Deformation quantization of Fréchet-Poisson algebras: Convergence of the Moyal product

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Keywords: deformation quantization, star-product, noncommutative geometry

Mathematics Subject Classifications (2000): 53D55, 81S10, 53D17

Dedicated to the memory of Professor Moshé Flato

1. Introduction

Beyond formal deformation quantization, i.e., deformation quantization of Poisson algebras with a formal deformation parameter [1], one can ask whether the formal parameter converges. In this direction, Rieffel [7] presented a notion of strict deformation quantization: Deformation quantization with a convergent product in the C^*-algebras sense. This suggests us opportunities of finding various kind of notions of deformation quantizations with a convergent product for suitable categories of algebras.

The purpose of this paper is to give a notion of deformation quantization corresponding to Rieffel's work [7] in the Fréchet categories, and to show that several strange phenomena occur when we treat exponential functions of quadratic forms.

Let $\mathscr{F}$ be a commutative, associative Fréchet algebra over $\mathbb{C}$, i.e., $\mathscr{F}$ has a metrizable complete topology defined by a family of semi-norms, and a product operation denoted by dot $\cdot$ which is continuous.

G. Dito and D. Sternheimer (eds.), Conférence Moshé Flato 1999, Vol. 2, 233 – 245.

$\mathscr{F}$ is called a *Fréchet-Poisson algebra* if $\mathscr{F}$ has a continuous bilinear operation $\{\ ,\ \}: \mathscr{F} \times \mathscr{F} \to \mathscr{F}$ (called a *Poisson bracket* on $\mathscr{F}$) such that for any $f, g, h \in \mathscr{F}$:

(P1) $\{f,g\} = -\{g,f\}$ (skew symmetry);

(P2) $\sum_{\text{cyclic sum}}\{f,\{g,h\}\} = 0$ (Jacobi identity);

(P3) $\{f,gh\} = \{f,g\}h + g\{f,h\}$ (bi-derivation).

Generally, if $\mathscr{F}$ is a Fréchet space and has an associative product $*$ on $\mathscr{F}$ such that the operation $* : \mathscr{F} \times \mathscr{F} \to \mathscr{F}$ is continuous, we call $(\mathscr{F}, *)$ a *Fréchet algebra.*

We now give a notion of *deformation quantization of Fréchet-Poisson algebra* as a family of associative product $*_\hbar$ on $\mathscr{F}$ parameterized by $\hbar \in \mathbb{R}$.

Definition 1.1 *Let $\hbar \in \mathbb{R}$. Let $\mathscr{F}$ be a Fréchet-Poisson algebra. $(\mathscr{F}, *_\hbar)$ is called a deformation quantization of a Fréchet-Poisson algebra $\mathscr{F}$ if the following conditions hold:*

(FD1) For any $\hbar$, there exists an associative product $*_\hbar$ on $\mathscr{F}$ so that $(\mathscr{F}, *_\hbar)$ is a Fréchet algebra.

(FD2) $f *_\hbar g$ is continuous in $\hbar \geq 0$ for every $f, g \in \mathscr{F}$ independent of $\hbar$, and $f *_0 g = f \cdot g$.

(FD3) $\frac{1}{\hbar i}(f *_\hbar g - f \cdot g) \to \frac{1}{2}\{f,g\}$ as $\hbar \to 0$ for every $f, g \in \mathscr{F}$ independent of $\hbar$.

We now give an example of the above notion, which is based on the Moyal product for functions as follows: Let $\mathbb{C}^2$ be the complex 2-plane, and x, y the coordinates on $\mathbb{C}^2$.

Denote by $\mathscr{P}(\mathbb{C}^2)$ the set of all polynomials of x and y, and denote by $\mathscr{E}(\mathbb{C}^2)$ the set of all entire functions on $\mathbb{C}^2$. $\mathscr{E}(\mathbb{C}^2)$ is a complete topological vector space under the compact open topology.

For entire functions $f = f(x,y)$ and $g = g(x,y)$ on $\mathbb{C}^2$, we set

$$\{f,g\} = \partial_y f \cdot \partial_x g - \partial_x f \cdot \partial_y g. \tag{1.1}$$

This gives a Poisson bracket on $\mathbb{C}^2$, which is canonical in the sense that the coordinates x, y turn to be Darboux coordinates.

The Moyal product formula is given as follows [5]:

$$f *_\hbar g = \sum_k \left(\frac{\hbar i}{2}\right)^k \frac{1}{k!} f(\overleftarrow{\partial}_y \cdot \overrightarrow{\partial}_x - \overleftarrow{\partial}_x \cdot \overrightarrow{\partial}_y)^k g. \tag{1.2}$$

In general, $f *_\hbar g$ is not always defined for any $f, g \in \mathscr{E}(\mathbb{C}^2)$, but the following properties hold:

- $*_\hbar$ defines an associative product on the space $\mathscr{P}(\mathbb{C}^2)$.
- For every fixed $p \in \mathscr{P}(\mathbb{C}^2)$, the left- and the right-multiplication $p*_\hbar$, $*_\hbar p$ extend to continuous linear mappings

$$p*_\hbar,\ *_\hbar p:\ \mathscr{E}(\mathbb{C}^2) \to \mathscr{E}(\mathbb{C}^2).$$

Using polynomial approximation, we see easily that the associativity

$$f *_\hbar (g *_\hbar h) = (f *_\hbar g) *_\hbar h$$

holds, if two of f, g, h are polynomials. We call such a system $(\mathscr{E}(\mathbb{C}^2), \mathscr{P}(\mathbb{C}^2); *_\hbar)$ a $\mathscr{P}(\mathbb{C}^2)$-*bimodule*.

