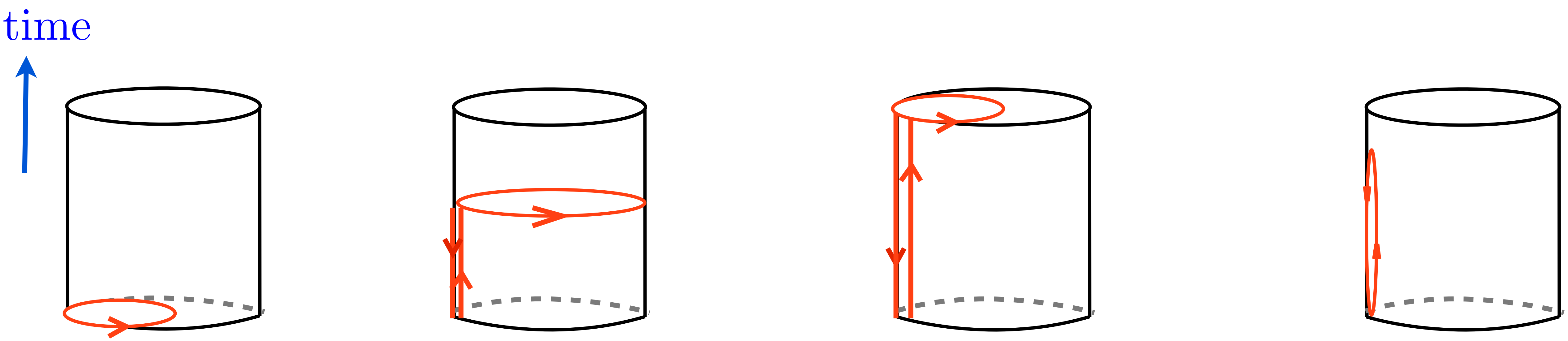
L.A. Ferreira and G. Luchini
1) [arXiv:1205.2088 [hep-th]], Phys. Rev. D 86, 085039 (2012)
2) [arXiv:1109.2606 hep-th]], Nuclear Physics B 858PM (2012) 336-365

$$J^\mu = \frac{1}{3!}\varepsilon^{\mu\nu\rho\lambda}\widetilde{J}_{\nu\rho\lambda}$$

Conserved Charges

$$P_2 e^{ie\int_{\partial\Omega} d\tau d\sigma\left[\alpha F^W_{\mu\nu} + \beta \widetilde{F}^W_{\mu\nu}\right]\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}} = P_3 e^{\int_\Omega d\zeta d\tau V \mathcal{J} V^{-1}}$$

If Ω_c is a closed volume ($\partial\Omega_c = 0$) $\longrightarrow$ $P_3 e^{\oint_{\Omega_c} d\zeta d\tau V \mathcal{J} V^{-1}} = \mathbb{1}$



$P_3 e^{\int_{\Omega_0} d\zeta d\tau V \mathcal{J} V^{-1}}$ $\quad$ $P_3 e^{\int_{S^2_\infty \times I} d\zeta d\tau V \mathcal{J} V^{-1}}$ $\quad$ $P_3 e^{\int_{\Omega_t^{-1}} d\zeta d\tau V \mathcal{J} V^{-1}}$ $\quad$ $P_3 e^{\int_{S^2_0 \times I} d\zeta d\tau V \mathcal{J} V^{-1}}$

Iso-spectral evolution:

$$V(\Omega_t) = U(t) \cdot V(\Omega_0) \cdot U^{-1}(t)$$

Eigenvalues of $V(\Omega_t)$ are constant in time

Conserved charges are eigenvalues of the operator

$$V(\Omega_t) = P_2\, e^{ie\int_{S^{2,(t)}_{\infty}} d\tau\, d\sigma\, \left(\alpha F^{W}_{\mu\nu} + \beta \widetilde{F}^{W}_{\mu\nu}\right) \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau}} = P_3\, e^{\int_{\Omega_t} d\zeta\, d\tau\, V \mathcal{J} V^{-1}}$$

