Star-products, spectral analysis, and hyperfunctions

Carlos Moreno
Laboratoire Gevrey de Mathématique physique
Université de Bourgogne, BP 47870, 21078 Dijon Cedex, France
and
Departamento de Física Teórica, Universidad Complutense, E-28040 Madrid, Spain
e-mail: cmoreno@eucmos.sim.ucm.es

José António Pereira da Silva*
F.C.T., Departamento de Matemática, Universidade de Coimbra, 3000 Coimbra, Portugal

Abstract. We study the $\star$-*exponential* function $\mathscr{U}(t;X)$ of any element X in the affine symplectic Lie algebra of the Moyal $\star$-product on the symplectic manifold $(\mathbb{R}\times\mathbb{R};\omega)$. When X is a compact element, a natural specific candidate for $\mathscr{U}(t;X)$ to be the exponential function is suggested by the study we make in the non-compact case. $\mathscr{U}(t;X)$ has singularities in the t variable. The analytic continuation $\mathscr{U}(z;X), z=t+iy$, defines two boundary values $\delta_+\mathscr{U}(t;X)=\lim_{y\downarrow 0}\mathscr{U}(z;X)$ and $\delta_-\mathscr{U}(t;X)=\lim_{y\uparrow 0}\mathscr{U}(z;X)$. $\delta_+\mathscr{U}(t;X)$ is a *distribution* while $\delta_-\mathscr{U}(t;X)$ is a Beurling-type, Gevrey-class $s=2$ *ultradistribution*. We compute the Fourier transforms in t of $\delta_\pm\mathscr{U}(t;X)$. Both Fourier spectra are discrete but different (e.g. opposite in sign for the harmonic oscillator). The Fourier spectrum of $\delta_+\mathscr{U}(t;X)$ coincides with the spectrum of the selfadjoint operator in the Hilbert space $L^2(\mathbb{R})$ whose Weyl symbol is X. Only the boundary value $\delta_+\mathscr{U}(t;X)$ should be considered as the $\star$ -exponential function for the element X, since $\delta_-\mathscr{U}(t;X)$ has no interpretation in the Hilbert space $L^2(\mathbb{R})$.

Keywords: deformation of algebras, hyperfunction, quantization

Mathematics Subject Classifications (2000): 53D55, 81S30, 58G07

Soon after the founding papers [4, 5] appeared, Moshé Flato suggested that further study was needed for the $\star$-*exponential* and its Fourier transform, for functions defined on a symplectic manifold endowed with a $\star$-product. This led us in 1980 to the computations presented here [18, 19]. In particular, the idea of looking at the $\star$-exponential as a boundary value of some holomorphic function, that is to say as a hyperfunction, was also suggested to us by Moshé.

We dedicate this paper to Moshé with deep gratitude and affection.

1. Introduction

We consider the canonical symplectic manifold $(\mathbb{R}\times\mathbb{R};\omega)$, the Moyal $\star$-product on this manifold, and the affine symplectic Lie algebra $\mathscr{A}$ of Hamiltonians corresponding to the infinitesimal automorphisms of the Moyal star-product. This Lie algebra is the vector space of polynomials of degree ≤ 2 in the coordinates p and q endowed with the Lie algebra structure induced by the Poisson bracket defined by ω.

* Programa Praxis XXI 2 /2.1/MAT/458/94

G. Dito and D. Sternheimer (eds.), Conférence Moshé Flato 1999, Vol. 2, 211 – 224.

The aim of this paper is to study the $\star$-*exponential* of any element of $\mathscr{A}$:

$$\mathscr{U}(t;X) = \sum_0^\infty \frac{1}{n!}(it)^n (X*)^n, \tag{1}$$

and in particular its Fourier transform in t.

In Section 2 we introduce the Moyal $\star$-product. Since $X \in \mathscr{A}$, we may expect the function (1) to be of the type $f(t;X(p;q))$. Indeed the operator $X*$ acting on functions of this type becomes a second-order differential operator in the X variable. This is proved in Lemma 2. We also specify the two cases, that will be separately considered: non-compact elements of $\mathscr{A}$, corresponding to the notation $d < 0$ in this paper, and compact elements corresponding to $d > 0$.

In Sections 3 and 4, when $d < 0$, we solve the eigenvalue problem [5] :

$$X \star \pi(\lambda;X) = \lambda \pi(\lambda;X), \quad \lambda \in \mathbb{R}. \tag{2}$$

We see that this equation is nothing but the Kummer equation [23] for which the hypergeometric confluent function ${}_1F_1[a;b;X]$ is a solution. The study of this equation leads us in Section 4 to the following expression for the function (1):

$$\mathscr{U}(t;X) = \left(\cosh \frac{t}{|s|}\right)^{-1} e^{i(|s|(X+c)\tanh(\frac{t}{|s|})-ct)}. \tag{3}$$

Using an integral representation for the Kummer function, similar to those found in [23] and given in Section 3, it is possible to compute the Fourier transform in t of (3): It is just $\frac{1}{2\pi}\pi(\lambda;X)$. From the point of view of the corresponding Weyl quantization, this exponential function is the Weyl symbol of a one-parameter group of unitary operators on the space $L^2(\mathbb{R})$ generated by the selfadjoint operator whose Weyl symbol is $X(p\,;q)$ [9, 14, 18, 19]. The above Fourier transform of (3) determines the spectral resolution of that one-parameter group of unitary operators.

