Improved perturbation expansion in quantum chromodynamics

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Abstract. The problem of improving the perturbation expansion in quantum chromodynamics at low energies is considered. Certain modification of the conventional running coupling constant is proposed, which removes the singularity of the effective coupling at nonzero energy. The effect of such modification on the QCD predictions is discussed. It is argued, that such modification may be considered as an unconventional choice of the renormalization scheme.

Keywords: quantum chromodynamics, perturbation expansion, renormalization schemes

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Despite advances in nonperturbative methods the perturbation expansion remains an important tool in practical applications of quantum chromodynamics (QCD). In most cases the so called renormalization group (RG) improved perturbation expansion is used, i.e., an expansion in terms of the running coupling constant (RCC). In the simplest case of a totally inclusive quantity that depends on a single variable Q the RG-improved expression for the QCD correction δ takes the form:

$$\delta(Q) = a(Q)[1 + r_1 a(Q) + r_2 a^2(Q) + \dots], \tag{1}$$

where $a = g^2/(4\pi^2)$ is the RCC satisfying the equation:

$$Q\frac{da}{dQ} = \beta(a) = -ba^2\,(1 + c_1 a + c_2 a^2 + \dots). \tag{2}$$

The appearance of RCC is a consequence of partial resummation of the perturbation expansion to all orders. Such a resummation is useful for expressing QCD predictions at high energies, because the RCC is decreasing with increasing energy due to the asymptotic freedom property of QCD. On the other hand, the RCC increases with decreasing energy, and it may even become singular at nonzero value of energy. Obviously, this type of behavior seriously limits the reliability of the RG-improved perturbation expansion in QCD at low energies, which manifests itself for example via substantial renormalization scheme (RS) dependence of finite order predictions. This fact is usually taken as unavoidable feature of QCD and the indication, that at low energies in QCD it is necessary to use nonperturbative methods.

It is of course beyond doubt that QCD requires a nonperturbative treatment in the low energy region. However, the breakdown of the RG-improved perturbation expansion in QCD at low energies may result to some extent from the fact, that at certain energies the conventional RCC ceases to be a useful expansion param-

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eter. Let us note that the infrared behavior of RCC (in particular, the type and the location of the singularity) is determined by the *highest* term retained in the perturbative approximant for the β-function, and thus it strongly depends on the order of the expansion and the choice of the renormalization scheme. It seems therefore worthwhile to investigate other ways to define the effective QCD expansion parameter, which would lead to a better behavior at low energies. It is also important to study, what is exactly the range of energies, for which the modified perturbation expansion may be still meaningfully applied: is it $Q \geq \Lambda$ (where Λ is the QCD scale parameter), or is it perhaps only, say, $Q \geq 5\Lambda$. In many applications this would make an important difference.

A natural way to define a modified effective expansion parameter is to specify a modified form of the β-function. In order to obtain an effective expansion parameter which would be free from the singularity at nonzero value of energy we must introduce a β-function which deviates from the simple polynomial resulting from a truncation of the perturbation expansion. For example, it has been noted already some time ago, that it is possible to avoid singularity in RCC at nonzero energy by introducing exponentially small — i.e., essentially nonperturbative — terms into the β-function [1, 2]. (For other propositions to modify the RCC see [3, 4, 5, 6, 7].) However, in our approach we would like to avoid inserting the essentially nonperturbative effects by hand, and we would like to stay as close as possible to the standard perturbative framework. In fact, as we shall explain later, we would like to exploit in a slightly unconventional way the arbitrariness in the β-function resulting from the freedom of choice of the renormalization scheme.

Our proposition is to replace the polynomial approximant by a nonpolynomial expression which is analytic in some neighborhood of the real positive axis in the complex-a plane. In order to preserve consistency with the conventional perturbation expansion we shall require, that the expansion coefficients of this expression around $a = 0$ should coincide to the considered order with the coefficients in the conventional expression. Additionally, in order to avoid the singularity of the RCC at nonzero energy we shall require, that for large a the modified β-function should typically behave as a^k, where $k \leq 1$. These are of course very general requirements, that by no means specify the modified β-function uniquely. Further constraints on the modification of the β-function would come from the requirement, that the modified evolution of RCC should match closely the behavior emerging from nonperturbative considerations.