Conserved charges are:

1) Gauge invariant $\quad V(\Omega_t) \rightarrow g_R\, V(\Omega_t)\, g_R^{-1}$

2) Independent of reference point

$$V_{x_R}(\Omega_t) \rightarrow W^{-1}(\widetilde{x}_R, x_R) V_{\widetilde{x}_R}(\Omega_t) W(\widetilde{x}_R, x_R)$$

3) Independent of parameterization

$P_3\, e^{\int_{\Omega^{(t)}_{\infty}} d\zeta\, d\tau\, V \mathcal{J} V^{-1}}$ is path independent

4) Gives non-trivial dynamical magnetic charges to monopoles

5) Relevant for the global aspects of Yang-Mills theory

Wu-Yang and 't Hooft-Polyakov monopoles

At spatial infinity they are the same:

$$A_i = -\frac{1}{e}\,\varepsilon_{ijk}\,\frac{\hat{r}_j}{r}\,T_k \qquad F_{ij} = \frac{1}{e}\,\varepsilon_{ijk}\,\frac{\hat{r}_k}{r^2}\,\hat{r}\cdot T \qquad [T_i\,,\,T_j] = i\,\varepsilon_{ijk}\,T_k$$

Important property:

$$\frac{d}{d\,\sigma}\left(W^{-1}\,\hat{r}\cdot T\,W\right) = W^{-1}\,D_i(\hat{r}\cdot T)\,W\,\frac{d\,x^i}{d\,\sigma} \xrightarrow{D_i(\hat{r}\cdot T)=0} W^{-1}\,\hat{r}\cdot T\,W = T_R$$

$$W^{-1}\,F_{ij}\,W = \frac{1}{e}\,\varepsilon_{ijk}\,\frac{\hat{r}_k}{r^2}\,T_R \qquad T_R \equiv \left(\hat{r}\cdot T\right)_{\text{at }x_R}$$

Charge operator:

$$Q_S = e^{i\,e\,\alpha\,\int_{S^2_\infty} d\sigma\,d\tau\,W^{-1}\,F_{ij}\,W\,\frac{dx^i}{d\sigma}\,\frac{dx^j}{d\tau}} = e^{-i\,e\,\alpha\,\int_{S^2_\infty} d\vec{\Sigma}\cdot\vec{B}^R} = e^{i\,4\,\pi\,\alpha\,T_R}$$

Conserved charges are the eigenvalues of Q_S

Text book charges: $$Q_{\text{old}} = \int_{S^2_\infty} d\sigma\,d\tau\,F_{ij}\,\frac{dx^i}{d\sigma}\,\frac{dx^j}{d\tau} = 0$$

Note however that one must have (for $\beta = 0$)

$$Q_S = P_2 e^{\alpha\, ie \int_{S^2_\infty} d\tau d\sigma F^W_{ij} \frac{dx^i}{d\sigma}\frac{dx^j}{d\tau}} = P_3 e^{e^2 \alpha(\alpha-1)\int_{R^3} d\zeta d\tau V \mathcal{C} V^{-1}}$$

with

$$\mathcal{C} \equiv \int_0^{2\pi} d\sigma \int_0^{\sigma} d\sigma' \left[F^W_{\kappa\rho}(\sigma') \, , \, F^W_{\mu\nu}(\sigma) \right] \frac{d\,x^\kappa}{d\,\sigma'} \frac{d\,x^\mu}{d\,\sigma} \left(\frac{d\,x^\rho(\sigma')}{d\,\tau} \frac{d\,x^\nu(\sigma)}{d\,\zeta} - \frac{d\,x^\rho(\sigma')}{d\,\zeta} \frac{d\,x^\nu(\sigma)}{d\,\tau} \right)$$

density of magnetic charge

Integral Bianchi identity implies: $Q_S^{(\alpha=1)} = e^{-i\,e \int_{S^2_\infty} d\vec{\Sigma}\cdot\vec{B}^R} = \mathbb{1}$

eigenvalues of $\int_{S^2_\infty} d\vec{\Sigma} \cdot \vec{B}^R = \frac{2\,\pi\,n}{e}$