In Section 5 we handle the case $d > 0$ in a similar way. We begin with the Kummer equation (14) and transform it by substituting formally $X \to iX$; $t \to it$; $c \to ic$; $i \to -i$. We then arrive at the formal result that the exponential function for this case would be [5, 6, 14]

$$\mathscr{U}(t;X) = \left(\cos \frac{t}{|s|}\right)^{-1} e^{i(|s|(X+c)\tan(\frac{t}{|s|})-ct)}. \tag{4}$$

The exponential function cannot be completely defined $\forall (t;X) \in \mathbb{R} \times \mathbb{C}$ by our procedure because of singularities appearing in the t variable of (4). It is therefore natural [18, 19][3] to take the function (4) as a boundary value [10, 15, 17] of the holomorphic function

$$\mathscr{U}(z;X) = \left(\cos \frac{z}{|s|}\right)^{-1} e^{i(|s|(X+c)\tan(\frac{z}{|s|})-cz)}, \quad \text{where } z = t + iy. \tag{5}$$

[3] It has been pointed out to the authors that the idea of taking the boundary values of functions in the convergent $\star$-product problem was used independently in a recent paper [20].

In Section 6 we prove that for $y \to 0$, $y > 0$, the boundary value of (5) is a distribution that we denote by $\delta_+\mathscr{U}(t;X)$; and for $y \to 0$, $y < 0$, the boundary value is an ultradistribution that we denote by $\delta_-\mathscr{U}(t;X)$, the type of which we determine in this Section 6. We also prove some properties of $\delta_\pm\mathscr{U}(t;X)$ in Proposition 7.

The computation of the Fourier transform in t of the hyperfunctions $\delta_\pm\mathscr{U}(t;X)$ is presented in Section 7. The Fourier spectrum is discrete in both cases. For $\delta_+\mathscr{U}(t;X)$, we get [9] what corresponds to the Weyl quantization of a selfadjoint operator on the space $L^2(\mathbb{R})$, whose Weyl symbol is $X(p;q)$, and $\delta_+\mathscr{U}(t;X)$ is the Weyl symbol of the one-parameter group of unitary operators generated by the same selfadjoint operator [5, 14, 18, 19]. Furthermore, the expressions (34) and (32) determine the Weyl symbol of the spectral resolution of this operator, the spectrum of which is (37): $\lambda_{N_0} = \frac{2}{|s|}(N_0 + \frac{1}{2}) - c$, $N_0 \in \mathbb{N}$. There is no Hilbert-space interpretation in $L^2(\mathbb{R})$ for $\delta_-\mathscr{U}(t;X)$. Nevertheless [18, 19] we can obtain Fourier expansions similar to those for $\delta_+\mathscr{U}(t;X)$, such as (35) and (33) which give the spectrum (38): $\lambda_{N_0} = -\frac{2}{|s|}(N_0 + \frac{1}{2}) - c$, $N_0 \in \mathbb{N}$. In particular, for the harmonic oscillator, these spectral values are *precisely the opposite* of the correct ones obtained with $\delta_+\mathscr{U}(t;X)$.

As we see in this paper, in order to obtain the set of spectral values of the classical observable $X(p;q)$ in $\star$-quantization, we must complexify time, $t+iy$, and then choose one of the above boundary values. At this point it is worthwhile mentioning that complexifying t appears also in a formulation of both classical and quantum-statistical mechanics in terms of star-products, [2]. Schematically, we may say that the structure involved is defined as follows: A first ingredient is a $\star$-product with deformation parameter $\hbar$ and the associated formal algebra of the power series. A second ingredient is another star-product with deformation parameter $\beta = \frac{1}{T}$ that defines a new product on the formal algebra. A third ingredient is a *conformal symplectic structure*, defined by β, on some manifold [3]. In physics, the new parameter β denotes the inverse of the temperature T of the system. Given this algebraic and geometric setting, a suitable internal automorphism of the algebra of observables can now be defined (in particular, by complexifying t to $t+i\hbar\beta$) by $\exp(-(t+i\hbar\beta)H)$. This automorphism then serves to define the *physical* classical and quantum KMS states, or KMS boundary condition, in terms of $\star$-products. Here H is a classical observable.