We consider the following subspace of $\mathscr{E}(\mathbb{C}^2)$: For every positive $p > 0$, set

$$\mathscr{E}_p(\mathbb{C}^2) = \{f \in \mathscr{E}(\mathbb{C}^2) \mid \|f\|_{p,s} = \sup |f|\, e^{(-s|\xi|^p)} < \infty, \forall s > 0\}, \tag{1.3}$$

where $|\xi| = (|x|^2 + |y|^2)^{1/2}$. The family $\{\|\ \|_{p,s}\}_{s>0}$ induces a topology on $\mathscr{E}_p(\mathbb{C}^2)$ and $(\mathscr{E}_p(\mathbb{C}^2), \cdot)$ is an associative commutative Fréchet algebra, where the dot $\cdot$ is the ordinary multiplication for functions in $\mathscr{E}_p(\mathbb{C}^2)$. It is easily seen that for $0 < p < p'$, we have a continuous embedding

$$\mathscr{E}_p(\mathbb{C}^2) \subset \mathscr{E}_{p'}(\mathbb{C}^2) \tag{1.4}$$

as commutative Fréchet algebras [3].

It is obvious that every polynomial is contained in $\mathscr{E}_p(\mathbb{C}^2)$ and $\mathscr{P}(\mathbb{C}^2)$ is dense in $\mathscr{E}_p(\mathbb{C}^2)$ for any $p > 0$. The Poisson bracket (1.1) is also well-defined on $\mathscr{E}_p(\mathbb{C}^2)$, and $(\mathscr{P}(\mathbb{C}^2), \{,\}, \cdot)$ is a dense Poisson subalgebra of $(\mathscr{E}_p(\mathbb{C}^2), \{,\}, \cdot)$. We remark that every exponential function $e^{\alpha x + \beta y}$ is contained in $\mathscr{E}_p(\mathbb{C}^2)$ for any $p > 1$, but not in $\mathscr{E}_1(\mathbb{C}^2)$, and functions such as $e^{ax^2 + by^2 + 2cxy}$ are contained in $\mathscr{E}_p(\mathbb{C}^2)$ for any $p > 2$, but not in $\mathscr{E}_2(\mathbb{C}^2)$.

Our main result in this paper is as follows:

Theorem 1.2 *The Moyal product formula* (1.2) *gives the following:*

(i) *For* $0 < p \leq 2$, $(\mathscr{E}_p(\mathbb{C}^2), *_\hbar)$ *is a deformation quantization of* $(\mathscr{E}_p(\mathbb{C}^2), \cdot, \{,\})$.

(ii) *For* $p > 2$ *and a fixed* $\hbar \in \mathbb{R}$*, the Moyal product formula gives a continuous bilinear mapping of*

$$\begin{aligned} &\mathscr{E}_p(\mathbb{C}^2) \times \mathscr{E}_{p'}(\mathbb{C}^2) \to \mathscr{E}_p(\mathbb{C}^2), \\ &\mathscr{E}_{p'}(\mathbb{C}^2) \times \mathscr{E}_p(\mathbb{C}^2) \to \mathscr{E}_p(\mathbb{C}^2), \end{aligned} \tag{1.5}$$

for every p' *such that* $\frac{1}{p} + \frac{1}{p'} \geq 1$.

We remark here about the statement (ii). Since $p > 2$, p' must be $p' \leq 2$, hence the statement (i) gives that $(\mathscr{E}_{p'}(\mathbb{C}^2); *_\hbar)$ is a Fréchet algebra. So the statement (ii) means that every $\mathscr{E}_p(\mathbb{C}^2)$, $p > 2$, is a topological $\mathscr{E}_{p'}(\mathbb{C}^2)$-bimodule.

We remark also that if $\hbar > 0$, then $e^{\pm\frac{2i}{\hbar}xy} \in \mathscr{E}_p(\mathbb{C}^2)$ for every $p > 2$, but $e^{\frac{2i}{\hbar}xy} *_\hbar e^{-\frac{2i}{\hbar}xy}$ diverges. This means in a sense the above result is the best possible.

We define a complex conjugate $\bar{a}$ of $a \in \mathscr{E}(\mathbb{C}^2)$ as an anti-homomorphism of the bimodule by setting $\bar{x} = x, \bar{y} = y, \bar{i} = -i$ on generators. It is obvious that $\mathscr{E}_p(\mathbb{C}^2)$ is closed under the complex conjugation, and in general this looks as a very nice property.

