Homotopy Gerstenhaber algebras

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Abstract. The purpose of this paper is to complete Getzler-Jones' proof of Deligne's Conjecture, thereby establishing an explicit relationship between the geometry of configurations of points in the plane and the Hochschild complex of an associative algebra. More concretely, it is shown that the B_∞-operad, which is generated by multilinear operations known to act on the Hochschild complex, is a quotient of a certain operad associated to the compactified configuration spaces. Different notions of homotopy Gerstenhaber algebras are discussed: One of them is a B_∞-algebra, another, called a homotopy G-algebra, is a particular case of a B_∞-algebra, the others, a G_∞-algebra, an $\overline{E}^1$-algebra, and a weak G_∞-algebra, arise from the geometry of configuration spaces. Corrections to the paper of Kimura, Zuckerman, and the author related to the use of a nonextant notion of a homotopy Gerstenhaber algebra are made.

Keywords: Gerstenhaber algebra, Hochschild cohomology, homotopy algebra, little disks operad, moduli space of punctured spheres

Mathematics Subject Classifications (2000): 16E40, 18C99, 55P99, 14H10, 17A30, 17A42

Dedicated to the memory of Moshé Flato

In an unpublished paper of E. Getzler and J. D. S. Jones [10], the notion of a homotopy n-algebra was introduced. Unfortunately the construction that justified the definition contained an error, which passed unnoticed in subsequent work, in spite of being heavily used in it. That work included the solution by Getzler and Jones [10] of Deligne's Conjecture, whose weak version had been proven in [28]; the construction by T. Kimura, G. Zuckerman, and the author [21] of a homotopy Gerstenhaber algebra structure (called a G_∞-algebra therein) on the state space of a topological conformal field theory (TCTF); the extensions of the above work by Akman [1, 2] and Gerstenhaber and the author [28]; a few papers delivered at the Workshop on Operads in Osnabrück in June 1998 [29]. The purpose of this paper is to correct the error in the original construction of [10], complete Getzler-Jones' proof of Deligne's Conjecture accordingly, and make appropriate corrections in [21].

First, let us describe the problem. A *Gerstenhaber (G-) algebra* is defined by two operations, a (dot) product ab and a bracket $[a,b]$, on a graded vector space V over a ground field k of characteristic zero, so that the product defines a graded commutative algebra structure on V and the bracket a graded Lie algebra structure on $V[1]$, the desuspension of the graded vector space $V = \bigoplus_n V^n$: $V[1]^n = V^{n+1}$.

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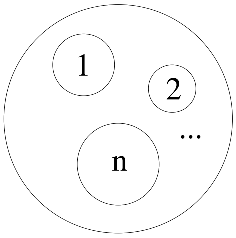

Figure 1. The little disks operad.

The bracket must be a graded derivation of the product in the following sense:

$$[a,bc] = [a,b]c + (-1)^{(|a|-1)|b|} b[a,c],$$

where $|a|$ denotes the degree of an element $a \in V$. In other words, a G-algebra is a specific graded version of a Poisson algebra.

A G-algebra may be equivalently defined as an algebra over the operad $e_2 = H_\bullet(D) = H_\bullet(D;k)$ of the homology of the *little disks operad*. This is a collection of manifolds $D(n)$, $n \geq 1$, where each $D(n)$ is the configuration space of n nonoverlapping little disks inside the standard unit disk in the plane, see Figure 1.

The space $D(n)$ is obviously an open set in $\mathbb{R}^{3n}$ (whence a manifold structure), each configuration being uniquely determined by the position of the centers of the disks and their radii. It is assumed that each little disk is labeled by a number from 1 through n, which defines the action of the permutation group Σ_n on $D(n)$. The operad composition

$$\gamma : D(k) \times D(n_1) \times \cdots \times D(n_k) \to D(n_1 + \cdots + n_k)$$

is given by scaling down given configurations in $D(n_1), \ldots, D(n_k)$, gluing them into the k holes in a given configuration in $D(k)$, and erasing the seams. Thus $D = \{D(n) \mid n \geq 1\}$ becomes an operad of manifolds.

The way the little disks operad D has relevance to G-algebras is through the following theorem.

Theorem 0.1. **(F. Cohen [5])** *The structure of a G-algebra on a $\mathbb{Z}$-graded vector space is equivalent to the structure of an algebra over the homology little disks operad $H_\bullet(D)$.*

In view of this theorem, we will refer to the operad $e_2 := H_\bullet(D)$ as the *G-operad.*

On the other hand, a purely algebraic example of a G-algebra was given by the following result.

Theorem 0.2. **(M. Gerstenhaber [8])** *The Hochschild cohomology of an associative algebra has the natural structure of a G-algebra with respect to the cup product and the Gerstenhaber (G-) bracket.*

Thus the Hochschild cohomology is naturally an algebra over the homology little disks operad e_2. In a letter [6], P. Deligne pointed out that there must be more than this formal relationship between the little disks operad and the Hochschild cohomology, which later became quoted as the following conjecture.

Conjecture 0.3. **(Deligne's Conjecture)** *The structure of an algebra over the homology little disks operad e_2 on the Hochschild cohomology may be naturally lifted to the (co)chain level.*

The goal of this paper is to prove a version of this conjecture, see Corollary 3.2. The conjecture has found several interpretations, both algebraic and geometric. Algebraically, the conjecture would imply the structure of a homotopy G-algebra on the Hochschild cochain complex. Here is an account of what is known about it up to date.

- A homotopy G-algebra structure on the Hochschild complex was defined by Gerstenhaber and the author [28]. Apart from the cup product and the G-bracket, we used higher operations (called *braces*) on the Hochschild complex constructed by T. Kadeishvili [14] and Getzler [9], and wrote down a set of identities for the operations interpreted as homotopies for the G-algebra identities and a hierarchy of homotopies between homotopies. The bilinear and trilinear braces providing homotopies for the commutativity of the cup product and the distributivity of the G-bracket, respectively, had been known since the original paper of Gerstenhaber [8].

- Getzler and Jones [10] showed that the same operations on the Hochschild complex defined the structure of a B_∞-*algebra*, a more general version of a homotopy G-algebra introduced by H. Baues in the study of the double bar construction in algebraic topology, see Definition 2.1 below. The B_∞-algebra structure on the Hochschild complex is obtained by setting one of the B_∞-operations to be the dot product, some others to be the braces, and the others to zero. In terms of operads, the B_∞-structure on the Hochschild complex is obtained by realizing the homotopy G-operad of the previous paragraph as a quotient of the B_∞-operad and using the homotopy G-structure on the Hochschild complex.

- Tamarkin [26] extended the cup product and the G-bracket on the Hochschild complex to the structure of a G_∞-*algebra*, which is the most canonical notion of a homotopy G-algebra. It may be defined in terms of the corresponding operad, which is the minimal model of the G-operad in the sense of M. Markl [23]. The G_∞-algebra structure of Tamarkin was again constructed by defining a morphism of operads $G_\infty \to B_\infty$ and using the B_∞-algebra structure described in the previous two paragraphs. The construction of the morphism $G_\infty \to B_\infty$ is very involved: It uses the existence of Drinfeld's associator and Etingof-Kazhdan's quantization theorem. Tamarkin [26] used this construction in his algebraic proof of Kontsevich's Formality Theorem [17], which implied the Deformation Quantization Conjecture [4].

Remark 1 Note that we used the same notation "G_∞-algebra" for a different object in our paper [21]. Since as it has turned out, that object does not exist, this should not create any confusion.

The above results are purely algebraic. However they bring more evidence to the relationship between the geometry of the little disks operad and the algebra of the Hochschild complex hinted by Deligne's Conjecture. Moreover, to establish this relationship, one just needs to relate the above operads (the homotopy G-operad, B_∞, and G_∞) to the little disks operad. Here is what is known in this direction.

- Getzler and Jones [10] noticed that the G_∞-operad is isomorphic to the the first term of a spectral sequence associated to the so-called "topological" filtration of the moduli space operad of configurations of points in the plane. This operad is homotopy equivalent to the little disks operad. For the moduli space operad, Getzler and Jones also offered the geometric construction of a cellular model which mapped surjectively to the B_∞-operad. Unfortunately, there was an error in the construction: The "cellular model" was, strictly speaking, not cellular, because its components were not cell complexes. A counterexample was found by Tamarkin.

