On the deformation aspects of the replica thermalisation of the $SU(2)$-invariant Thirring model

Mathias Pillin
Department of Mathematics, King's College, Strand, London WC2R 2LS, U.K.
e-mail: `map@mth.kcl.ac.uk`

Abstract. We introduce an S-matrix preserving replica-thermalisation of integrable massive quantum field theories in $1+1$-dimensions within the context of form factors. The deformation character of the methods used therein is being highlighted. We then solve a deformed version of the $SU(2)$-invariant Thirring model employing hypergeometric solutions of the rational sl_2-type quantum Knizhnik-Zamolodchikov equation at generic level and multiperiodic Barnes functions.

Keywords: form factors, replica, deformation, qKZ-equations

Mathematics Subject Classifications (2000): 81R50, 33C60

1. Introduction

It has been mentioned by many speakers during the conference that a *leitmotif* in the scientific work of Moshé Flato is the concept of deformation. He has been advocating that in physics one should look whether a well established structure admits a deformation. One of the most important applications of this idea is of course *deformation quantisation* [1], the fundamental insight being that we actually can understand quantum mechanics as a deformation of the algebraic setup of classical mechanics. The deformation parameter is nothing else than Planck's constant $\hbar$. Once we have accepted this framework (or the category in which the deformation is carried out, to be precise) one could try to deform quantum mechanics even further. However, there are severe cohomological obstructions prohibiting to do this. This means that quantum mechanics is stable with respect to deformations. I am grateful for many enlightening discussions with Moshé on this topic resulting in the work [2], where this rigidity has been established and applied to a certain structure arising in the theory of quantum groups.

Very much in the same spirit this kind of logic appears also in another of Moshé's important contributions, which is the theory of singletons in anti-de Sitter space, see [3]. Here the starting point is Wigner's classification irreducible representations of the Poincaré group and their identification with particle states. One, however, realises that the Poincaré group can be obtained from the anti-de Sitter group by group contraction and therefore the former can be understood as a deformation of the latter, which is stable in the aforementioned sense. One could then ask whether the irreducible representations of the anti-de Sitter group could also play a fundamental role in physics. In fact, this has been established in many

G. Dito and D. Sternheimer (eds.), Conférence Moshé Flato 1999, Vol. 2, 247 – 255.

circumstances as can be read off a recent article [3] and the list of literature cited therein, and also by the results in the context of M-theory.

In this contribution we would like to argue that a deformation absolutely parallel in spirit to the ones described above, does appear within the context of massive integrable field theories in $1+1$-dimensional space-time. First we are going to describe the framework or the category in which we deform a given quantum field theory (QFT). Then we show that this deformed theory can actually be solved exactly using the deformed (quantum) Knizhnik-Zamolodchikov (qKZ) equations.

Before starting off we introduce the $SU(2)$-invariant Thirring model (ITM) which will provide the playground in this paper. The details may be found e.g. in [4]. We should, however, say that the general results to be mentioned below are not restricted to this particular model. For the sake of orientation let us display the classical Lagrangian. It is given in terms of a two-component spinor ψ, valued in the fundamental representation of $SU(2)$.

$$\mathscr{L} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - g(\bar{\psi}\sigma^{a}\gamma^{\mu}\psi)(\bar{\psi}\sigma_{a}\gamma_{\mu}\psi)\,. \tag{1}$$

Here σ^a are the Pauli matrices. The QFT associated with this model exhibits dynamical mass generation and is believed to be asymptotically free. The physical particles of the theory are an SU(2) doublet of massive kinks. The index $\varepsilon \in \{\pm\}$ refers to either of these kinks. Moreover it is known that the model is integrable, and the factorised S-matrix is given as a function of the kink-rapidity z by the following expression:

$$S_{\varepsilon_1\varepsilon_2}^{\varepsilon_1'\varepsilon_2'}(z) = S_0(z)\frac{z\ \delta_{\varepsilon_1}^{\varepsilon_1'}\delta_{\varepsilon_2}^{\varepsilon_2'} - i\pi\ \delta_{\varepsilon_1}^{\varepsilon_2'}\delta_{\varepsilon_2}^{\varepsilon_1'}}{z-i\pi}, \tag{2}$$