Since its introduction in [4, 5], the $\star$-*exponential* function (or distribution) has played an important role in a variety of problems, such as $\star$-representation theory of Lie groups. Also, [7] solves the problem of determining whether the mathematical equivalence of two star-products is the same notion as that of the equivalence of their quantizations. Looking at the boundary value aspect may prove useful in all such domains, but we shall not develop this further here. Nevertheless, remaining at the computational level, it is worth mentioning that, in particular, [6] contains interesting developments from which further computations, similar to those shown here, can be achieved in the $2n$-dimensional case, $\mathbb{R}^n \times \mathbb{R}^n$.

2. The Moyal $\star$-product and its infinitesimal automorphisms

Let $(V_{2n};\omega)$ be a symplectic manifold. Let us assume that it is non-compact and connected. Any $\star$-product on this manifold has a finite-dimensional Lie algebra $L_{\mathscr{A}}$ of infinitesimal automorphisms. Whenever the $\star$ -product is canonical, i.e., defined by a flat, symmetric, symplectic connection on the manifold, $L_{\mathscr{A}}$ coincides with the affine symplectic Lie algebra of the triple $(V_{2n};\omega;\Gamma)$, where Γ is the form of the connection [4]. A Hamiltonian $a \in C^\infty(V_{2n})$ corresponding to the vector field $X_a \in L_{\mathscr{A}}$ verifies (denoting by P the *Poisson bracket* defined by ω)

$$P(a;f\star g) = P(a;f)\star g + f\star P(a;g). \tag{6}$$

Any canonical $\star$-product also satisfies the equality

$$a\star b - b\star a = i\hbar P(a;b), \quad \forall a,b \in \mathscr{A}. \tag{7}$$

Throughout this paper, $\hbar$ denotes the deformation parameter in the $\star$-product.

Definition 1. **([4])** *A $\star$-product on the manifold $(V_{2n};\omega)$ is said $L_{\mathscr{A}}$-invariant, or invariant under the Poisson Lie algebra $\mathscr{A}$, if it verifies* (6) *and* (7).

For simplicity we consider only the case where $n = 1$. See [6] for $n > 1$. Let $(\mathbb{R}\times\mathbb{R};\omega)$ be the symplectic manifold, $\psi = (\psi_1;\psi_2)$ a generic point, and $\omega(\psi;\eta) = \psi_1\eta_2 - \psi_2\eta_1$, the symplectic form. Let f,g be two elements in $\mathbb{C}^\infty(\mathbb{R}\times\mathbb{R})$. The Moyal $\star$-product [4] on this manifold is

$$(f\star g)(\xi) = \sum_{k=0}^{\infty} \big(\frac{i\hbar}{2}\big)^k \frac{1}{k!} P^k(f;g)(\xi), \tag{8}$$

where $P^k(f;g) = \big(\partial_1\otimes\partial_2 - \partial_2\otimes\partial_1\big)^k(f;g)$ is the k-th power of P, considered as a bidifferential operator.

Expression (8) is the asymptotic expansion in powers of $\hbar$ of the integral

$$(f\star g)(\xi) = (\pi\hbar)^{-2}\int \exp^{\frac{2i}{\hbar}\big(\omega(\xi;\eta)+\omega(\eta;\zeta)+\omega(\zeta;\xi)\big)} f(\eta)g(\zeta)\,d\eta\,d\zeta. \tag{9}$$

This integral should always be understood in the sense of distributions.

When f and g are polynomials, there are only a finite number of terms in series (8) whose sum coincides with the one obtained from (9). The main property of the formal series (8) being that it is a non-trivial formal deformation of the usual product of functions on the manifold $\mathbb{R}\times\mathbb{R}$.

It is easy to see [4] that for the Moyal $\star$-product given by (8) or (9), the set of polynomials of the form

$$X(p;q) = a_1q^2 + a_2p^2 + 2a_3pq + a_4q + a_5p + a_6, \quad \forall a_i \in \mathbb{R}, \tag{10}$$

where $\xi = (p;q) \in \mathbb{R}\times\mathbb{R}$, is a Lie algebra under the Poisson bracket. It is the affine symplectic Lie algebra of Definition 1 for the manifold $(\mathbb{R}\times\mathbb{R};\omega;\Gamma)$, which corresponds to the Moyal $\star$-product on this manifold. This Lie algebra is isomorphic to a semidirect product of $\mathfrak{sl}(2)$ and $\mathfrak{h}_1$, where $\mathfrak{h}_1$ is a realization of the Heisenberg Lie algebra constituted by all of the polynomials of degree ≤ 1.