- Another interesting relation between the G_∞-operad and the little disks operad is the idea of a recent preprint of Tamarkin [27], who showed that the singular chain operad $C_\bullet(D)$ of little disks is *formal*, i.e., quasi-isomorphic to its homology $H_\bullet(D)$. Kontsevich [18] found a very simple geometric proof of this result. Both results imply that there exists a morphism $G_\infty \to C_\bullet(D)$ unique up to homotopy.

Since all the algebraic solutions of Deligne's Conjecture so far used the explicit B_∞-structure on the Hochschild complex, finding an explicit relationship between the B_∞-operad and the little disks operad seems to be most crucial for understanding the relationship between algebra and geometry suggested by Deligne's Conjecture. Moreover, the operads G_∞ and $C_\bullet(D)$, which have obvious geometric meaning, do not act explicitly on the Hochschild complex. In our understanding, placing the B_∞-operad within the topology of the little disks operad will provide the most complete solution of Deligne's Conjecture.

In this paper, we show that the B_∞-operad is a quotient of a certain operad $\overline{E}^1$ arising geometrically from the moduli space operad of configurations of points in the plane. The operad $\overline{E}^1$ is the first term of the spectral sequence associated to a filtration of the moduli space operad. We show that the operad $\overline{E}^1$ is free as an operad of graded vector spaces and quasi-isomorphic to the G-operad. We construct an explicit surjection from the operad $\overline{E}^1$ to the B_∞-operad, which will imply an explicit $\overline{E}^1$-algebra structure on the Hochschild complex. We find it truly remarkable that the object B_∞ of the algebraic world (with its natural action on the Hochschild complex) is ruled by the geometry of the configuration spaces (through the operad $\overline{E}^1$). For example, see Section 2.3, where the defining relations of a B_∞-algebra are read off from the incidence relations for the strata in the moduli spaces.

Since $\overline{E}^1$ is quasi-isomorphic to the G-operad, it follows from homotopy theory of operads that there exists a morphism $G_\infty \to \overline{E}^1$. This also implies the existence of a G_∞-algebra structure on the Hochschild complex.

At the end we will indicate which changes are to be made to the paper [21] by Kimura, Zuckerman, and the author, which utilized the incorrect notion of a G_∞-algebra. Briefly, the changes are that this notion should be replaced by the correct one, which brings corrections to the identities described implicitly in [21]. However, all the identities written out in [21] explicitly do not require corrections.

Remark 2 While this paper was in preparation, there were made two announcements of results of similar nature. J. McClure and J. Smith [24] constructed a cellular operad acting on the Hochschild complex and announced that it was homotopy equivalent to the little disks operad. Kontsevich [18] announced the construction of an operad with an explicit quasi-isomorphism to the G-operad and an explicit action on the Hochschild complex, along with the proof of his multi-dimensional generalization of Deligne's Conjecture. The proof of Deligne's Conjecture along these lines is published in this volume, see [19].

We will use the following *terminology* regarding basic notions of topology. All *topological spaces* considered will be Hausdorff, except when referring to complex algebraic curves, we will use standard terminology of Zariski topology, such as an irreducible component. For a topological space X, let $X^\bullet$ denote its *one-point compactification*, which is $X^\bullet = X$ if X is compact and $X^\bullet = X \cup \{\infty\}$ otherwise. A (p*-dimensional*) *cell* in a topological space X is a subset $E \subset X$ along with a continuous map $f : I^p \to \bar{E}$, where I^p is the closed unit cube in $\mathbb{R}^p$ with the boundary ∂I^p, such that f is a homeomorphism in the interior of I^p. A *cellular partition of* X is a partition of X into the disjoint union of cells. A *cell-complex structure on* X, or equivalently, a *cellular decomposition of* X, is a cellular partition such that for each p-dimensional cell $E \subset X$, its boundary ∂E is contained in X^{p-1}, where X^{p-1} is the $p-1$ *skeleton of* X, the union of $p-1$-dimensional cells. All cell complexes considered in the paper will be *finite*, i.e., consist of finitely many cells, and therefore automatically be *CW-complexes*. Cell complexes form a tensor category with respect to cellular maps and direct products. A *cellular operad* is an operad of cell complexes. A *stratification* of a manifold is a decomposition of the manifold into the disjoint union of connected submanifolds, called *strata*, so that the boundary of a stratum is the union of strata of lower dimensions.

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1. The G_∞-operad

1.1. Getzler-Jones' cellular partition

Let us recall the construction of Getzler and Jones [10] of a cellular partition of compactified configuration spaces.

1.1.1. *Fox-Neuwirth cells*

Consider Fox-Neuwirth's cellular decomposition [7] of the one-point compactification of a configuration space $F(\mathbb{C},n) = \mathbb{C}^n \setminus \Delta$, where Δ is the fat diagonal $\cup_{i\neq j}\{x_i = x_j\}$, of $n \geq 1$ distinct points in the complex plane $\mathbb{C}$: The cells, which we will call *Fox-Neuwirth cells*, will be labeled by ordered partitions of the set $\{1,\dots,n\}$ into ordered subsets. For example, $\{3\}\{2,1\}$ denotes a partition of $\{1,2,3\}$ into two subsets: the first subset is $\{3\}$ and the second subset is $\{2,1\}$, ordered so that 2 precedes 1. Partitioning labels $\{1,\dots,n\}$ into ordered subsets reflects grouping points lying on common vertical lines on the plane. Ordering between subsets is the left-to-right order between the vertical lines; ordering within a subset is the bottom-to-top order within the vertical line. For each $n \geq 1$, take the quotient space

$$\mathscr{M}(n) = F(\mathbb{C},n)/\mathbb{R}^2 \rtimes \mathbb{R}_+^*$$

by the action of translations and dilations on $F(\mathbb{C},n)$. The dimension of $\mathscr{M}(n)$ is equal to $2n-3$ for $n \geq 2$ and 0 for $n = 1$. The Fox-Neuwirth cells are obviously invariant under this action, and their quotients, which we will also call *Fox-Neuwirth cells*, make up a cellular decomposition of the one-point compactification $\mathscr{M}(n)^\bullet$ of $\mathscr{M}(n)$. The spaces $\mathscr{M}(n)$ do not form an operad, but one can glue lower $\mathscr{M}(k)$'s to the boundaries of higher $\mathscr{M}(n)$'s to form a topological operad $\underline{\mathscr{M}} = \{\underline{\mathscr{M}}(n) \mid n \geq 2\}$. In fact, the underlying spaces $\underline{\mathscr{M}}(n)$ are smooth manifolds with corners compactifying $\mathscr{M}(n)$, see next section.

All the spaces $D(n)$, $F(\mathbb{C},n)$, $\mathscr{M}(n)$, and $\underline{\mathscr{M}}(n)$ are homotopy equivalent. Moreover D and $\underline{\mathscr{M}}(n)$ are homotopy equivalent operads, see [10].

1.1.2. *Compactified moduli spaces*

The resulting space $\underline{\mathscr{M}}(n)$ is an S^1-bundle over the real compactification $\underline{\mathscr{M}}_{0,n+1}$ of the moduli space $\mathscr{M}_{0,n+1}$ of $n+1$-punctured curves of genus zero, see [10, 12]. The space $\underline{\mathscr{M}}(n)$ can also be interpreted as a "decorated" moduli space, see [12]. Indeed, it can be identified with the moduli space of data $(C;x_1,\dots,x_{n+1};\tau_1,\dots,\tau_m,\tau_\infty)$, where C is a stable complex complete algebraic curve with $n+1$

punctures $x_1, \dots, x_{n+1}$ and m double points. For each i, $1 \le i \le m$, τ_i is the choice of a tangent direction at the ith double point to the irreducible component that is farther away from the "root", i.e., from the component of C containing the puncture $\infty := x_{n+1}$, while τ_∞ is a tangent direction at ∞. The stability of a curve is understood in the sense of Mumford's geometric invariant theory: Each irreducible component of C must be stable, i.e., admit no infinitesimal automorphisms. The operad composition is given by attaching the ∞ puncture on a curve to one of the other punctures on another curve, keeping the tangent direction at each new double point.