However, this property causes some trouble for $p > 2$ as follows: We fix $\hbar > 0$. Then, the coordinate function y has a right-inverse $\frac{1}{y}(1 - e^{\frac{2i}{\hbar}xy})$, and a left-inverse $\frac{1}{y}(1 - e^{-\frac{2i}{\hbar}xy})$ as the *complex conjugate* of the right-inverse. If $(\mathscr{E}_p(\mathbb{C}^2); *_\hbar)$ were an associative algebra, then these would be the same genuine inverse, and we must set the nontrivial element $\frac{1}{y}\sin\frac{2}{\hbar}xy = 0$. Hence, it seems inconvenient to treat the right and the left inverse of y in the framework of C^* algebras. We discuss such phenomena more closely in sections 5 and 7.

2. Completion of the free tensor algebra

Let $\mathscr{T}$ be the free tensor algebra generated by X_1, X_2. We introduce a topology into $\mathscr{T}$ so that $\mathscr{T}$ becomes a topological algebra. By definition, an element of $\mathscr{T}$ is written in the form

$$T = \sum c_\alpha X_\alpha \quad \text{(finite sum)}$$

where $X_\alpha = X_{\alpha_1} \otimes X_{\alpha_2} \otimes \cdots \otimes X_{\alpha_n}$, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$. We call X_α a *word* and we denote by $|\alpha|$ the *length* n of the word X_α. We define a system of semi-norms $\|\ \|_{\tau,s}$ $(\tau \geq 0, s > 0)$ by

$$\|T\|_{\tau,s} = \sum |c_\alpha| |\alpha|^{\tau|\alpha|} s^{\tau|\alpha|}. \tag{2.1}$$

For a fixed $\tau > 0$, we consider a topology introduced by a system of semi-norms $\{\|\cdot\|_{\tau,s}\}_{s>0}$. We set $\mathscr{T}_\tau$ the completion of $\mathscr{T}$ with respect to this topology. Then one easily sees that

$$\mathscr{T}_\tau = \{T = \sum_\alpha c_\alpha X_\alpha \text{ (infinite sum)};\ \|T\|_{\tau,s} < \infty \text{ for any } s > 0\}. \tag{2.2}$$

Then we have

Lemma 2.1 *For $\tau > 0$, $\mathscr{T}_\tau$ becomes a Fréchet algebra satisfying*

$$\|T \otimes T'\|_{\tau,s} \leq \|T\|_{\tau,2s} \|T'\|_{\tau,2s}.$$

Proof. It follows easily from the inequality

$$m^{\tau m} n^{\tau n} \leq (m+n)^{\tau(m+n)} \leq 2^{\tau(m+n)} m^{\tau m} n^{\tau n} \tag{2.3}$$

for non-negative integers m,n. ■

Furthermore, we have

Lemma 2.2 *Let $T = \sum c_\alpha X_\alpha$ be an element of $\mathscr{T}$ and let N be the maximum length of words X_α of T. Then $e^T = \sum \frac{1}{k!} T^{\otimes k}$ is an element of $\mathscr{T}_\tau$ for $0 < \tau < \frac{1}{N}$.*

Proof. It suffices to see the convergence of $\sum_{k=0}^{\infty} \frac{1}{k!}(kN)^{k\tau N} s^{k\tau N}$ under the condition $\tau N < 1$. Since

$$R_k = \frac{1}{(k+1)!} \ ((k+1)N)^{(k+1)\tau N} s^{(k+1)\tau N} / \tfrac{1}{k!}(kN)^{k\tau N} s^{k\tau N}$$
$$= s^{\tau N} \frac{((k+1)N)^{\tau N}}{k+1} (1+\tfrac{1}{k})^{k\tau N}$$

one sees easily that $\lim_{k\to\infty} R_k = 0$ if $\tau N < 1$. ■

3. Subspaces of symmetric elements

Now, we consider a subspace $\mathscr{S}_\tau \subset \mathscr{T}_\tau$ equipped with the symmetric product.

First we introduce a product $a \circ b = \frac{1}{2}(a \otimes b + b \otimes a)$ in $\mathscr{T}$. We set [5]

$$(a\circ)^p \cdot c = a \circ (a \circ (\cdots (a \circ c) \cdots))$$

and

$$(a\circ)^p \cdot (b\circ)^q \cdot c = a \circ (a \circ (\cdots (a \circ (b\circ)^q \cdot c) \cdots)).$$

Using this notation, we define a linear subspace $\mathscr{S}$ as

$$\mathscr{S} = \{\sum c_{\alpha,\beta} (X_1\circ)^\alpha \cdot (X_2\circ)^\beta \cdot 1\}.$$

We also introduce a commutative product $\odot$ on $\mathscr{S}$ by

$$\begin{aligned} ((X_1\circ)^k \cdot (X_2\circ)^l \cdot 1) &\odot ((X_1\circ)^m \cdot (X_2\circ)^n \cdot 1) \\ &= (X_1\circ)^{k+m} \cdot (X_2\circ)^{l+n} \cdot 1. \end{aligned}$$

For simplicity, we write

$$X_1^\alpha \odot X_2^\beta = (X_1\circ)^\alpha \cdot (X_2\circ)^\beta \cdot 1. \tag{3.1}$$

Then the set $\mathscr{S}$ with the multiplication $\odot$ becomes a commutative associative algebra.