where the scalar part S_0 of the S-matrix is

$$S_0(z) = \frac{\Gamma\left(\frac{1}{2}+\frac{z}{2i\pi}\right)\Gamma\left(-\frac{z}{2i\pi}\right)}{\Gamma\left(\frac{1}{2}-\frac{z}{2i\pi}\right)\Gamma\left(\frac{z}{2i\pi}\right)}. \tag{3}$$

We mention that without S_0 the expression in (2) coincides with the rational R-matrix of the quantum group sl_2 in the defining representation.

2. Deformation-thermalisation and the form factor approach

The deformation we are going to describe here is actually associated to a well-known thermalisation phenomenon in QFT. We consider the Rindler wedge W in a $1+1$-dimensional Minkowski space, with K being the Lorentz boost generator. Let $\mathscr{O}_i$ for $i = 1,\dots,n$ be local operators with support inside W. The Bisognano-Wichmann theorem [5] then states that the vacuum of a Minkowski space QFT inside W looks like a thermal state of inverse temperature $\beta = 2\pi$ with respect to the Killing time in W. This temperature is both physically and mathematically linked to the Unruh temperature $T_U = 1/2\pi$, which is seemingly of a very different origin. Symbolically we can encode this fact into

$$\langle 0|\mathscr{O}_1(x_1)\cdots\mathscr{O}_n(x_n)|0\rangle \sim \mathrm{Tr}[\exp(2\pi K)\mathscr{O}_1(x_1)\cdots\mathscr{O}_n(x_n)]\,. \tag{4}$$

Note that we use the symbol Tr only as a mnemonic to remind of the thermal structure of the matrix element. Actually for a continuum QFT the object we are dealing with is meaningful only within the framework of modular theory and our operators $\mathscr{O}_i$ will in general not be of trace-class and moreover the non-compact nature of K would spoil the naive game anyway. A detailed treatment of this, however, would be beyond the scope of this paper.

Now we bring the deformation idea into play. Obviously the value $\beta = 2\pi$ is very special in our physical system. Even though one mainly deals with this particular value in standard QFT we will not be in a stable context or in the most general setup would we not allow arbitrary values for β. However, we would like to constrain the category we are working with a bit. Namely we want to vary the value of the "inverse temperature" β which amounts to replace

$$\exp(2\pi K) \to (\exp(2\pi K))^{\beta/2\pi} = \exp(\beta K), \tag{5}$$

while leaving the factorised scattering property, hence in particular the S-matrix as in (2), of the model under consideration unaffected. This kind of situation arises also in the broader context of quantum gravity. The procedure outlined can be considered as a concrete realisation of the *replica* idea of Callan and Wilczek [6].

What has been said so far in this section remains an empty application of the deformation concept if we were not able to implement our idea into a realistic QFT. Or in concrete terms the question is how we can define correlation functions of local operators consistent with the replica idea (5) and avoiding mathematical ambiguities.

The solution of this problem was given by Niedermaier in a series of papers [7]. He showed that the replica deformation outlined before can be consistently implemented on the level of form factors [4]. A form factor is a matrix element of a local operator $\mathscr{O}(x)$ between the vacuum and an n-particle scattering state. These states depend on rapidities z_i and are created from the vacuum by means of Faddeev-Zamolodchikov operators $A_{a_i}(z_i)$, where the index a_i refers to the charge of the state. Encoding this in a formula means

$$\begin{aligned} &F_{a_1\ldots a_n}(z_1,\ldots,z_n) = \\ &\qquad \langle 0|\mathscr{O}(x)|z_1,\ldots,z_n\rangle_{a_1\ldots a_n} \sim \mathrm{Tr}\left[e^{2\pi K}\mathscr{O}(x)A_{a_1}(z_1)\cdots A_{a_n}(z_n)\right], \end{aligned} \tag{6}$$

where we are still using the trace mnemonic described before. Employing the standard Källén-Lehmann spectral series this form factor enables us in principle to compute the Wightman functions of the QFT by the following expression (omitting for the moment the charge indices of the form factor).