1.1.3. *Getzler-Jones cells*
Fox-Neuwirth's cell decomposition of each space $\mathscr{M}(n)^\bullet$ induces a cell partition of the compactification $\underline{\mathscr{M}}(n)$ in the following way. First of all, by an *n-tree* we mean a directed rooted tree with n labeled initial edges, the *leaves*, and one terminal edge, the *root*, each of these $n+1$ edges incident to only one vertex of the tree, such that the number $n(v)$ of the incoming edges for any vertex v is at least two. Define the *tree degree* of an n-tree T as $n - v(T) - 1$, where $v(T)$ is the number of vertices in T. The cells, which we will call *Getzler-Jones cells*, in $\underline{\mathscr{M}}(n)$ are enumerated by pairs (T, p), where T is a tree, labeling a stratum of the boundary of $\underline{\mathscr{M}}(n)$, and p is a function $p(v)$ on the vertices v of the tree T, such that each $p(v)$ is an ordered partition, as in 1.1.1 above, of the set $\mathrm{in}(v)$ of incoming edges for a vertex v of the T. These partitions $p(v)$ label cells in the corresponding open moduli spaces $\mathscr{M}(n(v))$, whose products make up the stratum. However generally speaking, it is not true that this cellular partition is a cell complex: The boundary of a q-dimensional cell does not always lie in the $q-1$-skeleton. We will take the union of certain Getzler-Jones cells to form a stratification of $\underline{\mathscr{M}}(n)$ compatible with the operad structure.

1.2. Stratification of $\underline{\mathscr{M}}$

1.2.1. *Stratification of $\mathscr{M}$*
Consider the following subsets in $\mathscr{M}(n)$.

- For each ordering of the set $\{1, \dots, n\}$, consider the corresponding Fox-Neuwirth cell. It consists of configurations of n points on a vertical line in the prescribed order going from bottom to top.
- For each ordered partition of the set $\{1, \dots, n\}$ into two parts, consider the corresponding Fox-Neuwirth cell. It consists of configurations of n points on two vertical lines in the prescribed order going from bottom to top and from left to right.
- The complement to the union of subsets of the above two types.

These subsets obviously form a stratification of the manifold $\mathscr{M}(n)$. We will also need the following filtration of $\mathscr{M}(n)$ into three closed subsets: The closure J_1 of the union of the strata of the first type, the closure J_2 of the union of the strata of the second type, and J_3 which is all of $\mathscr{M}(n)$.

Adding the point at ∞ to each filtration component and setting $J_0 = \{\infty\}$, we get a filtration of the pointed space $\mathscr{M}(n)^\bullet$ for $n \geq 3$:

$$J_0 \subset J_1 \subset J_2 \subset J_3 = \mathscr{M}(n)^\bullet. \tag{1}$$

For $n = 2$, the space $\mathscr{M}(2)$ is diffeomorphic to S^1 and thereby compact. In this case we will not add a point ∞ to any filtration components and will set $J_0 = \emptyset$.

Consider the corresponding homological spectral sequence converging to $H_\bullet(\mathscr{M}(n)^\bullet, \infty)$, which is naturally isomorphic to $H^\bullet(\mathscr{M}(n))$ by Poincaré-Lefschetz duality, all coefficients being taken in k. The term E^1 may be identified with the sum of $E^1_{p,q} = H_{p+q}(J_p, J_{p-1})$, $1 \leq p \leq 3$, $-p \leq q \leq 2n - p - 3$.

Proposition 1.1 *The homological spectral sequence associated to the filtration* (1) *collapses at* E^2.

Proof. Since $J_2 \setminus J_1$ and $J_1 \setminus J_0$ are disjoint unions of $n!(n-1)$ and $n!$ cells of dimension $n-1$ and $n-2$, respectively,

$$\dim H_{2+q}(J_2, J_1) = \begin{cases} n!(n-1) & \text{for } 2+q = n-1, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$\dim H_{1+q}(J_1, J_0) = \begin{cases} n! & \text{for } 1+q = n-2, \\ 0 & \text{otherwise}. \end{cases}$$

Note also that with respect to the Fox-Neuwirth cellular decomposition of J_3, the subspace J_2 is a cell subcomplex. The dimensions of cells in the complement of J_2 run from n through $2n-3$. Thus, $H_{p+q}(J_3, J_2) = 0$, unless $n \leq p+q \leq 2n-3$. Observe that for any pair (p,q) and integer $r \geq 2$, either $E^r_{p,q}$ or $E^r_{p-r,q+r-1}$ is 0. Thus the spectral-sequence differentials $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ will all vanish for $r \geq 2$, which implies the collapse of the spectral sequence at E^2. ∎

1.2.2. *Stratification of $\underline{\mathscr{M}}$*

First of all, stratify $\underline{\mathscr{M}}(n)$ by the topological type of the stable algebraic curve. We will refer to this stratification as *coarse*. Each stratum S_T will correspond to a tree T and be isomorphic to the product of the spaces $\mathscr{M}(n(v))$ over the vertices v of the tree as in 1.1.3. Then subdivide the coarse stratification as follows. For each space $\mathscr{M}(n(v))$, take the stratification into three types of strata as in the previous section. The products of these strata over the set of vertices v of T will form a finer partition of the space $\underline{\mathscr{M}}(n)$. Figure 2 pictures the part obtained as the product of two strata $J_2 \setminus J_1$ in $\mathscr{M}(3)$ and $J_1 \setminus J_0$ in $\mathscr{M}(2)$.

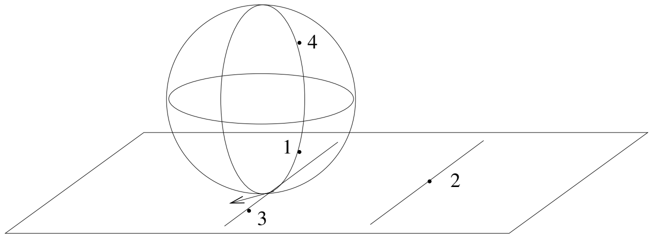


Figure 2. A stratum: a 3D picture.

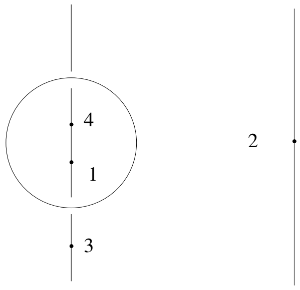


Figure 3. The same stratum: a 2D picture.

Here the plane denotes the irreducible component of the stable curve with the puncture ∞ placed at ∞. The other irreducible component is the sphere. The arrow at the double point determines the positive direction of the real axis on the sphere. We prefer to work with the following replacement of the above figure, which one may think of as projection of the three-dimensional figure onto the plane after rotating the sphere with the arrow, so that the arrow points in the direction of the positive real axis on the plane. The circle may be thought of as a magnifying glass through which the observer living on the plane sees what happens in the infinitesimal world at the double point, see Figure 3.
We will usually think of either of these figures as not a single configuration of points, but rather the set of all configurations of points which are in the relative position determined by the figure.

The above partition of the space $\underline{\mathscr{M}}(n)$ will in fact be a stratification. The reason is that the boundary of each part may be split into the union of the boundary within the coarse stratum S_T and the boundary within the boundary of S_T. The boundary of the first type is the union of other strata within the coarse stratum S_T, because we used a stratification of each $\mathscr{M}(n(v))$ to construct our partition of $\underline{\mathscr{M}}(n)$. The boundary of the second type is obtained by letting a few groups of points on the irreducible components of the stable curve bubble off, forming new components attached at double points. If all the points on a component lie on a single vertical line, the points on the limiting components will also group on a single vertical

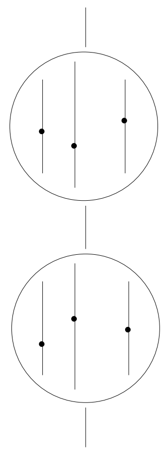


Figure 4. A proper subset of this cell lies in the boundary of another cell.

line within each bubble, and again, the whole fine part will be in the boundary. If all the points on a component lie on two vertical lines, then the points on the components that bubble off will lie on two or one vertical lines, and again, the whole fine part will be in the boundary, because by translations and dilations, one can always match up the vertical lines on different components (if less than three on each), for example, pass through points 0 or 1 on the real axis. Since $J_3 \setminus J_2$ comprises the top strata in $\mathscr{M}(n(v))$, the closure of this component of the stratum will be all $\underline{\mathscr{M}}(n(v))$, therefore, the boundary is $J_2 \cup \partial \underline{\mathscr{M}}(n(v))$, which is the union of lower-dimensional strata.

Remark 3 Note that if we extended our partition to a cell partition by taking all the Getzler-Jones cells, the boundary of a cell would not be the union of other cells, in general. For example, take the Fox-Neuwirth cell formed by six points on two vertical lines, three points on each. Its boundary has a nonempty intersection with the Getzler-Jones cell sketched on Figure 4.

