Linearization problems for Lie algebroids and Lie groupoids

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Abstract. In this survey, we discuss a series of linearization problems — for Poisson structures, Lie algebroids, and Lie groupoids. The last problem involves a conjecture on the structure of proper groupoids. Attempting to prove this by the method of averaging leads to problems concerning almost actions of compact groups and almost invariant submanifolds for compact group actions. The paper ends with a discussion of possible extensions of the convexity theorems for momentum maps of hamiltonian actions of compact groups.

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In memory of Moshé Flato, 1937-1998

1. Introduction

Why is it so hard to prove the linearizability of Poisson structures with semisimple linear part? Conn published proofs about 15 years ago in a pair of papers [5, 6] full of elaborate estimates. Except for the somewhat more conceptual reformulation by Desolneux-Moulis [9] in the smooth case, no simplification of Conn's proofs has appeared.

This is a mystery to me, because analogous theorems about the linearizability of actions of semisimple groups near their fixed points were proven in the compact (smooth or analytic) case by Bochner [3] using a simple averaging method and in the noncompact analytic case by Guillemin-Sternberg [13] and Kushnirenko [18], who used analytic continuation from the compact case — a nonlinear version of "Weyl's unitary trick". Hermann [16] established formal linearization for actions of general semisimple algebras, using cohomological methods similar to those which will appear several times in the present report.

After Conn's work appeared, I tried without success to prove his results by simple averaging. In a conversation in 1994 over coffee in the Jardin du Luxembourg, Moshé Flato and Daniel Sternheimer revived my interest in the subject. Their experience with Jacques Simon on other linearization problems (see for instance [10]) led them to expect simpler proofs of Conn's theorems. In addition, they pointed out that Poisson linearization could be seen as an infinite-dimensional

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Levi decomposition. Although I expressed strong skepticism at the time, Moshé and Daniel's optimism has remained in my mind and kept me from abandoning the problem. In particular, it has resulted in new investigations which, although they have not produced a Poisson linearization theorem (and maybe never will!), have led to a few results and many questions which are interesting in their own right.

These new results and problems form the content of the present paper, which I offer as a small memorial to Moshé. The paper also serves to record a talk which I gave at the Dijon Mathematical Physics Seminar in June, 1999, at Daniel's invitation.[1] The occasion was much saddened by Moshé's absence, but it has been gratifying to see the Seminar continue as an ongoing international "workshop" for discussion of many of Moshé's favorite scientific issues.

2. Linearization and Levi decomposition

The linearization of a Poisson structure with semisimple linear part can be seen, as mentioned above, as something like a Levi decomposition of a Lie algebra of functions with the Poisson bracket operation. Here are the details.

Let $\pi = \frac{1}{2}\sum(\sum c_{ij}^k x_k + \mathbf{O}(x^2))\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}$ be a Poisson structure defined on a neighborhood $\mathscr{U}$ of the origin in $\mathbb{R}^n$, i.e., $\{x_i, x_j\} = \sum c_{ij}^k x_k + \mathbf{O}(x^2)$. In the Lie algebra of germs at 0 of smooth functions, the germs of functions vanishing at 0 form a Lie subalgebra $\mathfrak{m}$ in which those vanishing to order at least 2 form a Lie algebra ideal $\mathfrak{m}^2$. The quotient $\mathfrak{m}/\mathfrak{m}^2 = \mathfrak{g}$ may be identified with the cotangent space of $\mathbb{R}^n$ at 0, the (finite-dimensional) **cotangent Lie algebra** whose structure constants are just the first Taylor coefficients c_{ij}^k of the Poisson structure π. The **linearization problem** is to find (perhaps after shrinking the neighborhood $\mathscr{U}$) new coordinates $(y_1, \ldots, y_n)$ centered at 0 such that $\{y_i, y_j\} = \sum c_{ij}^k y_k$ without any higher order terms.