$$\langle 0|\mathscr{O}_1(x)\mathscr{O}_2(0)|0\rangle = \sum_n \frac{1}{n!}\int F^{\mathscr{O}_1}(z_1,\ldots,z_n)(F^{\mathscr{O}_2}(z_1,\ldots,z_n))^* e^{-m|x|\sum_{i=1}^n \cosh z_i}\prod_{i=1}^n \frac{\mathrm{dz_i}}{2\pi}. \tag{7}$$

One of the most fascinating aspects of integrable massive quantum field theories in $1+1$-dimensions is that one can in principle compute the form factors (6) exactly. In concrete terms this means that there exists a set of equations which enables to compute the expression (6) in a non-perturbative way. This was first realised in [8] but essentially carried out by Smirnov [4] for $\beta = 2\pi$. It was also Smirnov who proved that once one has these equations at hand, they do imply locality of the QFT.

It is one of the achievements of Niedermaier [7] that he could prove the results of Smirnov within the context of axiomatic QFT using modular theory and thereby generalising the setup to the deformed framework outlined above.

The first two conditions the form factor has to satisfy are the following

$$\begin{aligned} F_{a_1,a_2\ldots a_n}(z_1+i\beta,z_2,\ldots,z_n) &= \eta\ F_{a_2\ldots a_n,a_1}(z_2,\ldots,z_n,z_1),\\ F_{a_1,\ldots a_n,a_{n-1}}(z_1,\ldots,z_n,z_{n-1}) &= S^{a'_{n-1}a'_n}_{a_{n-1}a_n}(z_{n-1}-z_n)F_{a_1,\ldots,a'_{n-1}a'_n}(z_1,\ldots,z_{n-1},z_n). \end{aligned} \tag{8}$$

The parameter η is a pure phase, its actual value depends on the particular operator $\mathscr{O}$ described by the form factor.

It is interesting to mention already at this stage that because the S-matrix entering (8) in our case corresponds to the rational R-matrix of the quantum group sl_2 it can be shown that Equations (8) imply a quantum Knizhnik-Zamolodchikov equation. From this point of view one can realise another aspect of our β-deformation. If $\beta = 2\pi$ we are in the particular case of a qKZ equation at level zero [9]. This is exactly the special case studied by Smirnov [4]. Hence, it is absolutely natural from the qKZ point of view to ask whether there is a physical interpretation of the qKZ-solutions at generic level, which means for $\beta \neq 2\pi$. However, the physics of the model in question requires further properties of the form factor associated with its analytic structure. Let $C = i\sigma^2$ be the charge conjugation matrix of the ITM. We then get the following residue equations for the form factor

$$\begin{aligned} \operatorname*{res}_{z_n=z_{n-1}+i\pi} F_{a_1,\ldots a_{n-1},a_n}(z_1,\ldots,z_{n-1},z_n) &= i\,C_{a_{n-1}a_n}F_{a_1,\ldots a_{n-2}}(z_1,\ldots,z_{n-2}),\\ \operatorname*{res}_{z_n=z_{n-1}-i\pi} C^{a_{n-1}a_n}F_{a_1,\ldots a_{n-1},a_n}(z_1,\ldots,z_{n-1},z_n) &= i\,F_{a_1,\ldots a_{n-2}}(z_1,\ldots,z_{n-2}). \end{aligned} \tag{9}$$