However, only part of the this Getzler-Jones cell will be in the boundary of the Fox-Neuwirth cell. This part will consist of those positions of three vertical lines on the two bubbles that can be brought together by the action of $\mathbb{R}^2 \rtimes \mathbb{R}_+^*$ on a single copy of the Riemann sphere. This is exactly the problem with Getzler-Jones' partition not being a cell complex. Our stratification is designed to get around this problem.

Now form two filtrations $\dots \subset F^p \subset F^{p+1} \subset \dots$ and $\dots \subset \overline{F}^p \subset \overline{F}^{p+1} \subset \dots$ by taking F^p (respectively, $\overline{F}^p$) to be the closure of the union of coarse (respectively, fine) strata of dimension p. Consider the corresponding spectral sequences, which converge to $H_\bullet(\underline{\mathscr{M}}(n))$ and whose first terms are

$$E^1_{p,q} = H_{p+q}(F^p, F^{p-1}),$$
$$\overline{E}^1_{p,q} = H_{p+q}(\overline{F}^p, \overline{F}^{p-1}).$$

1.2.3. *Operad properties of the first spectral sequence E^r*
Note that the operad composition respects the two filtrations of spaces $\underline{\mathscr{M}}(n)$, $n \geq 1$. Therefore, we are getting two operads of spectral sequences, cf. [20]. In particular, the first terms E^1 and $\overline{E}^1$ of the spectral sequences form operads of complexes. There is a purely algebraic interpretation of the first operad E^1, noticed by Getzler and Jones in [10].

Proposition 1.2. **(Getzler and Jones)** *The operad E^1 is naturally isomorphic to the cobar construction of the Gerstenhaber operad $e_2 = H_\bullet(D)$.*

Proof. Notice that for the space $\underline{\mathscr{M}}(n)$, $n \geq 2$, the group $E^1_{p,q} = H_{p+q}(F^p, F^{p-1})$ is naturally isomorphic to $H^{-q}(F^p \setminus F^{p-1})$ by Poincaré-Lefschetz duality. The space $F^p \setminus F^{p-1}$ is the disjoint union of the strata S_T, T running over the set of n-trees T which have exactly $2n-2-p$ vertices. Each stratum S_T is naturally isomorphic to the product of spaces $\mathscr{M}(n(v))$ over the set of vertices v of the tree T. Each space $\mathscr{M}(n(v))$ is homotopy equivalent to the configuration space $D(n(v))$ of $n(v)$ little disks. Thus

$$E^1_{p,q} = \bigoplus_{n\text{-trees } T} \left(\bigotimes_{v \in T} H^\bullet(D(n(v))) \right)^{-q}, \tag{2}$$

where the superscript $-q$ means the component of degree $-q$. The differential

$$d^1 : E^1_{p,q} \to E^1_{p-1,q}$$

takes the component $\bigotimes_{v\in T} H^\bullet(D(n(v)))$ corresponding to a n-tree T to the sum of components $\bigotimes_{v\in \widehat{T}} H^\bullet(D(n(v)))$ over all n-trees $\widehat{T}$ such that the tree T may be obtained by contracting an interior edge of $\widehat{T}$, merging two adjacent vertices v_1 and v_2 on the tree $\widehat{T}$ into a single vertex v_3 of T. The matrix element $d^1_{T\widehat{T}}$ of the differential is induced (up to a sign, which is treated below) by the map

$$\circ_i^* : H^\bullet(D(n(v_3))) \to H^\bullet(D(n(v_1))) \otimes H^\bullet(D(n(v_2)))$$

which is the dual of the corresponding operad structure map

$$\circ_i : H_\bullet(D(n(v_1))) \otimes H_\bullet(D(n(v_2))) \to H_\bullet(D(n(v_3))).$$

This map is induced on homology by the map

$$\circ_i : D(n(v_1)) \times D(n(v_2)) \to D(n(v_3))$$

gluing the unit disk with a configuration of $n(v_2)$ little disks into the ith little disk in a configuration of $n(v_2)$ little disks, i corresponding to the contracted edge in $\mathrm{in}(v_1)$, if we assume that the contracted edge is directed from v_2 to v_1.

The sign for $d^1_{T\widehat{T}}$ comes from the choice of orientation of the strata S_T. This orientation may be chosen by ordering all the edges of each n-tree T, except the root edge. The orientation on the stratum S_T is then given by ordering the x and the y coordinates of the points in $\mathbb{C}$ corresponding to the edges of T, according to the order of the edges, skipping for each vertex v the coordinates of the point corresponding to the first edge and the x coordinate of the point corresponding to the second edge — remember that $\mathscr{M}(n) = F(\mathbb{C},n)/\mathbb{R}^2 \rtimes \mathbb{R}^*_+$. Then the compatibility of orientations on S_T and $S_{\widehat{T}}$ implies that $d^1_{T\widehat{T}}$ is $\circ_i^*$ multiplied by the sign of the permutation from the ordered set of edges of $\widehat{T}$ to the ordered set $\{e, \text{edges of } T\}$, where e is the contracted edge in $\widehat{T}$.

This description of E^1 in terms of trees and the operad $H_\bullet(D)$ of vector spaces means that E^1 is the cobar construction of the operad $H_\bullet(D)$, just by the definition of Ginzburg-Kapranov [11]. ■

Corollary 1.3. **(Getzler and Jones, Markl)** *The operad E^1 is a free resolution of the G-operad e_2, i.e., there is a morphism of operads $E^1 \to e_2$, inducing an isomorphism on homology and E^1 is free as an operad of graded vector spaces. Moreover E^1 is a minimal model of e_2.*

Remark 4 Here we sketch a proof due to Markl [22], which is different from that of Getzler and Jones [10]. The fact that E^1 is a minimal model of e_2 was first noticed by Markl [22].

Proof. Using the description of the operad e_2 in terms of generators and relations, it is a straightforward exercise to check that the quadratic dual of e_2 is again e_2 up to a shift of grading and the change of it to the opposite. Then from the proof of Proposition 1.2, one can identify E^1 with the cobar construction of $e_2^!$. The natural homomorphism, see Lemma 4.1.2 of [11], which easily generalizes to the graded case, from the cobar construction of an operad $\mathscr{P}$ to the quadratic dual $\mathscr{P}^!$ gives for $\mathscr{P} = e_2^!$ a morphism of operads

$$E^1 \to e_2. \tag{3}$$

It is also known that the operad e_2 is Koszul, see [10] or a purely algebraic proof by Markl [22]. This means (by definition of [11]) that the morphism (3) is a quasi-isomorphism.

According to Markl [23], the cobar construction of the quadratic dual of a Koszul operad (see [11]) is a minimal model of that operad. Applied to e_2, this implies that E^1 is a minimal model of e_2. ■

Definition 1.1 *The G_∞-*operad *is the operad E^1. A G_∞-*algebra *is an algebra over the G_∞-operad.*

1.2.4. *Operad properties of the second spectral sequence $\overline{E}^r$*

Theorem 1.4 *The operad $\overline{E}^1$ is free as an operad of graded vector spaces, and its homology is isomorphic to e_2.*

Proof. 1. First of all, let us prove that $\overline{E}^1$ is free. Recall that

$$\overline{E}^1_{p,q} = H_{p+q}(\overline{F}^p, \overline{F}^{p-1}) = H^{-q}(\overline{F}^p \setminus \overline{F}^{p-1}).$$

Since $\overline{F}^p$'s make up a stratification of $\underline{\mathscr{M}}(n)$ (see Section 1.2.2), we have

$$\coprod_p \overline{F}^p \setminus \overline{F}^{p-1} = \coprod_{n\text{-trees T}} \prod_{v \in T} \coprod_{i=1}^{3} J_i(n(v)) \setminus J_{i-1}(n(v)),$$

where for each $l \geq 2$,

$$J_0(l) \subset J_1(l) \subset J_2(l) \subset J_3(l) = \mathscr{M}(l)^\bullet \tag{4}$$

is the filtration from Section 1.2.1. Therefore, passing to cohomology, we have

$$\overline{E}^1_{\bullet,\bullet} = \bigoplus_{n\text{-trees T}} \bigotimes_{v \in T} \bigoplus_{i=1}^{3} H^\bullet(J_i(n(v)) \setminus J_{i-1}(n(v))),$$

which by definition means that $\overline{E}^1_{\bullet,\bullet}$ is a free operad generated by the collection

$$\bigoplus_{i=1}^{3} H^\bullet(J_i(n) \setminus J_{i-1}(n)) = \bigoplus_{i=1}^{3} H_\bullet(J_i(n), J_{i-1}(n)) \tag{5}$$

of graded vector spaces with an action of the symmetric group S_n, $n \geq 2$.