For 'tHooft-Polyakov monopole both sides match.
Commutator plays the role of density of magnetic charge

For Wu-Yang monopole $\mathcal{C} = 0$
One needs a source of magnetic charge

$$\vec{D} \cdot \vec{B} = -\frac{1}{e}\, \hat{r} \cdot \vec{T}\, \frac{\delta(r)}{r^2}$$

It satisfies Bianchi identity in the sense of distribution theory

C.P. Constantinidis, LAF, G. Luchini; JPA 52, 155202 (2019)

One can expand both sides of the integral equation in powers of α

$$Q_S = P_2 e^{\alpha\, ie \int_{S^2_\infty} d\tau d\sigma F^W_{ij} \frac{dx^i}{d\sigma} \frac{dx^j}{d\tau}} = P_3 e^{e^2 \alpha(\alpha-1) \int_{R^3} d\zeta d\tau V \mathcal{C} V^{-1}}$$

It implies that

$$1 + \alpha \int_0^\tau d\tau' T(\tau') + \alpha^2 \int_0^\tau d\tau' \int_0^{\tau'} d\tau'' T(\tau'')\, T(\tau') + \ldots$$
$$= 1 + \alpha(\alpha - 1) \int_0^\zeta d\zeta' K(\zeta') + \alpha^2 (\alpha - 1)^2 \int_0^\zeta d\zeta' \int_0^{\zeta'} d\zeta'' K(\zeta')\, K(\zeta'') + \ldots$$

with $\quad T \equiv ie \int_0^{2\pi} d\sigma W^{-1} F_{\mu\nu} W \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau} \qquad K \equiv e^2 \int_0^{2\pi} d\tau V \mathcal{C} V^{-1}$

The 'tHooft-Polyakov monopole satisfies it for any α (checked up to second order)

C.P. Constantinidis, LAF, G. Luchini, [1710.03359], PRD 97 (085006) (2018)

Example: (multi) Monopoles

At spatial infinity the magnetic field has the form:

$$F_{ij} = \frac{1}{e}\,\varepsilon_{ijk}\,\frac{\hat{r}_k}{r^2}\,G(\theta\,,\,\varphi)$$

But $\frac{d}{d\sigma}\left(W^{-1}\,G\,W\right) = W^{-1}\,D_i G\,W\,\frac{d\,x^i}{d\,\sigma} \xrightarrow{D_i G = 0} W^{-1}\,G\,W = G_R$

In such a case the charge operator becomes:

$$Q_S = e^{i\,e\,\alpha\,\int_{S^2_\infty} d\sigma\,d\tau\,W^{-1}\,F_{ij}\,W\,\frac{dx^i}{d\sigma}\,\frac{dx^j}{d\tau}} = e^{i\,4\,\pi\,\alpha\,G_R}$$

Eigenvalues of G_R are the charges

For BPS multi-monopole solutions $\quad G_R = \frac{1}{2}\sum_{a=1}^{\mathrm{rank}\,G} n_a\,\frac{2\,\vec{\alpha}_a}{\alpha_a^2}\cdot\vec{h}$

In a d-dimensional representation the eigenvalues of G_R are:

$$\frac{1}{2}\sum_{a=1}^{\mathrm{rank}\,G} n_a\,\left(m_a^{(1)}\,,\,m_a^{(2)}\,,\,\ldots\,,\,m_a^{(d)}\right)$$

The case of $SU(3)$

For the triplet representation the charges are

$$\frac{1}{2\,e}\;(n_1\,,\,n_2-n_1\,,\,-n_2)$$

and for the adjoint representation

$$\frac{1}{2\,e}\;(n_1+n_2\,,\,2\,n_1-n_2\,,\,-n_1+2\,n_2\,,\,0\,,\,0\,,\,-2\,n_1+n_2\,,\,n_1-2\,n_2\,,\,-n_1-n_2)$$