Note that the residue equation for $z_n = z_{n-1}+i\pi$ is similar to the condition satisfied by traces of vertex operators in lattice models [10]. Without the second equation (9) however the recursion $n-2 \to n$ would be highly ambiguous. Taking advantage of the qKZ equation (8) to implement the analytic continuation $z_n \to z_n + i\beta$, the second equation (9) can be rewritten as follows

$$\begin{aligned} \mathrm{res}_{z_n=z_{n-1}-i\pi+i\beta} F_{a_1\ldots a_n}(z_1,\ldots,z_{n-1},z_n) &= \eta F_{b_1\ldots b_{n-2}}(z_1,\ldots,z_{n-2})\times \\ &\times C_{a_n c} S_{c_1 a_1}^{c\, b_1}(z_{n-1}-z_1)\cdots S_{a_{n-1}a_{n-2}}^{c_{n-1}b_{n-2}}(z_{n-1}-z_{n-2}). \end{aligned} \tag{10}$$

If we are in the deformed context $\beta \neq 2\pi$ is is easy to see that the form factor (6) can have a very rich pole structure in the region $|Im(z)| < 2\pi$. In contrast to that, if $\beta = 2\pi$ then period of analytic continuation is 2π and it can be shown that (9) and (10) combine to give the standard kinematical residue equations introduced by Smirnov [4].

To summarise this section we have shown that it is possible to deform an integrable massive QFT according to the prescription (5) preserving the S-matrix of the model. This deformation can be carried out in the context of the form factor approach and in fact one can find non-perturbative solutions of the QFT in the deformed region via (8), (9), and (10). From the point of view of the qKZ equations it seems absolutely natural to have form factors at $\beta = 2\pi$. We should, however, stress that the physics in the deformed model shows essentially new features. For example the structure of the conserved charges will be different. Then physically notable is the expected change in the kinematics, which for instance causes an increase of mass eigenvalues in the asymptotic states, or there may be circumstances where not the entire phase space is accessible by the model. These and other interesting features are discussed in [7].

3. Minimal form factors and Barnes' multi-periodic functions

We are now going to support the argument that it is mathematically absolutely natural to have the deformation parameter $\beta \neq 2\pi$. For that we consider the equations (8) in the case of only two particles present (so called minimal form factor equations). This problem essentially reduces to the following equations involving only the scalar part of the S-matrix (3).

$$F^m(z) = S_0(z)F^m(-z), \qquad F^m(z+i\beta) = F^m(-z). \tag{11}$$

The standard solution procedure of this equation is that one brings $S_0(z)$ into an integral representation and then solves (11). This has been done in [4] for the undeformed case of $\beta = 2\pi$. However, as was shown in [11] for a whole class of integrable models, this approach does not reveal the mathematical beauty of the solution spaces of (11). Upon assuming certain minimality conditions on F^m a natural solution can be found in terms of functions which were introduced by Barnes about one hundred years ago, see [11]. In our case we use Barnes' diperiodic Γ-function. This object is defined by a contour integral of Hankel type, where γ is the usual Euler constant. The periods are ω_1 and ω_2, and do enter symmetrically into the definition.

$$\log\Gamma_2(x|\omega_1,\omega_2) = \frac{1}{2\pi i}\int_{C_H} \frac{\exp(-xt)(\log(-t)+\gamma)}{\prod_{i=1}^{2}(1-\exp(-\omega_i t))}\frac{\mathrm{d}t}{t}. \tag{12}$$

The most important property of this function is its periodicity in the sense of Barnes, which means that

$$\frac{\Gamma_2(x+\omega_1|\omega_1,\omega_2)}{\Gamma_2(x|\omega_1,\omega_2)} \sim \frac{1}{\Gamma(x|\omega_2)}. \tag{13}$$

It is now straightforward to check that

$$F^m(\theta) = \frac{\Gamma_2(\pi - i\theta|2\pi,\beta)\Gamma_2(\pi+\beta+i\theta|2\pi,\beta)}{\Gamma_2(-i\theta|2\pi,\beta)\Gamma_2(\beta+i\theta|2\pi,\beta)} \tag{14}$$

is a solution to (11) for generic β. And as we have two periods in (14) we realise at once that it is natural to have one period different from 2π, while the undeformed case studied in [4] is just the degeneration of (14) for coinciding periods $\omega_1 = \omega_2 = 2\pi$. Even though we stick to the ITM in this note, it is worth emphazising that we get equivalent results for other integrable models as well, see [11].