Let $\mathscr{S}_\tau$ be the closure of $\mathscr{S}$ in $\mathscr{T}_\tau$. It is easy to see the following:

Proposition 3.1 *For every $\tau > 0$, the bilinear product $\odot$ is continuous and $\mathscr{S}_\tau$ is a commutative Fréchet algebra.*

On the space $\mathscr{S}$ we consider a non-commutative product $*_\hbar$ by using the Moyal product formula with the parameter $\hbar$.

To obtain the norm estimate of the $*_\hbar$-product, we remark first the following

Lemma 3.2 *For every $\tau > 0$ and every $s >> 0$, we have an inequality*

$$\|\partial_{X_1}^k \partial_{X_2}^l X_1^m \odot X_2^n\|_{\tau,s} \le \|X_1^m \odot X_2^n\|_{\tau,s} m^{k(1-\tau)} n^{l(1-\tau)}$$

Proof. Obviously we have that $\partial_{X_1}^k \partial_{X_2}^l X_1^m \odot X_2^n = \frac{m!}{(m-k)!}\frac{n!}{(n-k)!} X_1^{m-k} \odot X_2^{n-l}$. By the definition of semi-norms, we see that

$$\begin{aligned}
\|\partial_{X_1}^k \partial_{X_2}^l X_1^m \odot X_2^n\|_{\tau,s} &= \frac{m!}{(m-k)!}\frac{n!}{(n-l)!} s^{\tau(m+n-k-l)} (m+n-k-l)^{\tau(m+n-k-l)} \\
&< m^{\tau k} m^{k(1-\tau)} n^{\tau l} n^{l(1-\tau)} s^{\tau(m+n-k-l)} (m+n)^{\tau(m+n-k-l)} \\
&< m^{k(1-\tau)} n^{l(1-\tau)} s^{\tau(m+n-k-l)} (m+n)^{\tau(m+n)}.
\end{aligned}$$

The desired estimate follows easily. ■

Using this estimate we have the following:

Theorem 3.3 *The product $*_\hbar$ extends to $\mathscr{S}_\tau$ for $\frac{1}{2} \le \tau$ to define $(\mathscr{S}_\tau, *_\hbar)$ a non-commutative Fréchet algebra.*

In this algebra we have also the inequality

$$\|f *_\hbar g\|_{\tau,s} \le \|f \odot g\|_{\tau, se^{|\hbar|/\tau}}. \tag{3.2}$$

Proof. For $f = \sum a_{k,l} X_1^k \odot X_2^l$, $g = \sum b_{m,n} X_1^m \odot X_2^n$, a rough estimate gives

$$\begin{aligned}
\|f *_\hbar g\|_{\tau,s} \le \sum_{k,l,m,n} |a_{k,l}| \ |b_{m,n}| s^{\tau(k+l+m+n)} (k+l)^{\tau(k+l)} (m+n)^{\tau(m+n)} \\
\times \quad \textstyle\sum_p \frac{1}{p!} |\hbar|^p (k+l)^{p(1-\tau)} (m+n)^{p(1-\tau)}.
\end{aligned}$$

Remarking

$$(k+l)^{p(1-\tau)} (m+n)^{p(1-\tau)} < (k+l+m+n)^{2p(1-\tau)}$$

and

$$\sum_p \frac{1}{p!} |\hbar|^p (k+l+m+n)^{2p(1-\tau)} = e^{|\hbar|(k+l+m+n)^{2(1-\tau)}},$$

we see that $\frac{1}{2} \le \tau$ gives $2(1-\tau) \le 1$ and hence

$$\|f *_\hbar g\|_{\tau,s} \le \|f \odot g\|_{\tau, se^{|\hbar|/\tau}}.$$

■

In the above estimate, we have no need to sum up p for all non-negative integers, but p is restricted in fact in the domain $0 \le p \le \min(k+l, m+n)$. Thus, if $0 < \tau < \tau'$ then we can set

$$(k+l)^{\tau(k+l)}(k+l)^{p(1-\tau)} \le (k+l)^{\tau'(k+l)}(k+l)^{p(1-\tau')}.$$

Now suppose $0 < \tau < \frac{1}{2}$. Then we have

Lemma 3.4 *For τ' such that $0 < \tau < \tau'$ and $\tau + \tau' \ge 1$, it holds*

$$\|f *_\hbar g\|_{\tau,s} \le \|f\|_{\tau, se^{|\hbar|/\tau}} \|g\|_{\tau', se^{|\hbar|/\tau'}}.$$

Thus, we have

Theorem 3.5 *If $0 < \tau < \frac{1}{2}$, the product $*_\hbar$ gives continuous bilinear products*

$$*_\hbar : \mathscr{S}_\tau \times \mathscr{S}_{\tau'} \to \mathscr{S}_\tau, \quad *_\hbar : \mathscr{S}_{\tau'} \times \mathscr{S}_\tau \to \mathscr{S}_\tau$$

for every τ' with $\tau < \tau'$ and $\tau + \tau' \ge 1$.