2. The next step is to show that the spectral sequence $\overline{E}^r$ collapses at the second term $\overline{E}^2$. In order to show that the cohomology of $\overline{E}^1$ is $\overline{E}^\infty = e_2$, regard $\overline{E}^1$ as a filtered complex, with the kth filtration component defined by the tree degree $n - v(T) - 1 \leq k$. We will compute the homology of $\overline{E}^1$ using the spectral sequence associated with this filtration. The first term of this spectral sequence is $\bigoplus_{n\text{-trees T}} \bigotimes_{v \in T} H^\bullet(\mathscr{M}(n(v)))$ with the Gysin homomorphism as the differential, because of Proposition 1.1, that is the first term E^1 of the spectral sequence associated with the filtration $F^\bullet$, see (2). Corollary 1.3 shows that the homology of E^1 is isomorphic to e_2. By construction this is the second term of the spectral sequence associated to the filtered complex $\overline{E}^1$ and the spectral sequence converges to the homology $\overline{E}^2$ of d^1. On the other hand, e_2 is the ∞ term $\overline{E}^\infty$. Thus e_2 is the second term of a spectral sequence converging to the second term of another spectral

sequence converging to the same e_2. This implies that both spectral sequences collapse at the second terms. Therefore, the homology of $\overline{E}^1$ is e_2. ■

2. The B_∞-operad

2.1. B_∞-algebras and the B_∞-operad

Let $V = \bigoplus_{n\in\mathbb{Z}} V^n$ be a graded vector space over a field k of characteristic zero, $V[1]$ its *desuspension*: $V[1] = \bigoplus_{n\in\mathbb{Z}} V[1]^n$, where $V[1]^n = V^{n+1}$, and $TV[1] = \sum_{p=0}^{\infty}(V[1])^{\otimes p}$ the *tensor coalgebra* on $V[1]$. We will adopt the standard notation $[a_1|\dots|a_p]$ for an element $a_1\otimes\cdots\otimes a_p \in (V[1])^{\otimes p} \subset TV[1]$. By definition $|[a_1|\dots|a_p]| = |a_1|+\cdots+|a_p| - p$, where $|a|$ denotes the degree of an element a in a graded vector space. The graded coalgebra structure on $TV[1]$ is given by the coproduct $\Delta: TV[1]\to TV[1]\otimes TV[1]$,

$$\Delta[a_1|\dots|a_p] = \sum_{i=0}^{p}[a_1|\dots|a_i]\otimes[a_{i+1}|\dots|a_p],$$

for which the natural augmentation $TV[1]\to k$ is a counit.

We will be interested in studying a certain DG bialgebra structure on $TV[1]$. Here a *DG bialgebra* is an algebra A with a unit, the structure of a coalgebra, and a differential $D: A\to A[1]$, $D^2 = 0$, such that D is a graded derivation and coderivation and the comultiplication $A\to A\otimes A$ is a morphism of algebras.

Definition 2.1. **(H. J. Baues [3])** *A* B_∞-algebra *structure on a graded vector space V is the structure of a DG bialgebra on the tensor coalgebra $TV[1]$, such that the element $[\,]\in (V[1])^{\otimes 0}\subset TV[1]$ is a unit element.*

Since the tensor coalgebra is cofree and both the differential $D: TV[1]\to TV[1]$ and the product $M: TV[1]\otimes TV[1]\to TV[1]$ are respect the coproduct, they are determined by the compositions

$$\mathrm{pr}\, D = \sum_{k=0}^{\infty} M_k : TV[1]\to V[2]$$

and

$$\mathrm{pr}\, M = \sum_{k,l=0}^{\infty} M_{k,l} : TV[1]\otimes TV[1]\to V[1]$$

with the natural projection $\mathrm{pr}: TV[1]\to V[1]$. The condition $D^2 = 0$ can be rewritten as a collection of identities for the operations M_k, the associativity condition for M and the unit axiom for $[\,]$ as a collection of identities for the operations $M_{k,l}$ and the derivation property for D with respect to M as a collection of identities between $M_{k,l}$ and M_k. The restriction M_0 of D to $V[1]^{\otimes 0}$ must vanish, because D

must annihilate the unit $[\,]$. The equation $D^2=0$ then implies $M_1^2=0$, which means $d=M_1$ must be a differential on the graded vector space V, defining the structure of a complex with a differential of degree 1. Thus, a B_∞-algebra structure on a graded vector space is equivalent to a differential d and a collection of multilinear operations M_k of degree $|M_k|=2-k$ and $M_{k,l}$ of degree $|M_{k,l}|=1-k-l$ satisfying certain identities, i.e., the structure of an algebra over the DG operad generated by M_k, $k\geq 2$, and $M_{k,l}$, $k,l\geq 0$, with those identities being the defining relations. We will call this operad the *B_∞-operad*.

2.2. The algebraic description of the B_∞-operad

Here we will describe the B_∞-operad explicitly. We will use this description in the next section to show that the B_∞-operad is a quotient of the operad $\overline{E}^1$, associated to the little disks operad. Just to make the formulas more transparent, we will describe the identities satisfied by the operations M_k, $k\geq 0$, and $M_{k,l}$, $k,l\geq 0$, in a B_∞-algebra V. As we already noticed, $M_0=0$ and M_1 is a differential d on V. We will adopt the following convention:

$$M_k(a_1,\dots,a_k) := (-1)^{(k-1)|a_1|+(k-2)|a_2|+\cdots+|a_{k-1}|}M_k[a_1|\dots|a_k],$$

which morally means that the vertical bar $|$ has degree one and on the left-hand side all the bars are moved between M_k and a_1. Here $|a_i|$ denotes the degree of a_i in V. However, we set

$$M_{k,l}(a_1,\dots,a_k;b_1,\dots,b_l) := M_{k,l}([a_1|\dots|a_k]\otimes[b_1|\dots|b_l]).$$

2.2.1. $D^2=0$

The condition $D^2=0$ is equivalent for the operations M_k, $k\geq 1$, to define an A_∞-structure:

$$\sum_{i+j=n+1}\sum_{k=0}^{n-j}(-1)^{\varepsilon}M_i(a_1,\dots,a_k,\\ M_j(a_{k+1},\dots,a_{k+j}),\dots,a_n)=0,\quad n\geq 1,\quad (6)$$

where $\varepsilon=(i+1)j+(j+1)k+i|a_1|+(i-1)|a_2|+\cdots+(i-k+1)|a_k|+(n-k-1)|a_{k+1}|+(n-k-2)|a_{k+2}|+\cdots+|a_{n-1}|$ and $a_1,\dots,a_n\in V$. In fact, the sign ε is obtained as $|a_1|+\cdots+|a_k|-k$ plus the sign coming from moving the vertical bars in any occurrence of $M_p[a_1|\dots|a_p]$ to the place between M_p and a_1, thinking of a bar as having degree 1.

2.2.2. $[\,]$ *is a unit for M*

This is equivalent to $M_{1,0}=M_{0,1}=\mathrm{id}$, $M_{k,0}=M_{0,k}=0$ for $k\neq 1$.