It may seem that replacing a polynomial by, say, a rational function we deviate in a substantial and arbitrary way from the original theoretical expression. We want to point out, however, that such a modification may be viewed as a specific choice of the renormalization scheme at a given order of perturbation expansion. As is well known, the β-function depends on the choice of RS. Although the first two expansion coefficients are universal in the class of mass and gauge independent schemes, the coefficient c_2 and all β-function coefficients of higher order are always renormalization scheme dependent. As was pointed out in [8] these

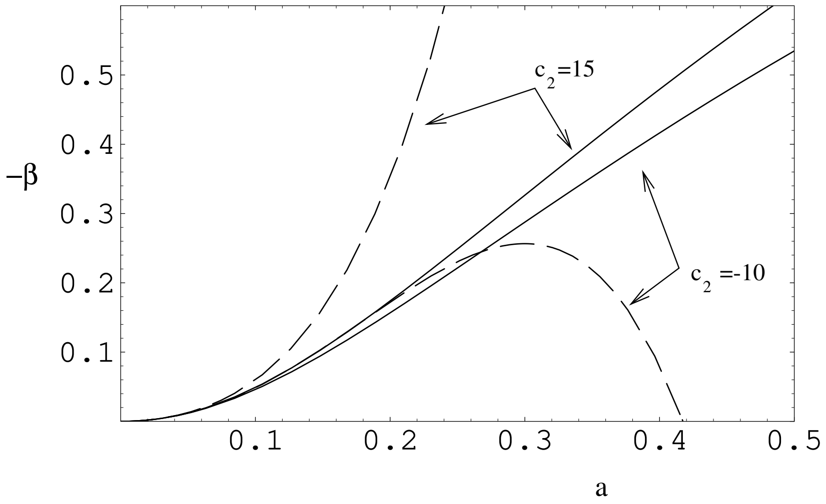

Figure 1. The behavior of the modified NNLO β-function for three "active" quark flavors (solid lines) compared with the conventional NNLO β-function (dashed lines) for $c_2 = 15, -10$ and $\eta = 4$.

coefficients may be even used to *label* the various schemes. In the conventional calculation of the next-to-next-to-leading order (NNLO) predictions one chooses some particular value for c_2 and one assumes $c_k = 0$ for $k \geq 3$. The use of a modified NNLO expression for the β-function corresponds to the situation where the c_2 coefficient is unchanged, but c_k coefficients for $k \geq 3$ take some nonzero value. With our assumption about the analyticity domain the modified β-function is uniquely determined by this infinite sequence of expansion coefficients. The idea to use the β-function with nonzero expansion coefficients at orders higher than the order of approximation for the considered physical quantity is somewhat unconventional, but it is completely consistent with the structure of the perturbation expansion. (Such an idea is also not entirely new - it has been discussed for example in [8, 6].) Under our assumptions the use of a modified β-function is therefore simply equivalent to a specific choice of the renormalization scheme.

Let us note that the QCD predictions obtained with the modified β-function would still depend on low order RS parameters, just as would be the case with the predictions in the conventional expansion: in NLO the choice of the scheme may be parametrized by r_1, in NNLO by r_1 and c_2. The difference between the choices of these scheme parameters is formally of higher order in the effective coupling than the order of the approximant, but numerical differences in the predictions may be significant. This ambiguity in the finite order predictions may be resolved by fixing the low order scheme parameters according to the so called Principle of Minimal Sensitivity [8]. For the purpose of this note we use a simpler choice of

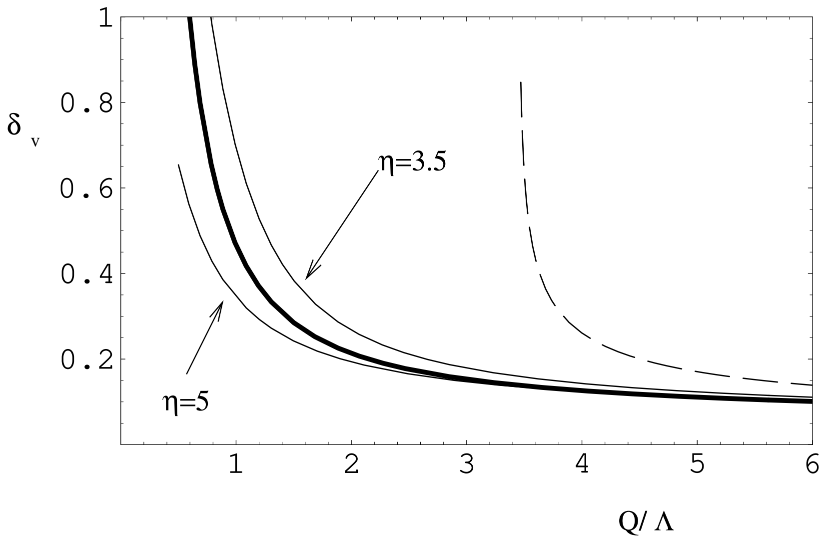

Figure 2. The improved NNLO QCD predictions for δ_V for three "active" quark flavors for $\eta = 3.5$ and $\eta = 5$, compared with the phenomenological expression given in [2] (thick line). The conventional NNLO QCD prediction for δ_V is also shown (dashed line).