To view a solution of the linearization problem for semisimple $\mathfrak{g}$ as a Levi decomposition, we note first that, in the exact sequence of Lie algebras

$$0 \to \mathfrak{m}^2 \to \mathfrak{m} \to \mathfrak{g} \to 0,$$

the quotient $\mathfrak{g}$ is (by assumption) semisimple, while the kernel $\mathfrak{m}^2$ has a certain nilpotency property. By this last statement, we mean that the spaces $\mathfrak{m}^k$ of functions vanishing to order at least k have the property that $[\mathfrak{m}^2, \mathfrak{m}^k] \subseteq \mathfrak{m}^{k+1}$. Unfortunately, the intersection $\mathfrak{m}^\infty = \bigcap_k \mathfrak{m}^k$ is not zero, but consists of the germs of functions which vanish to infinite order at the origin.[2]

[1] An earlier version of this paper was presented in a lecture at the Chern Symposium at MSRI [24].

[2] There are some interesting questions to be answered about the Lie algebra $\mathfrak{m}^\infty$. Does it have any semisimple quotients? Does it have any finite-dimensional quotients at all? Equivalently, does it have any finite-dimensional representations? What are the its finite-dimensional subalgebras?

A "Levi decomposition" of $\mathfrak{m}$ should be a vector space direct sum decomposition $\mathfrak{m} = \mathfrak{h} \oplus \mathfrak{m}^2$ in which $\mathfrak{h}$ is a subalgebra or, equivalently, a Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{m}$ which splits the exact sequence.

Given a linearization $(y_1, \ldots, y_n)$ of the Poisson structure, the subalgebra spanned by the germs of the y_i is the required $\mathfrak{h}$. Conversely, given a Levi decomposition and letting $(\eta_1, \ldots, \eta_n)$ be the preimages in $\mathfrak{h}$ of the basis $(x_1 + \mathfrak{m}^2, \ldots, x_n + \mathfrak{m}^2)$ of $\mathfrak{g}$, representatives y_i of the germs η_i have a common domain on which they solve the linearization problem.

Note that everything above remains true if the algebra of smooth germs is replaced by either that of real analytic germs or that of formal power series, with the additional feature that $\mathfrak{m}^\infty$ is now reduced to zero.

Unfortunately, I do not know of any proof of the standard Levi decomposition theorem which would apply here, so it seems that we have only reformulated the linearization problem. Nevertheless, this reformulation leads to some interesting new questions.

Finally, we note that A. Wade [22] has found a formal normalization of Poisson structures at general singular points which is based on the Levi decomposition of the cotangent Lie algebra itself.

3. Lie algebroids

The linearization problem for Poisson manifolds has analogues for Lie algebroids and Lie groupoids. To describe them, we recall that a Lie algebroid over a manifold M is a vector bundle $A \to M$ with a Lie algebra structure (over $\mathbb{R}$) on its space $\Gamma(A)$ of smooth sections and a bundle map $\rho : A \to TM$ inducing a Lie algebra homomorphism from sections of A to vector fields on M, such that $[a, fb] = f[a,b] + (\rho(a) \cdot f)b$ for sections a and b and functions f. Locally (which is all we need for this paper), a Lie algebroid over an open subset $\mathscr{U}$ of $\mathbb{R}^n$ is specified by the bracket relations $[e_i, e_j] = \sum c_{ij}^k(x) e_k$ among a basis of sections of A and the component description $\rho(e_i) = \sum b_{ij}(x) \frac{\partial}{\partial x_j}$ of the anchor map. The axioms of a Lie algebroid then become differential equations relating the **structure functions** c_{ij}^k and b_{ij}.

As with Poisson manifolds, the local classification of Lie algebroids can be reduced by a splitting theorem (modeled on the splitting theorem of Dazord [7] for singular foliations) to the totally singular case where the anchor map ρ vanishes at a point. In coordinates centered at such a point, we may write $c_{ij}^k(x) = c_{ij}^k(0) + \mathbf{O}(x)$ and $b_{ij}(x) = \frac{\partial b_{ij}}{\partial x_k}(0) x_k + \mathbf{O}(x^2)$. Our problem is to present the Lie algebroid in such a way that the higher order terms in the structure functions disappear. Although we have more conditions to satisfy than in the Poisson linearization problem, we also have more variables at our disposal, since we can change the basis of sections as well as the coordinates on the base.