2.2.3. *The associativity of M*

The associativity of $M = \sum_{k,l\geq 0} M_{k,l}$ is equivalent to the following identities

$$\begin{aligned}\sum_{r=1}^{l+m} \sum_{\substack{l_1+\dots+l_r=l\\ m_1+\dots+m_r=m}} &(-1)^{\varepsilon} M_{k,r}(a_1,\dots,a_k; M_{l_1,m_1}(b_1,\dots,b_{l_1};c_1,\dots,c_{m_1}),\\ &\dots, M_{l_r,m_r}(b_{l_1+\dots+l_{r-1}+1},\dots,b_l;c_{m_1+\dots+m_{r-1}+1},\dots,c_m))\\ &= \sum_{s=1}^{k+l} \sum_{\substack{k_1+\dots+k_s=k\\ l_1+\dots+l_s=l}} (-1)^{\delta} M_{s,m}(M_{k_1,l_1}(a_1,\dots,a_{k_1};b_1,\dots,b_{l_1}),\dots,\\ &M_{k_s,l_s}(a_{k_1+\dots+k_{s-1}+1},\dots,a_k;b_{l_1+\dots+l_{s-1}+1},\dots,b_l);c_1,\dots,c_m)\end{aligned} \tag{7}$$

for $a_1,\dots,a_k, b_1,\dots,b_l$, and $c_1,\dots,c_m$ in V. The sign $(-1)^{\varepsilon}$ is the sign picked up by reordering $[a_1|\dots|a_k|b_1|\dots|b_l|c_1|\dots|c_m]$ into $[a_1|\dots|a_k|b_1|\dots\ |b_{l_1}|c_1|\dots|c_{m_1}|\dots |b_{l_1+\dots+l_{r-1}+1}|\dots|b_l|c_{m_1+\dots+m_{r-1}+1}|\dots|c_m]$ in the graded vector space $TV[1]$. Similarly, $(-1)^{\delta}$ is the sign of reordering $[a_1|\dots|a_k|b_1|\dots|b_l|c_1|\dots|c_m]$ into $[a_1|\dots|a_{k_1}|b_1|\dots|b_{l_1}|\dots|a_{k_1+\dots+k_{s-1}+1}|\dots|a_k|b_{l_1+\dots+l_{s-1}+1}|\dots|b_l|c_1|\dots|c_m]$.

2.2.4. *The Leibniz rule for D with respect to M*

The fact that D is a derivation of the product M on $TV[1]$ is equivalent to the following identities

$$\begin{aligned}\sum_{n=1}^{k+l} \sum_{\substack{k_1+\dots+k_n=k\\ l_1+\dots+l_n=l}} &(-1)^{\varepsilon} M_n(M_{k_1,l_1}(a_1,\dots,a_{k_1};b_1,\dots,b_{l_1}),\dots,\\ &\qquad M_{k_n,l_n}(a_{k_1+\dots+k_{n-1}+1},\dots,a_k;b_{l_1+\dots+l_{n-1}+1},\dots,b_l))\\ = \sum_{r=1}^{k}\sum_{i=0}^{k-r} &(-1)^{\delta} M_{k-r+1,l}(a_1,\dots,a_i,M_r(a_{i+1},\dots,a_{i+r}),\dots,a_k;b_1,\dots,b_l)\\ &+(-1)^{|a_1|+\dots|a_k|-k} \sum_{s=1}^{l}\sum_{i=0}^{l-s}(-1)^{\eta} M_{k,l-s+1}(a_1,\dots,a_k;b_1,\dots,b_i,\\ &\qquad M_s(b_{i+1},\dots,b_{i+s}),\dots,b_l)\end{aligned} \tag{8}$$

for $a_1,\dots,a_k$ and $b_1,\dots,b_l$ in V. The sign $(-1)^{\varepsilon}$ is the sign of reordering $[a_1|\dots|a_k| b_1|\dots|b_l]$ into $[a_1|\dots|a_{k_1}|b_1|\dots|b_{l_1}|\dots|a_{k_1+\dots+k_{n-1}+1}|\dots|a_k|b_{l_1+\dots+l_{n-1}+1}|\dots|b_l]$ in $TV[1]$ multiplied by the sign of moving $n-1$ bars between $M_{k_1,l_1}(\dots),\dots, M_{k_n,l_n}(\dots)$ to the place between M_n and M_{k_1,l_1}. The sign $(-1)^{\delta}$ is equal to $|a_1|+\dots+|a_i|-i$ plus the sign coming from moving the vertical bars in $M_r[a_{i+1}|\dots|a_{i+r}]$ to the place between M_r and a_{i+1}. Similarly, the sign $(-1)^{\eta}$ is equal to $|b_1|+\dots|b_i|-i$ plus the sign coming from moving the vertical bars in $M_s[b_{i+1}|\dots|b_{i+s}]$ to the place between M_s and b_{i+1}.

Remark 5 A few "lower" identities including the derivation property of the differential $d = M_1$ with respect to the "dot" product M_2 and identities providing homotopies for such classical identities for binary operations as the commutativity and the associativity of the "dot" product, the homotopy left and right Leibniz rules for the "circle" product $M_{1,1}$ with respect to the "dot" product, and the homotopy Jacobi identity for the "bracket" $[a,b] = M_{1,1}(a,b) - (-1)^{(|a|-1)(|b|-1)}M_{1,1}(b,a)$ were written out explicitly in [21, Section 4.2]. Strictly speaking, those identities were claimed to be identities for another type of algebra, which later turned out to be nonextant. However, one can see from Theorem 2.1 of the next section, that those identities are satisfied in a B_∞-algebra.

2.3. Relation between the B_∞-operad and $\overline{E}^1$

The algebraic description above of the B_∞-operad might be a good exercise in tensor algebra, but is far from inspiring. However, everything falls into its place, when geometry comes into play. Since $\overline{E}^1$ is a homological operad (the degree of the differential is -1), if B_∞ is a cohomological operad (the degree of the differential is $+1$), let us change the grading on $\overline{E}^1$ to the opposite one, an element of degree k will be assigned degree $-k$, from now on, so that the differential on $\overline{E}^1$ is of degree $+1$.

Theorem 2.1 *There exists a surjective morphism of DG operads $\overline{E}^1 \to B_\infty$.*

Proof. The operad $\overline{E}^1$ is freely generated by the spaces $\bigoplus_{i=1}^3 H_\bullet(J_i(n), J_{i-1}(n))$, $n \geq 2$, see (5). Thus to define a morphism $\overline{E}^1 \to B_\infty$, it suffices to define maps

$$\bigoplus_{i=1}^{3} H_\bullet(J_i(n), J_{i-1}(n)) \to B_\infty(n), \qquad n \geq 2, \tag{9}$$

respecting the gradings, the symmetric group actions, and the differentials.

The complements $J_1 \setminus J_0$ and $J_2 \setminus J_1$ are disjoint unions of Fox-Neuwirth cells, and the spaces $H_\bullet(J_1, J_0)$ and $H_\bullet(J_2, J_1)$ have the Fox-Neuwirth cells of the types $\{i_1, \dots, i_n\}$ and $\{i_1, \dots, i_p\}\{i_{p+1}, \dots, i_n\}$, respectively, see 1.1.1, as natural bases. Define the maps (9) as follows:

$$\{1,2,\dots,n\} \mapsto M_n \qquad \text{for } n \geq 2, \tag{10}$$

$$\{1,2,\dots,k\}\{k+1,\dots,k+l\} \mapsto M_{k,l} \qquad \text{for } k,l \geq 1, \tag{11}$$

permutations of the cells mapping to permutations of the generators M_n and $M_{k,l}$ of the B_∞-operad. Finally define

$$H_\bullet(J_3(n), J_2(n)) \to B_\infty(n), \qquad n \geq 2,$$

as zero.

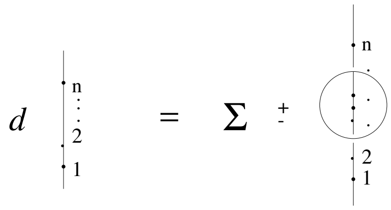


Figure 5. The boundary of a cell in J_1.

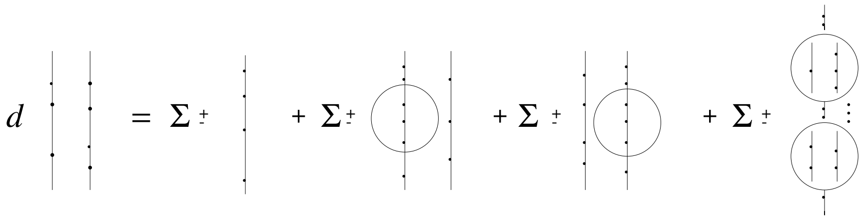


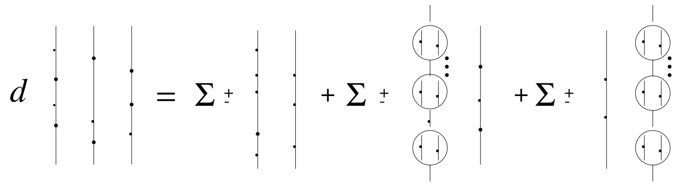
Figure 6. The boundary of a cell in $J_2 \setminus J_1$.

Since $\dim\{1,\ldots,n\} = n-2 = -|M_n|$ and $\dim\{1,\ldots,k\}\{1,\ldots,l\} = k+l-1 = -|M_{k,l}|$ and the action of the symmetric groups is respected by construction, the maps (10) and (11) define a morphism $\overline{E}^1 \to B_\infty$ of graded operads. The only thing which remains to be checked is the compatibility of this morphism with the differentials.