4. General solution and the qKZ equation

Up to now we have been explaining the naturalness of having a deformation parameter $\beta \neq 2\pi$ in the form factor approach. In order to make really sense of this we have to solve Equations (8)–(10) for an arbitrary kink configuration in the ITM.

I guess it would be in Moshé's spirit to say that once we have a mathematically firm fundament the rest is just computational technique. It turns out that this is true in the present case as we can take advantage of the classification of the solutions of the sl_2-qKZ equation with rational R-matrix carried out by Tarasov and Varchenko [12]. The details on how one can construct the form factor solutions from the qKZ ones can be found in [13], we will here merely sketch the results in order to indicate how the system works.

In the ITM we are concerned with kinks and anti-kinks which correspond to the fundamental representation V of $SU(2)$. Hence an n-particle state will be an element in $V^{\otimes n}$. Define the Pauli matrices acting in this tensor product space as $\Sigma^a = \sum_{i=1}^n \sigma^a$. The weight subspaces of $V^{\otimes n}$ are then given by:

$$(V^{\otimes n})_l = \{v \in V^{\otimes n} | \Sigma^3 v = (n-2l)v\}, \tag{15}$$

where we can interpret the quantum number $j = n/2 - l$ as the isospin carried by a local operator which is described by the form factor (6). Due to the intertwining properties of the local operator we can restrict attention to yet another subspace of $(V^{\otimes n})_l$, which is the subspace of singular vectors $(V^{\otimes n})_l^{sing}$ whose elements are annihilated by Σ^+. We will consider only those solutions of (8) which are valued in $(V^{\otimes n})_l^{sing}$. Following [9] we will use the following convention to label the vectors in this space. Let $M = \{m_1 < m_2 < \ldots < m_l\} \subseteq \{1,2,\ldots,n\}$, such that the number of elements of M is $\#M = l$. In other words this set labels the positions at which the form factor (6) has kinks of type "$-$" in the scattering state, i.e., $M = \{i|\varepsilon_i = -\}$. Hence, we can write $v_M := v_{\varepsilon_1} \otimes \ldots \otimes v_{\varepsilon_n}$.

This in turn means that for the indices at the form factor (6) we can make the identification $F_{a_1,\dots,a_n}(z_1,\dots,z_n) = F_M(z_1,\dots,z_n)$.

The solutions of the rational qKZ equation in the case of tensor products of fundamental representations of sl_2 can be found in [12]. According to [13] they can be rewritten along the lines indicated in [9] in the following way:

$$\begin{aligned}\mathscr{I}_M^P(z_1,\dots,z_n) \;=\; & \sum_{\#M=l} \int_{C_1} \cdots \int_{C_l} w_M(t_1,\dots,t_l) P(\xi_1,\dots,\xi_l;\zeta_1,\dots,\zeta_n) \times \\ & \times \prod_{a=1}^{l}\prod_{j=1}^{n} \Phi(t_a - z_j) \prod_{1\le a<b\le l} \psi(t_a - t_b)\, \mathrm{d}t_1 \cdots \mathrm{d}t_l\, v_M.\end{aligned} \tag{16}$$

We shall briefly explain the entities in this hypergeometric integral, which is a solution to the qKZ equation for the representation considered in this paper. The structure of the integration contours C_l is not really important in this paper and we refer to [13]. The analytic information on the kink-state involved is encoded in the function

$$w_M(t_1,\dots,t_l) = \mathrm{Asym}\left(\prod_{a=1}^{l}\left(\frac{1}{t_a - z_{m_a} - 3i\pi/2}\prod_{1\le k<m_a}\frac{t_a - z_k - i\pi/2}{t_a - z_k - 3i\pi/2}\right)\right), \tag{17}$$

where the antisymmetrisation has to be performed over all variables $t_1,\dots,t_l$. The following objects have a deep meaning in the geometry of hypergeometric functions [12]. We have

$$\Phi(y) := \Gamma\left(\frac{\pi/2 + iy}{\beta}\right)\Gamma\left(\frac{\beta - 3\pi/2 - iy}{\beta}\right)\exp(\frac{\pi}{\beta}y), \tag{18}$$

and

$$\psi(y) := \frac{1}{\pi^3\beta}\Gamma\left(\frac{\pi + iy}{\beta}\right)\Gamma\left(\frac{\pi - iy}{\beta}\right)\sinh(\frac{\pi}{\beta}\, y). \tag{19}$$