4. Two-sided ideal of the relations

Suppose $\hbar \in \mathbb{C}$. Let $\mathscr{R}$ be a two-sided ideal generated by the fundamental relation of the Weyl algebra:

$$X_1 \otimes X_2 - X_2 \otimes X_1 + \hbar i$$

Definition 4.1 *We call the quotient algebra $\mathscr{T}/\mathscr{R} = \mathscr{W}$ the Weyl algebra.*

An element of $\mathscr{S}$ is written as $T = \sum c_{\alpha,\beta}(X_1)^\alpha \odot (X_2)^\beta$. From the identity (3.1), we have the natural embedding

$$\iota : \mathscr{S} \to \mathscr{T}.$$

We consider now the replacement $\otimes$ sign in every word of an element of $\mathscr{T}$ by $*_\hbar$ sign in $\mathscr{S}$. For instance

$$\begin{aligned} X_2 \otimes X_1 \otimes X_2 \to X_2 *_\hbar X_1 *_\hbar X_2 &= X_2 \odot (X_1 *_\hbar X_2) + \frac{\hbar i}{2} X_2 \\ &= X_1 \odot X_2^2 - \frac{\hbar i}{2} X_2 + \frac{\hbar i}{2} X_2. \end{aligned}$$

Obviously X_1, X_2 are contained in $\mathscr{S}$, and this replacement gives a linear mapping φ from $\mathscr{T}$ onto $\mathscr{S}$. Since the product $*_\hbar$ is defined by using the Moyal product formula, it is not hard to see that φ is an algebra homomorphism of $(\mathscr{T}, \otimes)$ onto $(\mathscr{S}, *_\hbar)$. Remarking that $\varphi \circ \iota(X_1^m \odot X_2^n) = X_1^m \odot X_2^n$, we see that $\varphi \circ \iota = \text{identity}$.

Recall now that Weyl algebra can be naturally identified with the space of all polynomials with the Moyal product formula [5]. Hence, we see that the kernel of φ must be the ideal $\mathscr{R}$ of the relations of Weyl algebra.

Thus, we have a linear splitting

$$\mathscr{T} = \mathscr{S} \oplus \mathscr{R}.$$

Now, we consider the closures of these spaces $\mathscr{T}_\tau$, $\mathscr{S}_\tau$ and $\mathscr{R}_\tau$. By Proposition 3.1 and inequality (3.2), we have the following:

Theorem 4.2 *Suppose that $\tau \geq \frac{1}{2}$. Then, φ and ι extend to continuous mappings $\varphi : \mathscr{T}_\tau \to \mathscr{S}_\tau$ and $\iota : \mathscr{S}_\tau \to \mathscr{T}_\tau$. Hence if $\tau \geq \frac{1}{2}$, then $\mathscr{T}_\tau = \mathscr{S}_\tau \oplus \mathscr{R}_\tau$.*

5. Collapsing of algebras

In this section, we show the following:

Theorem 5.1 *If $0 < \tau < \frac{1}{2}$, then $\mathscr{T}_\tau = \mathscr{R}_\tau$.*

The above theorem is obtained by showing $\mathscr{R}_\tau \ni 1$ for $0 < \tau < \frac{1}{2}$.

Recall that for any $\tau > 0$, $\mathscr{R}_\tau$ is a closed two-sided ideal of $\mathscr{T}_\tau$ and the quotient algebra $\mathscr{T}_\tau/\mathscr{R}_\tau$ is a topological associative algebra.

Suppose $\mathscr{T}_\tau/\mathscr{R}_\tau$ is not a trivial algebra. $\mathscr{T}_\tau/\mathscr{R}_\tau$ is then an associative Fréchet algebra. Since the Weyl algebra $\mathscr{W}$ is simple and is identified with $\mathscr{T}/\mathscr{R}$, $\mathscr{W}$ is densely embedded in $\mathscr{T}_\tau/\mathscr{R}_\tau$.

Recall that $\mathscr{W}$ is also identified with the space of all polynomials with the product given by Moyal product formula. Thus, it is identified with $\mathscr{S}$ with the Moyal product $*_\hbar$.

Hence $(\mathscr{S}, *_\hbar)$ can be naturally embedded in $\mathscr{T}_\tau/\mathscr{R}_\tau$. If $a *_\hbar b = c$, then this implies that $a \otimes b = c \bmod \mathscr{R}_\tau$ in the space $\mathscr{T}_\tau$. We also note that the product $*_\hbar$ is continuous in the topology τ.

Remark that for every $\hbar \neq 0$, we see that

$$e_{\odot}^{\frac{t}{\hbar}X_1 \odot X_2} = \sum_k \frac{1}{k!}(\frac{t}{\hbar})^k X_1^k \odot X_2^k$$

is an element of $\mathscr{S}_\tau$ for every $0 < \tau < \frac{1}{2}$ and similarly,

$$X_2^\circ = \frac{1}{X_2}(1 - e_{\odot}^{\frac{2i}{\hbar}X_1 \odot X_2}) \in \mathscr{S}_\tau, \quad 0 < \tau < \frac{1}{2},$$

where the right-hand side stands for

$$\sum_{k=1}^{\infty} \frac{1}{k!}(\frac{2i}{\hbar})^k X_1^k \odot X_2^{k-1}.$$

Since $X_i \in \mathscr{S}_{\tau'}$, $\frac{1}{2} < \tau'$, $X_2 *_\hbar X_2^\circ$, $X_2^\circ *_\hbar X_2$ are well-defined, a direct calculation using the Moyal product formula gives that $X_2 *_\hbar X_2^\circ = 1$, but $X_2^\circ *_\hbar X_2 = 1 - 2e_{\odot}^{\frac{2i}{\hbar}X_1 \odot X_2}$.