We will compute the boundary (differential) $d := d^1$ on $\overline{E}^1$. Each space $J_i \setminus J_{i-1}$, $i = 1,2,3$, is a disjoint union of Fox-Neuwirth cells, which form a basis of the relative homology $H_\bullet(J_i, J_{i-1})$. We will study the action of d on this basis.

$i = 1$. Let us start with $i = 1$, when the points in a Fox-Neuwirth cell group on a single vertical line. For $n \geq 2$ the boundary of the cell $\{1,2,\ldots,n\}$ in $\overline{E}^1$ may be computed as shown on Figure 5.
where the left-hand side denotes the boundary of the cell $\{1,\ldots,n\}$ and the right-hand side denotes a linear combination of Getzler-Jones cells obtained as operad compositions $\{1,\ldots,k\} \circ_i \{1,\ldots,l\}$, $k,l \geq 2$, of two Fox-Neuwirth cells. This equation turns into equation (6), where all the terms with $i > 1$ or $j > 1$ are moved to the right-hand side. The signs here and henceforth in the proof are compatible with the signs in (6)–(8), if the orientations on Getzler-Jones cells are chosen as in the proof of Proposition 1.2.

$i = 2$. Cells for $i = 2$ are configurations of points on two vertical lines. The boundary of a cell $\{1,\ldots k\}\{k+1,\ldots,k+l\}$ may be described as shown on Figure 6.
This equation translates under the correspondence (10), (11) into the identity (8) in which all terms but those containing $d = M_1$ are moved to the right-hand side.



+ classes of GJ cells having at least one fragment |||

Figure 7. The boundary of a cell in $J_3 \setminus J_2$.

Note that the first sum on the figure corresponds to the terms $M_{k+l}(c_1, \dots, c_{k+l})$, where $c_1, \dots, c_{k+l}$ is a shuffle of $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_l\}$, which show up on the left-hand side of (8) for all pairs (k_i, l_i) being $(0,1)$ or $(1,0)$. The rest of the left-hand side of (8) is the last term on the figure above.

$i = 3$. Cells for $i = 3$ are configurations of points on at least three different vertical lines. The boundary of a cell with at least four vertical lines will produce the identity $0 = 0$ under the morphism $\overline{E}^1 \to B_\infty$, because of a dimension argument: The boundary of such cell has a dimension $\geq n$, while (multiple) operad compositions of Fox-Neuwirth cells from J_2 will have dimensions $\leq n-1$. Therefore, the boundary of a cell with at least four vertical lines will have no terms which are compositions of Fox-Neuwirth cells from J_2. Thus, the only nontrivial identity to be checked in B_∞ comes from the lowest Fox-Neuwirth cells in $J_3 \setminus J_2$, those made out of configurations of points on three vertical lines.

Figure 7 describes the differential of such cell in $\overline{E}^1$.
Under the morphism $\overline{E}^1 \to B_\infty$, this identity turns into (7) where all terms are moved to the right-hand side. Note that the first sum on the figure corresponds to the terms in (7) where all (l_i, m_i) or all (k_i, l_i) are either $(0,1)$ or $(1,0)$. ■

Corollary 2.2 *The operad $\overline{E}^1$ is formal, i.e., quasi-isomorphic to its homology e_2. There is a morphism of operads $E^1 \to \overline{E}^1$, unique up to homotopy.*

Proof. In [26] Tamarkin has constructed an operad morphism $B_\infty \to e_2$, which is surjective on homology. Composing it with our morphism $\overline{E}^1 \to B_\infty$, we get a morphism $\phi : \overline{E}^1 \to e_2$. To prove the formality of $\overline{E}^1$, it is enough to show that it is a quasi-isomorphism. Moreover, it suffices to show that ϕ is surjective on homology, because of a graded dimension argument.

Note that the operad e_2 is generated by $e_2(2)$, therefore we just need to show that the second component $\phi(2) : \overline{E}^1(2) \to e_2(2)$ of ϕ is surjective on homology. The morphism ϕ is a composition $\overline{E}^1 \to B_\infty \to e_2$. According to Tamarkin [26, Theorem 4.2.1], the induced homology morphism $H_\bullet(B_\infty(2)) \to e_2(2)$ is an isomorphism. On the other hand, notice that our morphism $\overline{E}^1(2) \to B_\infty(2)$ is an isomorphism, because $J_3(2) = J_2(2)$. In particular, it induces an isomorphism $H_\bullet(\overline{E}^1(2)) \to H_\bullet(B_\infty(2))$ of the homology. Thus, the composition $H_\bullet(\phi) : H_\bullet(\overline{E}^1(2)) \to e_2(2)$ is an isomorphism, which shows that $\overline{E}^1$ is formal.

The existence of a unique up to homotopy morphism $E^1 \to \overline{E}^1$ follows from the fact that both operads are quasi-isomorphic to e_2 and E^1 is a minimal model of e_2, see Corollary 1.3. ∎

Question 2.3 *Is the homology of the B_∞-operad isomorphic to the G-operad e_2? If yes, it will automatically be formal via Tamarkin's morphism $B_\infty \to e_2$.*

3. Action on the Hochschild complex

3.1. The Hochschild complex of an associative algebra

Let us recall some notions related to the Hochschild complex and some properties of it. Let A be an associative algebra and $C^n(A,A) = \operatorname{Hom}(A^{\otimes n}, A)$ its *Hochschild complex* with the *Hochschild differential*:

$$\begin{aligned}(dx)(a_1,\dots,a_{n+1}) := {}& a_1 x(a_2,\dots,a_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i x(a_1,\dots,a_{i-1},a_i a_{i+1},a_{i+2},\dots,a_{n+1}) \\ &- (-1)^n x(a_1,\dots,a_n)a_{n+1},\end{aligned} \tag{12}$$

for $x \in C^n(A,A)$, $a_1,\dots,a_{n+1} \in A$. The sign $(-1)^n$ above is equal to $(-1)^{|x|}$, $|x| := n$ being the *degree* of the cochain x in the Hochschild complex $C^\bullet = C^\bullet(A,A)$.

We will use the following operations on the Hochschild complex. The *dot product* is defined as the usual cup product:

$$(x \cdot y)(a_1,\dots,a_{k+l}) = x(a_1,\dots,a_k)y(a_{k+1},\dots,a_{k+l}) \tag{13}$$

for any k- and l-cochains x and y and $a_i \in A$. The following collection of multilinear operations, called *braces,* cf. Kadeishvili [14] and Getzler [9], on $C^\bullet$ is defined as

$$\begin{aligned}&\{x\}\{x_1,\dots,x_n\}(a_1,\dots,a_m) := \\ &\qquad \sum (-1)^\varepsilon x(a_1,\dots,a_{i_1},x_1(a_{i_1+1},\dots),\dots,a_{i_n},x_n(a_{i_n+1},\dots),\dots,a_m),\end{aligned}$$

for $x,x_1,\dots,x_n \in C^\bullet$, $a_1,\dots,a_m \in A$, where the summation runs over all possible substitutions of $x_1,\dots,x_n$ into x in the prescribed order and $\varepsilon := \sum_{p=1}^n (|x_p|-1)i_p$. The braces $\{x\}\{x_1,\dots,x_n\}$ are homogeneous of degree $-n$, i.e., $|\{x\}\{x_1,\dots,x_n\}| = |x| + |x_1| + \dots + |x_n| - n$. We will also adopt the following notation:

$$x \circ y := \{x\}\{y\}, \qquad [x,y] := x \circ y - (-1)^{(|x|-1)(|y|-1)} y \circ x.$$

The *G-bracket* $[x,y]$ defines the structure of a G-algebra on the Hochschild cohomology $H^\bullet(A,A)$. The bracket was introduced by Gerstenhaber [8] in order to describe the obstruction for extending a first-order deformation of the algebra A to the second order. The following definition of the bracket is due to Stasheff [25]. Considering the tensor coalgebra $T(A) = \bigoplus_{n=0}^{\infty} A^{\otimes n}$ with the comultiplication $\Delta(a_1 \otimes \cdots \otimes a_n) = \sum_{k=0}^{n} (a_1 \otimes \cdots \otimes a_k) \otimes (a_{k+1} \otimes \cdots \otimes a_n)$, we can identify the Hochschild cochains $\operatorname{Hom}(A^{\otimes n}, A)$ with the graded coderivations $\operatorname{Coder} T(A)$ of the tensor coalgebra $T(A)$. Then the G-bracket $[x,y]$ is defined as the (graded) commutator of coderivations. In fact, the Hochschild complex $C^\bullet$ is a differential graded Lie algebra with respect to this bracket.