Let us write expression (10) in the following matrix form

$$X(q;p) = \begin{pmatrix} q & p & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_3 & \frac{1}{2}a_4 \\ a_3 & a_2 & \frac{1}{2}a_5 \\ \frac{1}{2}a_4 & \frac{1}{2}a_5 & a_6 \end{pmatrix} \begin{pmatrix} q \\ p \\ 1 \end{pmatrix},$$

and use the notations:

$$d = a_1 a_2 - {a_3}^2, \quad \Delta = \text{determinant of the matrix}, \quad c = -\frac{\Delta}{d}. \tag{11}$$

Lemma 1 *Let* $r, s \in \mathbb{N}$; $P_i^r(\xi) = \xi_i^r$, $i = 1,2$, *and* $b \in \mathbb{C}^\infty(\mathbb{R}\times\mathbb{R})$. *Then*

$$\big((P_1^r P_2^s) \star b\big)(\xi) = \sum_{l=0}^{s}\sum_{m=0}^{r} \binom{s}{l}\binom{r}{m} \xi_1^m \xi_2^l \Big(\frac{\hbar}{2i}\frac{\partial}{\partial \xi_1}\Big)^{s-l}\Big(-\frac{\hbar}{2i}\frac{\partial}{\partial \xi_2}\Big)^{r-m} b(\xi).$$

Proof. $P_1^r P_2^s$ is a polynomial. An easy computation from (9) or (8) gives the desired expression.

Lemma 2 *Let* $\tilde{b} \in \mathbb{C}^\infty(\mathbb{R}^2)$ *be of the form* $\tilde{b}(\xi_1;\xi_2) = b(X(\xi_1;\xi_2))$, *where* $b \in \mathbb{C}^\infty(\mathbb{R})$. *Then*

$$X(p;q) \star b(X(p;q)) = \Big(-\hbar^2 d(X+c)\frac{d^2}{dX^2} - \hbar^2 d \frac{d}{dX} + X\Big) b(X). \tag{12}$$

Proof. We obtain the desired expression from Lemma 1 and definitions (11).

Definition 2 *We call Equation* (12) *the main equation.*

Two cases will be considered:

$$\begin{aligned} &(a)\ d < 0, \\ &(b)\ d > 0, \quad X(p_0;q_0) + c = 0,\ X(p;q) + c > 0,\ \forall (p;q) \neq (p_0;q_0). \end{aligned} \tag{13}$$

The point $(p_0;q_0)$ is uniquely determined by the equation $X(p;q) + c = 0$ under the hypothesis $d > 0$, because the determinant in (11), where a_6 is replaced by $a_6 + c$, is zero. The case $d = 0$ has no interest.

3. The main equation and the confluent hypergeometric function ${}_1F_1$

We shall solve Equation (2). According to Lemma 2, it can be written as

$$-\hbar^2 d\big((X+c)\frac{d^2}{dX^2}+\frac{d}{dX}-\frac{1}{\hbar^2 d}X\big)\pi(\lambda;X)=\lambda\pi(\lambda;X) \quad d\neq 0. \tag{14}$$

Let us introduce the new variables:

$$\rho=c+\lambda,\ Y=X+c,\ s^2=\frac{1}{\hbar^2 d},\ U=2sY,\ V(U)=e^{\frac{1}{2}U}\pi(\rho-c;Y-c). \tag{15}$$

Equation (14) then becomes

$$U\frac{d^2V}{dU^2}+(1-U)\frac{dV}{dU}-(\frac{1}{2}-\frac{s}{2}\rho)V=0, \quad d\neq 0. \tag{16}$$

The Kummer equation is [23]

$$U\frac{d^2V}{dU^2}+(b-U)\frac{dV}{dU}-aV=0, \tag{17}$$

where a and b are complex numbers. The following integral representation of the Kummer function ${}_1F_1[a;b;U]$ holds for $\Re(b)>\Re(a)>0$ [23]:

$${}_1F_1[a;b;U]=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 e^{Uz}(1-z)^{b-a-1}z^{a-1}\,dz. \tag{18}$$

Setting $z=\frac{1}{1+e^{-t}}$ (or equivalently $2z-1=\tanh\frac{t}{2}$) in the above integral, we obtain

$$\int_0^1 e^{Uz}(1-z)^{b-a-1}z^{a-1}dz=2^{-b}e^{\frac{1}{2}U}\int_{-\infty}^{\infty}e^{\frac{1}{2}U\tanh\frac{t}{2}}\big(e^{-\frac{1}{2}t}\big)^{b-2a}\big(\cosh\frac{t}{2}\big)^{-b}dt. \tag{19}$$

The above transformation is similar to those found in [23].

We now consider (17) for the following particular values:

$$b>0,\ a=\frac{b}{2}-\frac{|s|}{2}\rho i,\ \text{where } \rho\in\mathbb{R} \text{ and } s\in\mathbb{C}. \tag{20}$$

From expressions (19), (18), (20), and the substitution $t\to\frac{|s|}{2}t$, we get

$$\int_{-\infty}^{\infty}e^{\frac{1}{2}U\tanh(\frac{t}{|s|})}\big(\cosh\frac{t}{|s|}\big)^{-b}e^{-it\rho}dt= \tag{21}$$
$$\frac{2^{b-1}\,|s|\,\Gamma(\frac{b}{2}-\frac{|s|}{2}\rho i)\Gamma(\frac{b}{2}+\frac{|s|}{2}\rho i)}{\Gamma(b)}e^{-\frac{1}{2}U}{}_1F_1\big[\frac{b}{2}-\frac{|s|}{2}\rho i;b;U\big].$$

All functions in this expression are holomorphic for the values (20) and $\forall U\in\mathbb{C}$. We can prove [19]

Lemma 3 *The function* $F(t) = e^{m \tanh t}\big(\cosh t\big)^{-b}$*, where* $m \in \mathbb{C}$ *and* $b > 0$*, is an element of the Schwartz space* $S(\mathbb{R})$.