Considering a polynomial approximation of X_2° in $\mathscr{T}_\tau$ and using the continuity of the left multiplication $X_2 *_\hbar$ in $\mathscr{S}_\tau$, we see that $X_2 \circledast X_2^\circ = 1 \bmod \mathscr{R}_\tau$.

Taking the complex conjugate, we have $\overline{X_2^\circ} \circledast X_2 = 1 \bmod \mathscr{R}_\tau$. Since X_2° and $\overline{X_2^\circ}$ are a right-inverse and a left-inverse of X_2 respectively, we have $X_2^\circ - \overline{X_2^\circ} \in \mathscr{R}_\tau$. Hence, we have

$$\frac{1}{X_2} \sin_\odot (\frac{2}{\hbar} X_1 \odot X_2) \in \mathscr{R}_\tau.$$

Since

$$X_2 *_\hbar (\frac{1}{X_2} \sin_\odot \frac{2}{\hbar} X_1 \odot X_2) = \sin_\odot \frac{2}{\hbar} X_1 \odot X_2 + i \cos_\odot \frac{2}{\hbar} X_1 \odot X_2,$$

we have

$$X_2 \circledast (\frac{1}{X_2} \sin_\odot \frac{2}{\hbar} X_1 \odot X_2) - (\sin_\odot \frac{2}{\hbar} X_1 \odot X_2 + i \cos_\odot \frac{2}{\hbar} X_1 \odot X_2) \in \mathscr{R}_\tau.$$

Since the first term is in $\mathscr{R}_\tau$ and $\mathscr{R}_\tau$ is closed under complex conjugation, we see that $\sin_\odot \frac{2}{\hbar} X_1 \odot X_2$, $\cos_\odot \frac{2}{\hbar} X_1 \odot X_2 \in \mathscr{R}_\tau$. Hence we have $e_\odot^{\pm \frac{2i}{\hbar} X_1 \odot X_2} \in \mathscr{R}_\tau$.

Then we have

Lemma 5.2 $1 = e_\odot^{\frac{2i}{\hbar} X_1 \odot X_2} \odot e_\odot^{-\frac{2i}{\hbar} X_1 \odot X_2} \in \mathscr{R}_\tau$.

Proof. We first remark that

$$(X_1^m \odot X_2^n) \circledast (\partial_{X_1}^k \partial_{X_2}^l e_\odot^{\pm \frac{2i}{\hbar} X_1 \odot X_2}) \circledast (X_1^{m'} \odot X_2^{n'})$$

is an element of $\mathscr{R}_\tau$. This is because $\partial_{X_i} f$ can be written by using the commutator bracket.

Hence $(X_1^m \odot X_2^n) \odot e_\odot^{-\frac{2i}{\hbar} X_1 \odot X_2} \in \mathscr{R}_\tau$. This implies

$$(\sum_{k=0}^{n} \frac{1}{k!} X_1^k \odot X_2^k) \odot e_\odot^{-\frac{2i}{\hbar} X_1 \odot X_2} \in \mathscr{R}_\tau.$$

Taking the limit of the above quantity as $n \to \infty$, we see that $1 \in \mathscr{R}_\tau$, under the assumption that $\mathscr{T}_\tau / \mathscr{R}_\tau$ is not a trivial algebra. This is a contradiction, and hence we see that $\mathscr{R}_\tau \ni 1$. ■

6. $\mathscr{E}_p(\mathbb{C}^2)$ and $\mathscr{S}_\tau$

We first identify a polynomial on $\mathbb{C}^2$ and an element of the symmetric algebra $\mathscr{S}$ by

$$x^k y^l \longleftrightarrow X_1^k \odot X_2^l.$$

Proposition 6.1 *For every $p > 0$, $\mathscr{S}_{\frac{1}{p}}$ is linearly isomorphic to $\mathscr{E}_p(\mathbb{C}^2)$.*

Proof. Though the well-known estimates for entire functions gives the proposition [3], we will repeat the proof. First, we remark the following relation between seminorms of polynomials and the symmetric algebras:

$$\|x^k y^l\|_{p,s} \le \|X_1^k \circledcirc X_2^l\|_{\frac{1}{p},(sp)^{-1}} \le \|x^k y^l\|_{p,sp\delta} \quad \text{for every} \quad s >> 0, \tag{6.1}$$

where $\delta = e^{-(1+\frac{p}{2})}$. To prove these inequalities, we remark that the maximum point of

$$|x^k y^l| e^{-s|\xi|^p}, \quad |\xi|^2 = |x|^2 + |y|^2$$

satisfies $k+l = sp|\xi|^p$ and the maximum value is

$$k^{\frac{k}{2}} l^{\frac{l}{2}} e^{-\frac{1}{p}(k+l)} (sp)^{-\frac{1}{p}(k+l)} (k+l)^{(\frac{1}{p}-\frac{1}{2})(k+l)}.$$

It follows that

$$\|x^k y^l\|_{p,s} \le (sp)^{-\frac{1}{p}(k+l)} (k+l)^{\frac{1}{p}(k+l)}$$

and hence $\|x^k y^l\|_{p,s} \le \|X_1^k \circledcirc X_2^l\|_{\frac{1}{p},(sp)^{-1}}$.