Last but not least $P(\xi_1,\dots,\xi_l;\zeta_1,\dots,\zeta_n)$ is a polynomial in variables $\xi_a = \exp(-2\pi t_a/\beta)$ with complex coefficients and certain requirements on the partial degrees in the variables ξ, see [12, 13]; note that $\zeta_k = \exp(-2\pi z_k/\beta)$. The object P is yet undetermined as $\mathscr{I}_M^P$ is a solution of the qKZ equation independent of the actual form of P. In fact, P will serve to classify the local operator content of the QFT.

Using the results of the previous section, we can now prove that

$$F_M(z_1,\dots,z_n) = \frac{c_l}{(2\pi i)^l}\prod_{n\ge i>j\ge 1} F^m(z_i - z_j) \times \mathscr{I}_M^P(z_1,\dots,z_n), \tag{20}$$

satisfies the first bit (8) of the deformed form factor equations for some constant c_l. We would like to stress again, that the limiting value $\beta = 2\pi$ is naturally within the solutions (20), moreover from the viewpoint of qKZ solution spaces it would even be unnatural to restrict to the single value $\beta = 2\pi$.

5. Kinematical residue equations and the Noether currents

The harder part in proving that (20) really is a consistent form factor, is to establish conditions on the free parameters P arising from the kinematical residue equations (9) or (10). To evaluate the residues of (20) one has to carefully consider the analytic structure (poles and zeroes) of the hypergeometric integral (16). This structure has an important geometrical meaning [12] and is among the most beautiful features of the qKZ solutions.

The details of the calculation can be found in [13] and we will merely state one of the conditions on $P(\xi_1,\ldots,\xi_l;z_1,\ldots,z_n)=P_{l,n}$ in order to get a flavour of what it looks like.

$$P_{l,n}\big|_{\substack{t_l=z_{n-1}+3i\pi/2\\ z_n=z_{n-1}+i\pi}}\times\prod_{a=1}^{l-1}\frac{(-1)\;e^{2\pi(t_a-z_{n-1}-i\pi/2)/\beta}}{\sinh\frac{\pi}{\beta}(t_a-z_{n-1}-i\pi/2)\sinh\frac{\pi}{\beta}(t_a-z_{n-1}-5i\pi/2)}\times \\ \times\prod_{j=1}^{n-2}e^{\pi(z_{n-1}-z_j+3i\pi/2)/\beta}\qquad\sim\qquad P_{l-1,n-2}. \tag{21}$$

The "$\sim$" means that this is an equality up to a phase, which is permitted by the requirements on the form factors outlined before.

The main result obtained in [13] is then that once we find solutions of the recursion equation (21) (and another one of similar type arising from (10)) we have established that there exist infinite sequences of form factors (20), where each sequence is labelled by an invariant isospin $j=n/2-l$ and the solution spaces of (21) classify the local operator content of the QFT associated with the $SU(2)$-invariant Thirring model. The statements just made are true for any value of the deformation parameter β, and only for $\beta=2\pi$ are we in the context of the standard unthermalised Thirring model as it was described by Smirnov [4]. A discussion of the ambiguities of the solution spaces of (21) can be found in [13].

To conclude we would like to write down a solution [13] of (21) which is associated with the Noether current of the ITM for an arbitrary value of the deformation parameter.