In addition, the dot product (13) and the Hochschild differential (12) define the structure of a DG associative algebra on $C^\bullet(A,A)$.

3.2. The structure of a B_∞-algebra on the Hochschild complex

Define the structure of a B_∞-algebra on $C^\bullet(A,A)$ as follows:

$$\begin{aligned}
&M_0 := 0, && M_{0,1} = M_{1,0} := \mathrm{id},\\
&M_1 := d, && M_{0,n} = M_{n,0} := 0 \quad \text{for } n > 1,\\
&M_2(x_1,x_2) := x_1 \cdot x_2, && M_{1,n}(x; x_1,\ldots,x_n) := \{x\}\{x_1,\ldots,x_n\} \quad \text{for } n \geq 0,\\
&M_n := 0 \quad \text{for } n > 2, && M_{k,l} := 0 \quad \text{for } k > 1,
\end{aligned}$$

where $x, x_1, \ldots, x_n \in C^\bullet(A,A)$.

Theorem 3.1 *These operations define the structure of a B_∞-algebra on the Hochschild complex $C^\bullet$.*

Remark 6 The braces were defined by Kadeishvili [14] and Getzler [9], the identities among them and the dot product were written down in [28], where this structure was called a homotopy G-algebra structure. The fact that this algebraic data defines a B_∞-structure was noticed by Getzler and Jones in [10].

Proof. Taking into account the vanishing operations M_n and $M_{k,l}$ and rewriting the rest in terms of the dot product and braces, the identities (6) through (8) can be simplified as follows.

The identities (6) for $n = 1$, 2, and 3, are equivalent to

$$d^2 = 0, \tag{14}$$

$$d(x_1x_2) = (dx_1)x_2 + (-1)^{|x_1|} x_1 dx_2, \tag{15}$$

$$(x_1x_2)x_3 = x_1(x_2x_3), \tag{16}$$

respectively.

The identities (7) are nontrivial only for $k = 1$, when they turn into the following.

$$\sum_{0 \le i_1 \le \cdots \le i_l \le m} (-1)^{\varepsilon} \{x\}\{z_1, \ldots, z_{i_1}, \{y_1\}\{z_{i_1+1}, \ldots\}, \ldots, z_{i_l}, \{y_l\}\{y_{i_l+1}, \ldots\}, \ldots, y_m\} = \{\{x\}\{y_1, \ldots, y_l\}\}\{z_1, \ldots, z_m\}, \tag{17}$$

where $\varepsilon = \sum_{p=1}^{l}(|y_p| - 1)\sum_{q=1}^{i_p}(|z_q| - 1)$.

The identities (8) are nontrivial only when $k = 1$ and 2. For $k = 1$ they rewrite as the following family of identities:

$$\begin{aligned} d\{x\}\{y_1, \ldots, y_l\} &- (-1)^{|x|(|y_1|-1)} y_1 \cdot \{x\}\{y_2, \ldots, y_l\} \\ &+ (-1)^{|x|+|y_1|+\cdots+|y_{l-1}|+l-1}\{x\}\{y_1, \ldots, y_{l-1}\} \cdot y_l \\ = \{dx\}\{y_1, \ldots, y_l\} &- \sum_{i=0}^{l-1}(-1)^{|x|+|y_1|+\cdots+|y_i|-i}\{x\}\{y_1, \ldots, y_i, dy_{i+1}, \ldots, y_l\} \\ &- \sum_{i=0}^{l-2}(-1)^{|x|+|y_1|+\cdots+|y_{i+1}|-i}\{x\}\{y_1, \ldots, y_i, y_{i+1} \cdot y_{i+2}, \ldots, y_l\}, \end{aligned} \tag{18}$$

for each $l \ge 1$. For $k = 2$, Equations (8) turn into

$$\sum_{0 \le l_1 \le l} (-1)^{|x_2|(|y_1|+\cdots+|y_{l_1}|-l_1)}(\{x_1\}\{y_1, \ldots, y_{l_1}\}) \cdot (\{x_2\}\{y_{l_1+1}, \ldots, y_l\}) = \{x_1 \cdot x_2\}\{y_1, \ldots, y_l\}, \tag{19}$$

for each $l \ge 1$.

All these identities for the operations on the Hochschild complex may be checked directly. Some of the identities are classical, see e.g., Gerstenhaber [8], the others were not noticed until more recently, see [28]. One can find a detailed verification of the identities in Khalkhali's paper [16]. ■

Combining Theorems 2.1 and 3.1 and Corollary 2.2, we come to the following solution of Deligne's Conjecture.

Corollary 3.2 *The operad morphism $\overline{E}^1 \to B_\infty$ and the above action of B_∞ on the Hochschild complex $C^\bullet$ define on $C^\bullet$ the natural structure of an algebra over the operad $\overline{E}^1$, which is quasi-isomorphic to its homology e_2.*

Remark 7 This corollary along with Corollary 2.2 also yields the natural structure of a G_∞-algebra on $C^\bullet$, recovering a result of Tamarkin [26].

Remark 8 A complex of vector spaces with operations $x_1 \cdot x_2$ and $\{x\}\{x_1, \ldots, x_n\}$ for $n \ge 0$ satisfying identities (14)–(19) was called a *homotopy G-algebra* in [28, 12]. Kadeishvili rediscovered the same notion in [15] under the name of an *associative Hirsch algebra*.

4. Correction of [21]

As we have already mentioned, the paper [21] of Kimura, Zuckerman, and the author used the notion of a G_∞-algebra as an algebra over the Getzler-Jones' cellular operad, which was later noticed not to be cellular. The following changes have to be made to correct the resulting error.

In Section 4, after describing Getzler-Jones' cellular partition $K_\bullet \underline{\mathcal{M}}$, we should emphasize that it is not a cellular operad. However, there is a way to produce a DG operad out of it. Namely, notice that as a graded operad, $K_\bullet \underline{\mathcal{M}}$ is free on the collection of Fox-Neuwirth cells. The problem is that the differential is not well defined, in general. For those Fox-Neuwirth cells C_1 whose geometric boundary is the union of Getzler-Jones cells, the differential dC_1 is defined as the boundary operator. For those Fox-Neuwirth cells C_2 whose geometric boundary is not the union of Getzler-Jones cells, define the differential as a new generator dC_2. Take the free graded operad $K_\bullet$ generated by the Fox-Neuwirth cells and the differentials of the Fox-Neuwirth cells of the second type. The differential now is well defined on the generators: For a Fox-Neuwirth cell C_1 of the first type, the differential dC_1 is a linear combination of Getzler-Jones cells; for a Fox-Neuwirth cell C_2 of the second type, the differential is the generator dC_2; finally $d(dC_2) = 0$. This differential extends uniquely to the free graded operad $K_\bullet$. Definition 4.1 in [21] must be replaced by the following one.

Definition 4.1 *The* weak G_∞-operad *is the DG operad $K_\bullet$ constructed above. An algebra over it is called a* weak G_∞-algebra.

Every occurrence of the word "G_∞-algebra" in [21] must be replaced with the words "weak G_∞-algebra". Whenever the "operad" $K_\bullet \underline{\mathcal{M}}$ occurs in the sequel therein, it must be replaced with the above operad $K_\bullet$. All the lower identities written down in Sections 4.1 and 4.2 of [21] are satisfied in a weak G_∞-algebra and do not require corrections, because the Fox-Neuwirth cells used there are of the first type. For the same reason, the A_∞- and L_∞-operads map naturally to the weak G_∞-operad, therefore, a weak G_∞-algebra is naturally an A_∞- and L_∞-algebra. With the replacement of the G_∞-algebras by weak G_∞-algebras, all the results of the paper are correct with the same proofs. Moreover, Conjecture 2.3 of [21] with the above change has been proven in the meantime by Yi-Zhi Huang and Wenhua Zhao [13]. Thus it must be renamed to a theorem, as follows.

Theorem 4.1. **(Huang and Zhao)** *Let $V^\bullet[\cdot]$ be a TVOA satisfying $G(0)^2 = 0$. Then the dot product and the skew-symmetrization of the bracket defined in [21] can be extended to the structure of a weak G_∞-algebra on a certain completion of $V^\bullet[\cdot]$.*

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