$r_1 = 0$ and $r_2 = 0$, which approximates the minimal sensitivity value quite well. Detailed discussion of the optimal choice of the low order RS parameters is beyond the scope of the present note.

The general constraints on the form of the modified β-function mentioned so far are rather weak. In order to further restrict the class of allowed modifications we shall consider the predictions for δ_V, the QCD correction to the potential of interaction between static quarks [9]. The perturbative expansion for this quantity is known up to NNLO. On the other hand, this quantity has been extensively evaluated on the lattice [10], and there exists a useful phenomenological formula [2] (free from singularity at nonzero energy, of course), which seems to be confirmed by the lattice calculations. We shall require that the perturbative NNLO expression for this quantity evaluated using the modified RCC should match the phenomenological expression at low energies.

As a concrete example let us consider the following expression for the modified NNLO β-function:

$$\tilde{\beta}(a) = -\frac{ba^2}{(1+\eta a)^2}\left[1 + \frac{\tilde{c}_1 a}{1+\eta a} + \frac{\tilde{c}_2 a^2}{(1+\eta a)^2}\right], \tag{3}$$

where

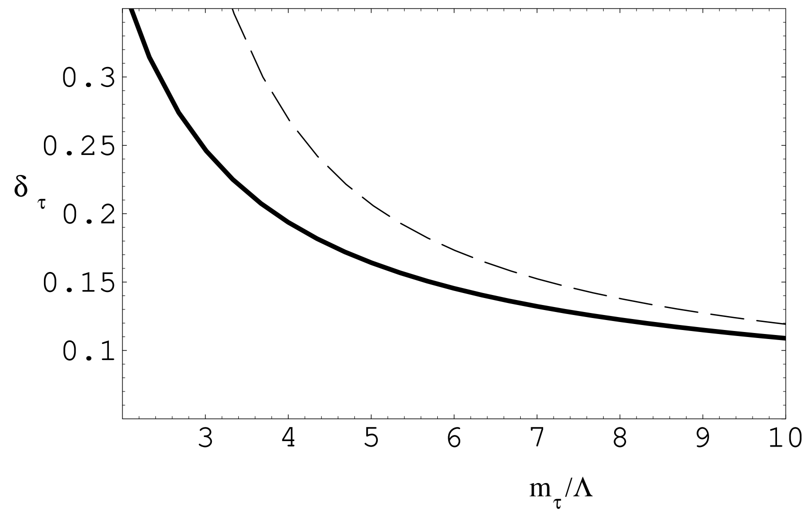

Figure 3. The improved NNLO prediction for the QCD correction to the total hadronic decay rate of the tau lepton (solid line), compared with the conventional NNLO prediction for this quantity (dashed line).

$$\tilde{c}_1 = c_1 + 2\eta, \;\; \tilde{c}_2 = c_2 + 3\eta c_1 + 3\eta^2, \tag{4}$$

and η is an adjustable parameter which controls the deviation of the function from the conventional polynomial form. (It is easy to imagine how a similar modified expression might be defined in arbitrary order.) We show in Figure 1, how the behavior of the modified NNLO β-function compares with the conventional NNLO β-function for two values of c_2 and $\eta = 4$ (we assume that there are three "active" quark flavors).

The numerical value of the improved RCC is determined from the implicit equation resulting from the integration of the RG equation with appropriate asymptotic condition. (All the relevant formulas and conventions may be found for example in [11].) In the case of the modified NNLO β-function (3) the relevant integral may be performed analytically, which greatly simplifies further calculations. The analytic form is too lengthy to be shown here.

In Figure 2 we show, how the NNLO predictions for δ_V depend on the value of η. The requirement that this NNLO prediction should match as closely as possible the phenomenological formula of [2] allows us to make a rough estimate of a sensible value for η; in the following we shall use $\eta = 4$.

We are now in position to obtain improved low energy predictions for other physical quantities of interest. As an example we shall consider improved predictions for QCD correction to the total hadronic tau lepton decay rate [12]. This correction plays important role in the determination of the strong coupling constant. (In fact, the value of the strong coupling constant implied by this quantity was a subject of some controversy [13].) As we may see from Figure 3, the use of a modified RCC results in a substantial reduction of the NNLO QCD correction at a given value of m_τ/Λ.

It is straightforward to apply our modified expansion to perturbative expressions for other physical quantities which are of interest in QCD.

Our discussion of the improved expansion has been very brief and we skipped many details. Full presentation of the method and its applications would be published elsewhere, together with a critical discussion of other approaches. We hope our proposition would be useful in improving the reliability of the QCD predictions at low energies. We also hope our analysis demonstrates, that there are some interesting problems to be solved in perturbation expansion of non-Abelian gauge theories.

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