We first try to change the basis to reduce the functions $c_{ij}^k(x)$ to the constants $c_{ij}^k(0)$. This is equivalent to making the Lie algebroid into one of a special kind. Namely, the constants $c_{ij}^k(0)$ define a Lie algebra structure on the fibre of A at 0, and the vector fields $\rho(e_i)$ give an action of this Lie algebra on $\mathscr{U}$. Conversely, when a Lie algebra $\mathfrak{g}$ acts on a manifold M, there is a Lie algebroid structure on the trivial bundle $\mathfrak{g} \times M$ for which the anchor is given by the action and the constant sections form a subalgebra on which the bracket is that of $\mathfrak{g}$. So our task is to determine when a given Lie algebroid is, near a point where the anchor vanishes, such an **action Lie algebroid**. If this task is accomplished, we can finish simplifying the Lie algebroid structure functions by linearizing the action, under the usual hypotheses of semisimplicity and, in the smooth setting, compact type.

Making A into an action Lie algebroid on a neighborhood amounts precisely to finding a Lie algebra homomorphism which splits the exact sequence:

$$0 \to \text{sections vanishing at } 0 \to \text{sections of } A \to \text{fibre at } 0 \to 0,$$

which is analogous to the sequence in the Poisson linearization problem. In particular, if we pass to germs, the kernel of the restriction map is once again "topologically nilpotent" in the sense discussed in Section 2. The analogy is so close that it is tempting to transfer the proofs of Poisson linearization theorems to the Lie algebroid case. In the formal category, this is no problem. The obstructions to stepwise lifting from the fibre $\mathfrak{g}$ at 0 to higher and higher jets of sections lie in the spaces $H^2(\mathfrak{g}, S^k(\mathbb{R}^n) \otimes \mathfrak{g})$, where $\mathfrak{g}$ acts on itself by the adjoint representation and on the symmetric tensor power $S^k(\mathbb{R}^n)$ via the action on $\mathbb{R}^n$ given by $e_i \mapsto \frac{\partial b_{ij}}{\partial x_k}(0)$. These cohomology spaces vanish when $\mathfrak{g}$ is semisimple.[3]

It seems likely that Conn's methods can be extended to establish linearizability in the smooth and analytic settings. I have not yet tried to do this, however, preferring another approach using averaging which might eventually lead to a new proof of Poisson linearization. The following sections describe this approach.

4. Lie groupoids

Averaging requires an action of a compact Lie group, not just of its Lie algebra. We therefore pose a linearization problem for Lie groupoids, which are the "integrated" form of Lie algebroids (see [4] or [19]). Note, though, that not every Lie algebroid is integrable to a groupoid [1], so that linearization of Lie groupoids will linearize only certain Lie algebroids.

Let Γ be a Lie groupoid over M with source and target maps β and α; the product gh is defined when $\beta(g) = \alpha(h)$. Γ is a **proper groupoid** if $(\alpha, \beta) : \Gamma \to M \times M$ is a proper mapping and if the source and target maps are locally trivial fibrations. This term, like many other terms in groupoid theory, comes from the following example.

[3] For formal Poisson linearization, the relevant cohomology spaces are $H^2(\mathfrak{g}, S^k(\mathfrak{g}))$.

If G is a group acting on M, then $G\times M$ is a Lie groupoid over M with $\alpha(g,x) = gx$, $\beta(g,x) = x$, and product $(g,hx)(h,x) = (gh,x)$. We call this the **action groupoid** associated to the group action. It is a proper groupoid if and only if the action is a proper action.

For a general groupoid Γ over M, the **isotropy group** Γ_x of x in M is defined as $\alpha^{-1}(x)\cap\beta^{-1}(x)$. Since $\Gamma_x = (\alpha,\beta)^{-1}(x,x)$, every isotropy group in a proper groupoid is compact.