This lemma shows that (21) is a Fourier transform on the space $S(\mathbb{R})$ for any fixed X.

4. Computation of $\mathscr{U}(t;X)$ for $d < 0$

We return to expression (21). In this case, $s = \mid s \mid i$ and $\mid s \mid = \frac{1}{\hbar\sqrt{|d|}}$. By (16) and (15), this expression becomes

$$\int_{-\infty}^{\infty} \big(\cosh \frac{t}{\mid s \mid}\big)^{-1} e^{i(|x|(X+c)\tanh(\frac{t}{|s|})-ct)} e^{it\lambda}\, dt = \pi(\lambda;X), \tag{22}$$

where

$$\pi(\lambda;X) = \frac{\pi \mid s \mid}{\cosh(\frac{|s|}{2}(\lambda+c))} e^{-|s|(X+c)i}\, {}_1F_1\big[\frac{1}{2} - \frac{\mid s \mid}{2}(\lambda+c)i; 1; 2 \mid s \mid (X+c)i\big]. \tag{23}$$

We have therefore proved

Proposition 1
1) When $d < 0$*, the function* $\pi(\lambda;X)$ *in* (23) *is a solution of the equation* $X * \pi(\lambda;X) = \lambda\pi(\lambda;X)$, $\lambda \in \mathbb{R}$.
2) The function $\mathscr{U}(t;X) = \big(\cosh \frac{t}{|s|}\big)^{-1} e^{i(|s|(X+c)\tanh(\frac{t}{|s|})-ct)}$ *is bounded, real analytic and* $\mathscr{U} \in S(\mathbb{R})\hat{\otimes}\mathscr{C}^\infty(\mathbb{R})$[22].
3) Equality (22) *is a partial Fourier transform, that is,*

$$((F^{-1}\hat{\otimes}I)\mathscr{U})(\lambda;X) = \frac{1}{2\pi}\pi(\lambda;X), \quad \big((F\hat{\otimes}I)\big(\frac{1}{2}\pi\big)\big)(t;X) = \mathscr{U}(t;X).$$

Proposition 2 *Let X be fixed. The following equalities are verified:*

$$\begin{aligned} \frac{1}{i}\frac{d}{dt}\mathscr{U}(t;X) &= X * \mathscr{U}(t;X), \\ \mathscr{U}(t;X) * \mathscr{U}(t';X) &= \mathscr{U}(t+t';X), \quad \forall t,t' \in \mathbb{R}. \end{aligned} \tag{24}$$

Let $t_0 \neq \frac{\pi|s|}{2}$*. The series* $\mathscr{U}(t;X) = \sum_0^\infty \big(\frac{1}{n!}(i(t-t_0))^n (X*)^n * \mathscr{U}(t_0;X)\big)$ *converges in a neighborhood of* t_0.

Proof. We get the first equality from Lemma 2 and after some easy computations. The second equality follows from (9) when we exploit the particular form of $X(p;q)$. The third equality thereby becomes obvious.

5. From the case $d < 0$ to the case $d > 0$

When $d > 0$, the function $X(p;q) + c$ is positive and has only one zero, (see (13)). We will now solve the equation (or (2) and (14))

$$-\hbar^2 d\big((X+c)\frac{d^2}{dX^2} + \frac{d}{dX} - \frac{1}{\hbar^2 d}X\big)\mathscr{U}(t;X) = \frac{1}{i}\frac{d}{dt}\mathscr{U}(t;X).$$

We perform in this equation the formal substitutions

$$(t \to it;\ \ X \to iX;\ \ c \to ic;\ \ i \to -i) \equiv (\mathbf{S}),$$

so it becomes

$$-\hbar^2(-d)\big((X+c)\frac{d^2}{dX^2} + \frac{d}{dX} - \frac{1}{\hbar^2(-d)}X\big)\mathscr{U}(\mathbf{S}) = \frac{1}{i}\frac{d}{dt}\mathscr{U}(\mathbf{S}).$$