Let j^a_μ be the Noether current associated with the Lagrangian (1); μ is the Lorentz index and a the $SU(2)$-index as introduced before. It is convenient to switch to lightcone coordinates both in Minkowski space and in internal space, i.e.,

$$j^a_\pm=j^a_0\pm j^a_1,\qquad j^\pm_\sigma=j^1_\sigma\pm i\,j^2_\sigma,\qquad \sigma=\pm. \tag{22}$$

Then we can introduce an index $\tau=\pm,3$, labelling the components of the lightcone currents as j^τ_σ. A solution of Equations (21) is then up to a constant given by

$$P^\sigma_{l,n}\sim e^{\pi(2l-n/2+\sigma)\sum z_j/\beta}\prod_{a=1}^{l}e^{-2\pi(l+\sigma)t_a/\beta}\times \\ \times\prod_{1\le a<b\le l}\sinh\frac{\pi}{\beta}(t_a-t_b+i\pi)\sinh\frac{\pi}{\beta}(t_a-t_b-i\pi), \tag{23}$$

It was shown in [13] that the form factors with the insertion of this object correspond to the form factors of the current j_σ^- in (22). The ones of the other currents are obtained from this one by obvious sl_2 action.

This at least gives an existence proof for solutions to (21) and hence we have established that we can exactly solve the replica deformed version of the $SU(2)$-invariant Thirring model using the method of form factors.

References

1. Bayen, F., Flato, M.. Fronsdal, C., Lichnerowicz, A., and Sternheimer, D.: Deformation Theory and Quantization I & II, *Ann. Physics* **111** (1978), 61–110 & 111–151.
2. Pillin, M.: On the deformability of Heisenberg algebras, *Comm. Math. Phys.* **180** (1996), 23–38.
3. Flato, M., Frønsdal, C., and Sternheimer, D.: Singletons, Physics in AdS universe and oscillations of composite neutrinos, *Lett. Math. Phys.* **48** (1999), 109–119.
4. Smirnov, F. A.: Lectures on integrable massive models of QFT, in: M. L. Ge and B. H. Zhao, (eds.), *Introduction to quantum group and integrable massive models of quantum field theory (Nankai, 1989)*, Nankai Lectures Math. Phys., World Sci. Publishing, River Edge, NJ, 1990, pp. 1–68.
5. Bisognano, J. and Wichmann, E.: On the duality condition for a Hermitian scalar field, *J. Math. Phys.* **16** (1975), 985–1007; On the duality condition for quantum fields, *J. Math. Phys.* **17** (1976), 303–321.
6. Callan, C. and Wilczek, F.: On geometric entropy, *Phys. Lett. B* **333** (1994), 55–61.
7. Niedermaier, M.: Varying the Unruh temperature in integrable QFTs, *Nucl. Phys. B* **535** (1998), 621–649; Form factors, thermal states and modular structures, *Nucl. Phys. B* **519** (1998), 517–550; A derivation of the cyclic form factor equation, *Comm. Math. Phys.* **196** (1998), 411–428.
8. Karowski, M. and Weisz, P.: Exact form factors in 1+1-dimensional field theoretic models with soliton behaviour, *Nucl. Phys. B* **139** (1978), 455–476.
9. Nakayashiki, A., Pakuliak, S., and Tarasov, V.: On solutions of the KZ and qKZ equations at level zero, preprint q-alg/9712002.
10. Jimbo, M. and Miwa, T.: *Algebraic analysis of solvable lattice models*, CBMS Regional Conference Series in Mathematics, 85, Amer. Math. Soc., Providence, RI, 1995.
11. Pillin, M.: Exact two-particle matrix elements in S-matrix preserving deformation of integrable QFTs, *Phys. Lett. B* **448** (1999), 227–233.
12. Tarasov, V. and Varchenko, A.: Geometry of q-hypergeometric functions as a bridge between Yangians and quantum affine algebras, *Invent. Math.* **128** (1997), 501–588.
13. Pillin, M.: Replica-deformation of the $SU(2)$-invariant Thirring model via solutions of the qKZ equation, preprint KCL-MTH-99-29 (1999), hep-th/9907147.