A subset $\mathscr{U}\subseteq M$ is called **invariant** under Γ if $\alpha(\beta^{-1}(u))\subseteq\mathscr{U}$. An invariant point is called a **fixed point**.

We show in [26] that every neighborhood of a fixed point of a proper groupoid contains an invariant neighborhood, so that the study of proper groupoids can be localized around fixed points. We make the following conjecture.

Proper Groupoid Structure Conjecture. *If x is a fixed point of a proper groupoid Γ, then x has an invariant neighborhood on which Γ is isomorphic to the action groupoid associated to some action of the compact group Γ_x.*

If the "PGS" conjecture above is true, the next step is to linearize the Γ_x action, which would give an equivalence between proper groupoids near their fixed points and linear actions of compact groups. From here it should be possible, using a slice theorem, to get a normal form for a proper groupoid in the neighborhood of any orbit $\beta(\alpha^{-1}(x))$. This program is described in [26].

The PGS conjecture fits into the Levi decomposition picture presented in Section 2 via the exact sequence of groups:

$$1\to\mathscr{G}^1_{\mathscr{U}}\to\mathscr{G}_{\mathscr{U}}\to\Gamma_x\to 1.$$

Here, $\mathscr{G}_{\mathscr{U}}$ is the group of **admissible sections** of Γ over the invariant set $\mathscr{U}$, i.e., the smooth submanifolds σ of the restricted groupoid $\Gamma_{\mathscr{U}}$ for which the restrictions $\alpha|_\sigma$ and $\beta|_\sigma$ are diffeomorphisms from σ to $\mathscr{U}$. The map $\mathscr{G}_{\mathscr{U}}\to\Gamma_x$ is evaluation at x, and the kernel $\mathscr{G}^1_{\mathscr{U}}$ consists of those sections which meet the unit section at x. Finding an isomorphism of $\Gamma_{\mathscr{U}}$ with an action groupoid amounts to finding a cross section $\Gamma_x\to\mathscr{G}_{\mathscr{U}}$ which is a group homomorphism.

Since the properness of Γ implies that Γ_x is compact, we could construct a homomorphic section by averaging if the kernel $\mathscr{G}^1_{\mathscr{U}}$ were the additive group of a vector space. Of course, this is not the case, but we do get a nice composition series for $\mathscr{G}^1_{\mathscr{U}}$ if we pass to infinite jets at x of admissible sections of Γ. We can then use the standard stepwise proof (this time using group cohomology) to find an action groupoid structure for Γ over a "formal neighborhood of x". Note that semisimplicity plays no role here — compactness of Γ_x is enough, so it could be a torus, for instance.

In the analytic or smooth categories, we could try to imitate Conn's proofs, but instead we propose another approach. We confine our discussion to the smooth case.

Since α is a locally trivial fibration, we can choose a cross section $\sigma_0 : \Gamma_x \to \mathscr{G}_{\mathscr{U}}$ for sufficiently small $\mathscr{U}$ such that the corresponding map from $\Gamma_x \times \mathscr{U}$ to $\alpha^{-1}(\mathscr{U})$ is a diffeomorphism. This σ_0 will not generally be a group homomorphism, so we will try to "improve" it to become one. By the formal normal form result, we can assume (here we imitate Conn [6]) that σ_0 is a homomorphism modulo flat sections; i.e., for each g and h in Γ_x, $\sigma_0(g)\sigma_0(h)$ and $\sigma_0(gh)$ are tangent to infinite order at the point gh where they meet Γ_x. Using our local trivialization of α and local coordinates on a sufficiently small open neighborhood $\mathscr{U}$ of x, we can assume that, for all g and h in Γ_x, $\sigma_0(g)\sigma_0(h)$ and $\sigma_0(gh)$ considered as maps from $\mathscr{U}$ to Γ_x are as close as we wish, together with as many derivatives as we desire.