This equation is identical to the one we solved for the case of $d < 0$, (14), (24), and Proposition 1. The solution that will be the most interesting for us is

$$\mathscr{U}(\mathbf{S}) = \big(\cosh\frac{t}{\mid s\mid}\big)^{-1} e^{i(|s|(X+c)\tanh(\frac{t}{|s|})-ct)},$$

that is to say,

$$\mathscr{U}(t;X) = \big(\cos\frac{t}{\mid s\mid}\big)^{-1} e^{i(|s|(X+c)\tan(\frac{t}{|s|})-ct)}. \tag{25}$$

The singularities of this function are located at the points $t = (2k+1)\frac{\pi}{2}\mid s\mid$, $k \in \mathbb{R}$. We need to find out what kind of generalized function this would be on $\mathbb{R}$. For that purpose, we may consider its analytic continuation in t (or in $(t;X)$ but after Proposition 3 below, it will not be hard to see that only the former actually matters):

$$\mathscr{U}(z;X) = \big(\cos\frac{z}{\mid s\mid}\big)^{-1} e^{i(|s|(X+c)\tan(\frac{z}{|s|})-cz)}, \quad \text{where } z = t+iy. \tag{26}$$

This function has no other singularities than the points $t = (2k+1)\frac{\pi}{2}\mid s\mid$, $k \in \mathbb{R}$, just mentioned.

6. Characterization of $\delta_+\mathscr{U}(t;X)$ and $\delta_-\mathscr{U}(t;X)$

Let $\Gamma^+ = \{z \mid \Im(z) > 0\}$ and $\Gamma^- = \{z \mid \Im(z) < 0\}$, and fix X for the time being. The function (25) can be considered as *either* of the following two boundary values [10, 11, 15, 17] of the analytic function (26):

$$\delta_+\mathscr{U}(t;X) = \lim_{y\downarrow 0}\mathscr{U}(z;X); \qquad \delta_-\mathscr{U}(t;X) = \lim_{y\uparrow 0}\mathscr{U}(z;X). \tag{27}$$

The limits are taken with respect to the topologies of the spaces involved (see below). The characterization of distributions as boundary values of holomorphic functions is given in [10, 15] and that of ultradistributions in [10, 11]. Let $(a;b)$, $(0;c) \subseteq \mathbb{R}$. Let F be an analytic function on the open set $(a;b) \times i(0;c)$.

Theorem 1. **([10, 15, 16])** *The hyperfunction $\delta_+ F$ on the interval $(a;b)$ is a distribution on this interval if and only if, for each compact $K \subset (a;b)$, there is an integer $N > 0$ and a constant $C > 0$ such that, $\forall t \in (a;b)$ and small $|\, y \,|$, we have $\sup_{t\in K} |F(t+iy)| \leq C \,|\, y \,|^{-N}$. If this is the case, $\delta_+ F(t) = \lim_{y\downarrow 0} F(t+iy)$, the limit being taken in the space of distributions $\mathscr{D}'((a,b))$.*

The theorem holds for holomorphic functions of an arbitrary number of complex variables [15, 16]. Let $(a;b)$, $(d;0) \subseteq \mathbb{R}$. Let F be an analytic function on the open $(a;b) \times i(d;0)$. The following theorems are well known.

Theorem 2. **([10, 11])** *The hyperfunction δ_F on the interval $(a;b)$ is a Beurling-type, Gevrey-class ultradistribution of order $s > 1$ on this interval if and only if, for every compact $K \subset (a;b)$, there are constants $L > 0$ and $C > 0$ such that, $\forall t \in (a;b)$ and small $|\, y \,|$, $[\sup_{t\in K} |F(t+iy)| \leq C \exp\big(\big(\frac{L}{y}\big)^{\frac{1}{s-1}}\big)$. If this is the case, $\delta\_F(t) = \lim_{y\uparrow 0} F(t+iy)$, the limit being taken in the topology of the space of Beurling-type, Gevrey-class ultradistributions $\mathscr{D}^{(s)\prime}((a;b))$ of order $s > 1$.*

This theorem holds for holomorphic functions of an arbitrary number of complex variables [10, 11, 17]. We now examine the limits in (27). By so doing we get

Proposition 3

$$\Big|\cos\big(\frac{t+iy}{|\, s\,|}\big)\Big|^2 = \frac{1}{2}\big(\cosh(\frac{2y}{|\, s\,|}) + \cos(\frac{2t}{|\, s\,|})\big), \;\; \forall t,y \in \mathbb{R}.$$

$$\frac{1}{\big|\cos\big(\frac{t+iy}{|s|}\big)\big|^2} \leq \frac{2}{\sum_{p\geq 1}\big(\frac{2y}{|s|}\big)^{2p}\frac{1}{(2p)!}} < \frac{|\, s\,|^2}{|\, y\,|^2}, \;\; \forall t \in \mathbb{R} \;\text{ and small } |\, y \,| \,.$$

$$\tan\big(\frac{t+iy}{|\, s\,|}\big) = \frac{1}{\big|\cos\big(\frac{t+iy}{|s|}\big)\big|^2}\frac{1}{2}\big(\sin\big(\frac{2t}{|\, s\,|}\big) + i\sinh\big(\frac{2y}{|\, s\,|}\big)\big), \;\; \forall t,y \in \mathbb{R}.$$