To obtain the next inequality, we remark that $2 < e$ and recall the inequality (2.3). Then, we have

$$(sp)^{-\frac{1}{p}(k+l)} (k+l)^{\frac{1}{p}(k+l)} \le k^{\frac{k}{2}} l^{\frac{l}{2}} e^{-\frac{1}{p}(k+l)} (sp\delta)^{-\frac{1}{p}(k+l)} (k+l)^{(\frac{1}{p}-\frac{1}{2})(k+l)},$$

where $\delta = e^{-(1+\frac{p}{2})}$. It follows that $\|X_1^k \circledcirc X_2^l\|_{\frac{1}{p},(sp)^{-1}} \le \|x^k y^l\|_{p,sp\delta}$.

The inequality (6.1) naturally yields the embedding

$$\mathscr{S}_{\frac{1}{p}} \hookrightarrow \mathscr{E}_p(\mathbb{C}^2),$$

since $T = \sum a_{k,l} X_1^k \circledcirc X_2^l$ and the corresponding $t = \sum a_{k,l} x^k y^l$ satisfy the inequality

$$\|t\|_{p,s} \le \|T\|_{\frac{1}{p},(sp)^{-1}}.$$

Next, we show that $\mathscr{E}_p(\mathbb{C}^2) \subset \mathscr{S}_{\frac{1}{p}}$.

Since every $f \in \mathscr{E}_p(\mathbb{C}^2)$ is an entire function, f is expressed as a convergent power series $f = \sum c_{m,n} x^m y^n$. Let $M(r) = \max_{|\xi|=r} |f(x,y)|$. By Cauchy's integration formula, there is a constant $C_\delta > 0$ such that

$$|c_{m,n}| \le \frac{M(r)}{r^{m+n}} \le C_\delta \frac{e^{\delta r^p}}{r^{m+n}}$$

holds for every $\delta > 0$, since every $f \in \mathscr{E}_p(\mathbb{C}^2)$ satisfies $|f(x,y)| \le Ce^{\delta|\xi|^p}$.

Taking the minimal value of the right-hand side with respect to r for a fixed δ, we have

$$|c_{m,n}| \le C_\delta (\delta e p)^{\frac{1}{p}(m+n)} (\frac{1}{m+n})^{\frac{1}{p}(m+n)}. \tag{6.2}$$

Using (6.2), we easily see that $f \in \mathscr{S}_{\frac{1}{p}}$, if δ is sufficiently small. This gives also that every $f \in \mathscr{E}_p(\mathbb{C}^2)$ is also an element of $\mathscr{S}_{\frac{1}{p}}$. Hence we see that $\mathscr{E}_p(\mathbb{C}^2)$ and $\mathscr{S}_{\frac{1}{p}}$ are linearly isomorphic. ■

6.1. Proof of Theorem 1.2.

Theorem 3.3 gives that if $p \leq 2$, then $(\mathscr{E}_p(\mathbb{C}^2); *_\hbar)$ is a Fréchet algebra. Theorem 3.5 gives (ii) of Theorem 1.2.

Hence, to complete the proof of Theorem 1.2, it is sufficient to show that $(\mathscr{E}_p(\mathbb{C}^2); *_\hbar)$ is a deformation quantization of $(\mathscr{E}_p(\mathbb{C}^2), \cdot, \{\ \})$.

Since ∂_u, ∂_v are continuous linear operators on $\mathscr{E}_p(\mathbb{C}^2)$, $(\mathscr{E}_p(\mathbb{C}^2), \cdot, \{\ \})$ is a Fréchet-Poisson algebra. By Theorem 3.3, the concrete formula (1.2) is absolutely convergent in $\hbar$. Thus, $f *_\hbar g$ is analytic in $\hbar$ for any $f, g \in \mathscr{E}_p(\mathbb{C}^2)$.

7. The product formula for e^{isxy}

As mentioned in the introduction, y has a right-inverse y° and a left-inverse $y^\bullet$ at the same time in the space $\mathscr{E}_p(\mathbb{C}^2)$ for $p > 2$. Hence the associativity of the product $*_\hbar$ fails if $\hbar > 0$:

$$y^\bullet *_\hbar (y *_\hbar y^\circ) \neq (y^\bullet *_\hbar y) *_\hbar y^\circ.$$

In this section, we show that if $p > 2$, and $\hbar > 0$, then $\mathscr{E}_p(\mathbb{C}^2)$ is not closed under the product $*_\hbar$. In fact, we see that $y^\bullet *_\hbar y^\circ$ diverges. See also [6] for details.