Thus, giving $\mathscr{G}_{\mathscr{U}}$ a C^k-metric, we can say that σ_0 is an "almost homomorphism" in the sense that $d(\sigma_0(g)\sigma_0(h), \sigma_0(gh)$ is small for all g and h in Γ_x, and our problem is to approximate this almost homomorphism by a homomorphism.

5. Almost homomorphisms and almost invariant submanifolds

There are quite a few theorems which assert that an almost homomorphism is near a homomorphism (a very general formulation of this problem was proposed by Ulam [21]), but none of them suits our needs, so we must prove new ones.

The following two theorems are closest to what we are looking for.

Almost Homomorphism Theorem. (Grove-Karcher-Ruh [12]). *Let G and H be compact Lie groups. Choose a bi-invariant metric on H such that: (1) the exponential map is an embedding when restricted to the open ball of radius π in the Lie algebra $\mathfrak{h}$; (2) the Lie bracket on $\mathfrak{h}$ satisfies $||[v,w]|| \leq ||v||\ ||w||$. (Such a metric always exists.) Let $\sigma_0 : G \to H$ be a continuous map such that $d(\sigma_0(g)\sigma_0(h), \sigma_0(gh)) \leq q \leq \pi/6$ for all g and h in G. Then there is a (continuous) homomorphism $\sigma : G \to H$ such that $d(\sigma_0(g), \sigma(g)) < 1.36q$ for all g in G.*

Almost Representation Theorem. (de la Harpe-Karoubi [8]) *Let T_0 be a continuous map from a compact group G to the group H of invertible bounded linear operators on some Hilbert space. Let $K \geq 1$ and $\varepsilon \leq 2^{-6}$ be real numbers such that $||T_0(gh) - T_0(g)T_0(h)|| \leq \varepsilon(2K)^{-9}$ for all g and h in G. Then there is a continuous homomorphism $T : G \to H$ such that $||T_0(g) - T(g)|| \leq \varepsilon$ for all g in G.*

An intermediate result between those theorems and the one we want for groupoid linearization would be the following, stated somewhat imprecisely.

Almost Action Conjecture. *Let G be a compact group and H the group of diffeomorphisms of a compact manifold M. If $\sigma : G \to H$ is a map such that the distance $d(\sigma_0(g)\sigma_0(h), \sigma_0(gh)$ is sufficiently small for all g and h in G, then there is a homomorphism ψ close to σ. Here, the distance between two diffeomorphisms is taken to be their C^1 distance defined using a riemannian metric μ, and the measure of "sufficient smallness" will depend on the size of the first and second derivatives* *of the maps in $\phi(G)$, as well as on upper bounds on the absolute value of the curvature and the reciprocal of the injectivity radius for the metric μ.*

Even this conjecture is currently beyond our reach. When G is finite, it should follow from the following result in [25].

Almost Invariant Submanifold Theorem. *Let N be a compact submanifold of a riemannian manifold M, and let G be a compact group acting on M. Choose an invariant metric on M such that: (1) the exponential map of M, restricted to the normal bundle of N, is (defined and) an embedding on the open ball bundle B of radius 1; (2) all the sectional curvatures of M on $\exp(B)$ have absolute value less than 1; (3) the exponential map of M is (defined and) an embedding on the ball of radius 1 in each tangent space of $\exp(B)$. (This is always possible: Multiply any invariant metric by a suitably large constant.) If the C^1 distance (defined below) $d(N,gN)$ is less than $\varepsilon < \frac{1}{20000}$ for each $g \in G$, then there is a G-invariant submanifold $\overline{N} \subseteq M$ such that $d(N,\overline{N}) < 136\sqrt{\varepsilon}$.*

The distance $d(N,N')$ between submanifolds in the theorem above is defined as follows. First of all, we assume that N' is the image under exp of a section s of the normal bundle of N. For each $x \in N$, we take the maximum $a(x)$ of the following two numbers: The length of the geodesic segment σ from x to $s(x)$, and the maximum angle between unit vectors in T_xN and $T_{x'}N'$, where vectors in one space are moved to the other by parallel transport along σ. Now $d(N,N')$ is defined as $\max_{x\in N} a(x)$.