From Proposition 3 and Theorem 1, we can state

Proposition 4 *$\forall t \in \mathbb{R}$, $y > 0$, small $|\, y \,|$, and fixed X, there is a constant k such that $|\mathscr{U}(t+iy;X)| < k\big(\frac{|y|}{|s|}\big)^{-1}$. Therefore $\delta_+\mathscr{U}(t;X) = \lim_{y\downarrow 0}\mathscr{U}(t+iy;X)$, the limit is taken in the topology of $\mathscr{D}'(\mathbb{R})$.*

From Proposition 3 and Theorem 2, we also have

Proposition 5 *$\forall t \in \mathbb{R}$, $y < 0$, small $|\, y \,|$, and fixed X there is a constant $\tilde{k}$ such that $|\mathscr{U}(t+iy;X)| < e^{\frac{\tilde{k}}{|y|}}$. Therefore $\delta\_\mathscr{U}(t;X) = \lim_{y\uparrow 0}\mathscr{U}(t+iy;X)$, the limit being in the space of Beurling-type, Gevrey-class ultradistributions of order $s = 2$, $\mathscr{D}^{(2)\prime}(\mathbb{R})$, [11]. $\delta\_\mathscr{U}$ is not a distribution.*

As a consequence one can prove [11, 15, 18, 19]

Proposition 6
i) $\delta_+\mathscr{U}(t;X) = \lim_{y\downarrow 0}\mathscr{U}(t+iy;X) \in \mathscr{D}'(\mathbb{R})\hat{\otimes}\mathscr{D}'(\mathbb{R})$ *where the limit is taken in the topology of* $\mathscr{D}'(\mathbb{R})\hat{\otimes}\mathscr{D}'(\mathbb{R})$.
ii) $\delta_-\mathscr{U}(t;X) = \lim_{y\uparrow 0}\mathscr{U}(t+iy;X) \in \mathscr{D}^{(2)\prime}(\mathbb{R})\hat{\otimes}\mathscr{D}'(\mathbb{R})$, *where the limit is taken in the topology of* $\mathscr{D}^{(2)\prime}(\mathbb{R})\hat{\otimes}\mathscr{D}'(\mathbb{R})$.

Proposition 7 *The following equalities hold:*
i) $\frac{1}{i}\frac{d}{dz}\mathscr{U}(z;X) = X * \mathscr{U}(z;X)$, $\forall z \neq \frac{\pi|s|}{2}, k \in \mathbb{R}$.
ii) $\frac{1}{i}\frac{d}{dt}\delta_+\mathscr{U}(t;X) = X * \delta_+\mathscr{U}(t;X)$.
iii) $\frac{1}{i}\frac{d}{dt}\delta_-\mathscr{U}(t;X) = X * \delta_-\mathscr{U}(t;X)$.
iv) $\delta_\pm\mathscr{U}(t;X) * \delta_\pm\mathscr{U}(t';X) = \delta_\pm\mathscr{U}(t+t';X)$.

Proof. Statement i) can be proved from the definition (26) of $\mathscr{U}(z;X)$ and the fact that the *differential operator* $(X*)$ is acting on functions that depend only on X. For statement ii), note that the operator $\frac{1}{i}\frac{d}{dt} - I\hat{\otimes}(X*)$ acts continuously on the space $\mathscr{D}'(\mathbb{R})\hat{\otimes}\mathscr{D}'(\mathbb{R})$. Similarly, statement iii) follows from the fact that the operator $\frac{1}{i}\frac{d}{dt} - I\hat{\otimes}(X*)$ acts continuusly on the space $\mathscr{D}^{(2)\prime}(\mathbb{R})\hat{\otimes}\mathscr{D}'(\mathbb{R})$. For statement iv), we have

$$\begin{aligned}\delta_\pm\mathscr{U}(t;X) * \delta_\pm\mathscr{U}(t';X) &= \lim_{y;y'}\big(\mathscr{U}(t+iy;X) * \mathscr{U}(t+iy';X)\big)\\ &= \lim_{y+y'}\mathscr{U}((t+t')+i(y+y');X) = \delta_\pm\mathscr{U}(t+t';X).\end{aligned}$$

The first equality is clearly the meaning of (9) for $\delta_\pm\mathscr{U}(t;X) * \delta_\pm\mathscr{U}(t';X)$.