Since xy is a polynomial, the product $(ixy) *_\hbar f(x,y)$ is well-defined for every $\mathscr{E}_p(\mathbb{C}^2)$. Thus, we consider a differential equation

$$\frac{d}{dt} f_t = (ixy) *_\hbar f_t(x,y), \quad f_0(x,y) = f(x,y).$$

By the Moyal product formula (1.2), this is written as the ordinary evolution equation

$$\frac{d}{dt} f_t = (ixy) f_t - \frac{\hbar}{2}(x\partial_x f_t - y\partial_y f_t) + i\frac{\hbar^2}{4}\partial_y \partial_x f_t.$$

The existence of the solution does not hold in general, but the uniqueness holds in the real analytic category in t. Hence if the initial function f_0 is a function $F(xy)$ of xy, we may suppose that the solution is also a function of xy.

Under this assumption, we consider the differential equation for $z = xy$:

$$\frac{d}{dt} f_t(z) = iz f_t(z) + i\frac{\hbar^2}{4}(f_t'(z) + z f_t''(z)). \tag{7.1}$$

The solution with initial condition 1 may be written as the $*_\hbar$-exponential function e_*^{itz}, which is given in fact by

$$e_*^{itz} = \frac{1}{\cosh \frac{\hbar t}{2}} e^{i(\frac{2}{\hbar}\tanh \frac{\hbar t}{2})z}, \quad t \in \mathbb{C} - \{(m + \tfrac{1}{2})\frac{i}{\hbar};\ m \in \mathbb{Z}\}. \tag{7.2}$$

The exponential law $e_*^{isz} *_\hbar e_*^{itz} = e_*^{i(s+t)z}$ holds by uniqueness whenever $s, t, s+t \in \mathbb{C} - \{(m+\frac{1}{2})\frac{i}{\hbar};\ m \in \mathbb{Z}\}$.

Rewriting the exponential law, we obtain the product formula

$$ae^{isz} *_\hbar be^{itz} = \frac{4ab}{4+\hbar^2 st} e^{\frac{4i(s+t)}{4+\hbar^2 st} z}. \tag{7.3}$$

It is easy to check the associativity when products are defined, and the inverse of ae^{isz} is $\frac{a^{-1}}{4}(4-\hbar^2 s^2)e^{-isz}$ for $4-\hbar^2 s^2 \neq 0$. If $s = \pm\frac{2}{\hbar}$, then $e^{\pm\frac{2i}{\hbar}z}$ are not invertible but have the idempotent property:

$$2e^{\pm\frac{2i}{\hbar}z} *_\hbar 2e^{\pm\frac{2i}{\hbar}z} = 2e^{\pm\frac{2i}{\hbar}z}. \tag{7.4}$$

However, $e^{\frac{2i}{\hbar}z} *_\hbar e^{-\frac{2i}{\hbar}z}$ is not defined. We refer to elements $2e^{\pm\frac{2i}{\hbar}z}$ as *vacuums*.

By the above argument, we see that

$$e_*^{itxy} = \frac{1}{\cosh\frac{\hbar t}{2}} e^{(\frac{2i}{\hbar}\tanh\frac{\hbar t}{2})xy}$$

is an element of $\mathscr{E}_p(\mathbb{C}^2)$, $p > 2$, for $t \in \mathbb{R}$. If $\hbar$ is a fixed positive real, then

$$\int_0^\infty e_*^{itxy} dt, \qquad \int_{-\infty}^0 e_*^{itxy} dt$$

are also elements of $\mathscr{E}^2(\mathbb{C}^2)$, $p > 2$ and $-i\int_0^\infty e_*^{itxy}dt$, and $i\int_{-\infty}^0 e_*^{itxy}dt$ are two different inverse elements of xy, that is

$$(xy) *_\hbar (-i\int_0^\infty e_*^{itxy} dt) = (xy) *_\hbar (i\int_{-\infty}^0 e_*^{itxy} dt) = 1.$$

Thus the associativity fails again: By denoting

$$(xy)_{0i+}^{-1} = -i\int_0^\infty e_*^{itxy} dt, \quad (xy)_{0i-}^{-1} = (i\int_{-\infty}^0 e_*^{itxy} dt)$$

we have

$$((xy)_{0i+}^{-1} *_\hbar (xy)) *_\hbar (xy)_{0i-}^{-1} \neq (xy)_{0i+}^{-1} *_\hbar ((xy) *_\hbar (xy)_{0i-}^{-1}).$$

This is because

$$(xy)_{0i+}^{-1} - (xy)_{0i-}^{-1} = -i\int_{-\infty}^\infty e_*^{itxy} dt \neq 0.$$

The right-hand side may be written as $-i\delta_*(xy)$ and the above identity corresponds to a relation of hyperfunctions [4].

If $\hbar > 0$, then by the well-known Hansen-Bessel formula (cf. [2], p. 1802), we see that

$$\int_{-\infty}^\infty e_*^{itxy} dt = \int_{-\infty}^\infty \frac{1}{\cosh\frac{\hbar t}{2}} e^{(\frac{2i}{\hbar}\tanh\frac{\hbar t}{2})xy} dt = \frac{\pi}{2} J_0(\frac{2}{\hbar}xy).$$

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