Like the proofs of the Almost Homomorphism Theorem and the Almost Representation Theorem, the proof of the Almost Invariant Submanifold Theorem uses averaging over the group G. The estimates involved are nontrivial, but they are somewhat simpler, and certainly more geometric, than the ones in Conn's proof. The challenge now is to work back from here to prove some linearization theorems without adding too much more complication.

6. Properness and convexity

Our interest in proper groupoids was originally motivated, not by linearization problems, but by an attempt to understand the convexity theorems of Atiyah [2], Guillemin-Sternberg [14, 15] and Kirwan [17] as results in Poisson geometry. Their theorems establish convexity properties of the image of the momentum map $J : S \to \mathfrak{g}^*$ for a hamiltonian action of a compact Lie group G on a symplectic manifold S. When G is a torus, the theorem states that $J(S)$ is a convex polyhedron, while for general compact G it is the intersection of $J(S)$ with a positive Weyl chamber which is convex.

When G is simply connected, a momentum map is simply a Poisson map from S to $\mathfrak{g}^*$, which suggests that there might be a convexity theorem for Poisson maps $J : S \to P$ from symplectic manifolds to a wider class of Poisson manifolds. The case of the torus $\mathbb{T}^n$ shows that the problem involves more than just Poisson manifolds: In this case, $P = \mathscr{T}^{n*}$ has the zero Poisson structure, so that the set of Poisson maps from S to P is closed under arbitrary diffeomorphisms of P, which usually destroy any convexity properties of the image. The point here is that J must the momentum map for an action of $\mathbb{T}^n$ and not just of its universal covering $\mathbb{R}^n$.

To express in geometric terms the choice of group associated to a given (dual of a) Lie algebra, we recall that hamiltonian actions of the Lie group G correspond to symplectic actions of the symplectic groupoid $T^*G \rightrightarrows \mathfrak{g}^*$ [20]. (The essentials of this result were presented in [23], before the advent of symplectic groupoids in the 1980's.) We therefore pose the following conjecture (having well in mind that some further hypotheses may be needed.)

Poisson Convexity Conjecture. *Let $\Gamma \rightrightarrows P$ be a proper symplectic groupoid over the Poisson manifold P, and let $J : S \to P$ be the moment map (see [20]) for an action of Γ on a symplectic manifold S. Then the image $J(S) \subseteq P$ has a convexity property with respect to some affine structure attached to the groupoid Γ.*

The problem here is to establish the appropriate notion of convexity, which requires an understanding of the structure of proper symplectic groupoids. Such an understanding would also tell us the extent to which our conjecture goes beyond the known convexity theorems for group actions.

Examples. Let G be a semisimple Lie group of noncompact type. The **elliptic** subject $E \subset \mathfrak{g}^*$ is defined to consist of those elements whose coadjoint isotropy group in G is compact. (Under the orbit method, E corresponds to the discrete series of representations of G.) It appears that E is an open subset on which the restriction of the coadjoint representation is a proper action. By restricting the symplectic groupoid $T^*G \rightrightarrows \mathfrak{g}^*$ to E, we obtain a proper symplectic groupoid for E. Now a hamiltonian action of G on S may be called "elliptic" if the image $J(S)$ of the momentum map lies in E. A special case of our problem would be to establish a convexity theorem for the momentum maps of such elliptic actions.

The previous example is still a transformation groupoid. Another example of a proper symplectic groupoid is the fundamental groupoid $\pi(M)$ of a compact, connected, symplectic manifold M with finite fundamental group. Symplectic actions of this groupoid correspond to symplectic actions of the fundamental group. (See Section 7.6 of [4].) In this case, there is no convexity problem, since momentum maps to the symplectic manifold M are surjective, but perhaps products $\pi(M) \times T^*G \rightrightarrows M \times \mathfrak{g}^*$ will give interesting examples.

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