So, charges are the magnetic weights n_a

The topological charges are

$$SU(3)\rightarrow U(1)\otimes U(1) \longrightarrow Q_B^{\rm Top.(max.)}=v\,\frac{\pi}{e}\;(n_1+n_2)$$

$$SU(3)\rightarrow SU(2)\otimes U(1) \longrightarrow Q_B^{\rm Top.(min.)}=v\,\frac{\pi}{e}\,n_2$$

[arXiv:1508.03049[hep-th]] JHEP12(2015)1376)

To think further...

- Yang-Mills is equivalent to the flatness condition on loop space $\mathcal{L}^{(2)}\ (S^2 \to M)$

$$\delta \mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0$$

$$\begin{aligned}\mathcal{A} \equiv & \int_0^{2\pi} d\tau \int_0^{2\pi} d\sigma\, V \left\{ ie\beta \widetilde{J}^{W}_{\mu\nu\lambda} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau} \delta x^\lambda \right. \\ & + e^2 \int_0^\sigma d\sigma' \left[\left((\alpha - 1)\, F^W_{\kappa\rho} + \beta \widetilde{F}^W_{\kappa\rho} \right)(\sigma'),\, \left(\alpha F^W_{\mu\nu} + \beta \widetilde{F}^W_{\mu\nu} \right)(\sigma) \right] \\ & \left. \times \frac{d\,x^\kappa}{d\,\sigma'} \frac{d\,x^\mu}{d\,\sigma} \left(\frac{d\,x^\rho(\sigma')}{d\tau} \delta x^\nu(\sigma) - \delta x^\rho(\sigma') \frac{d\,x^\nu(\sigma)}{d\,\tau} \right) \right\} V^{-1}\end{aligned}$$

(it solves parameterization problem!)

- The hidden symmetries are the gauge transformations on loop space $\mathcal{L}^{(2)}$

$$\mathcal{A} \to g\, \mathcal{A}\, g^{-1} + \delta g\, g^{-1}$$

$$g = P_2 e^{\int d\sigma\, d\tau\ \mathcal{W}^{-1} \beta_{\mu\nu} \mathcal{W} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau}}$$

$$\frac{d\mathcal{W}}{d\sigma} + \alpha_\mu \frac{dx^\mu}{d\sigma} \mathcal{W} = 0$$

- The conserved charges are eigenvalues of the holonomy

$$\mathcal{Q} = \mathcal{P} e^{\int_{\text{space}} \mathcal{A}}$$

- It connects to integrable field theories

Final Comments

- The important symmetries for many non-linear phenomena are not, in general, of the Noether type

- The concept of path independency plays an important role in iso-spectral evolution

- Soliton theory in two dimensions and its applications, are based on flat connections

- Gauge theories and integrable field theories share some very interesting structures

- After 60 years the integral Yang-Mills equations and their truly gauge invariant charges could be constructed using concepts of flat connections on loop spaces

- It paves the way to interesting new developments

Thank You

O. Alvarez, LAF, J.S. Guillen
1) [hep-th/9710147], NPB 529, 689 (1998)
2) [arXiv:0901.1654 [hep-th]], IJMPA 24, 1825 (2009)
LAF, G. Luchini
3) [arXiv:1205.2088 [hep-th]], PRD 86, 085039 (2012)
4) [arXiv:1109.2606 hep-th]], NPB 858PM (2012) 336-365
C.P. Constantinidis, LAF, G. Luchini
5) [arXiv:1508.03049 [hep-th]], JHEP 12 (2015) 137
6) [arXiv:1611.07041 [hep-th]] JPA 52, 155202 (2019)
7) [arXiv:1710.03359 [hep-th]], PRD 97 (085006) (2018)