7. Computation of the Fourier transforms $(F\hat{\otimes}I)\delta_\pm\mathscr{U}$

The following spaces will be used in this section [8]: $Z(\mathbb{R})$ denotes the topological vector space whose elements are the Fourier transforms of all the elements in $\mathscr{D}(\mathbb{R})$, endowed with the topology making the Fourier transform an isomorphism of the topological vector spaces: $\mathscr{D}(\mathbb{R}) \simeq Z(\mathbb{R})$, $Z(\mathbb{R}) \subset S(\mathbb{R})$. And the image of this continuous inclusion is dense. $Z'(\mathbb{R})$ will denote the strong dual of space $Z(\mathbb{R})$. The Fourier transform is a topological vector space isomorphism from space $\mathscr{D}'(\mathbb{R})$ to space $Z'(\mathbb{R})$: $\mathscr{D}'(\mathbb{R}) \simeq Z'(\mathbb{R})$. $S'(\mathbb{R}) \subset Z'(\mathbb{R})$, and the image of this continuous inclusion is dense.

Let $\mathscr{D}^{(2)\prime}(\mathbb{R})$ be the space of Beurling-type, Gevrey-class ultradistributions of order $s=2$. $Z^{(2)}(\mathbb{R})$ denotes the topological vector space whose elements are the Fourier transforms of all ultradifferentiable functions of the said class and type [10]. $\mathscr{D}^{(2)}(\mathbb{R}) \subset \mathscr{D}(\mathbb{R})$ is a continuous inclusion with a dense image. $Z^{(2)}(\mathbb{R})$ is endowed with the topology making the Fourier transform an isomorphism of the topological vector spaces: $\mathscr{D}^{(2)}(\mathbb{R}) \simeq Z^{(2)}(\mathbb{R})$. $Z^{(2)\prime}(\mathbb{R})$ denotes the strong dual topological vector space of space $Z^{(2)}(\mathbb{R})$. The Fourier transform is a topological vector space isomorphism from the space $\mathscr{D}^{(2)\prime}(\mathbb{R})$ to the space $Z^{(2)\prime}(\mathbb{R})$: $\mathscr{D}^{(2)\prime}(\mathbb{R}) \simeq Z^{(2)\prime}(\mathbb{R})$. $Z'(\mathbb{R}) \subset Z^{(2)\prime}(\mathbb{R})$, $\mathscr{D}'(\mathbb{R}) \subset \mathscr{D}^{(2)\prime}(\mathbb{R})$, the images of these continuous inclusions being dense.

From the definition of the Fourier transform [12, 13], we have

$$(F\hat{\otimes}I)\delta_{\pm}\mathscr{U} = \lim_{y\downarrow 0; y\uparrow 0} I_y, \quad \text{where } I_y(\lambda;X) = \int_{-\infty}^{\infty} \mathscr{U}(t+iy;X)e^{-i\lambda t}\, dt, \tag{28}$$

and the limit is in $Z'(\mathbb{R})\hat{\otimes}\mathscr{D}'(\mathbb{R})$ for $y>0$, or in $Z^{(2)\prime}(\mathbb{R})\hat{\otimes}\mathscr{D}'(\mathbb{R})$ for $y<0$. We can also write

$$I_y(\lambda;X) = \mid s \mid \sum_{K=-\infty}^{K=\infty} e^{-i2K\pi|s|(\lambda+c)} \times$$
$$\int_0^{2\pi} \left(\cos\left(t' + i\frac{y}{\mid s \mid}\right)\right)^{-1} e^{i(|s|(X+c)\tan(t'+i\frac{y}{|s|}))} e^{-it'|s|(\lambda+c)} e^{cy} dt', \quad K \in \mathbb{Z}.$$

Now $\sum_{K=-\infty}^{K=\infty} e^{iK(-2\pi|s|(\lambda+c))} = \frac{1}{|s|}\sum_{K=-\infty}^{K=\infty} \delta_{-(\frac{K}{|s|}+c)}(\lambda)$, which converges in $S'(\mathbb{R})$ and therefore also in $Z'(\mathbb{R})$ and $Z^{(2)\prime}(\mathbb{R})$. We therefore have

$$I_y(\lambda;X) = e^{cy} \sum_{K=-\infty}^{K=\infty} \delta_{-(\frac{K}{|s|}+c)}(\lambda) J_y(K;X), \tag{29}$$

where

$$J_y(K;X) = \int_0^{2\pi} \left(\cos\left(t' + i\frac{y}{\mid s \mid}\right)\right)^{-1} e^{i(|s|(X+c)\tan(t'+i\frac{y}{|s|}))} e^{it'K} dt'.$$

By setting $e^{it'} = z$, the computation of this integral gives

Lemma 4

$$J_y(2n;X) = 0 \quad n = 0, \pm 1, \pm 2, \dots$$

To compute the integral for $K = 2n+1$, $n \in \mathbb{Z}$, we write $J_y(2n+1;X) = J_y^+(2n+1;X) + J_y^-(2n+1;X)$, where $+$ and $-$ stand respectively for integration on Γ^+ and Γ^-. We set $z = e^{i\pi}v$ in the above integrals and thereby get $J_y(2n+1;X) = 2J_y^+(2n+1;X)$. We now need to compute

$$J_y^+(2n+1;X) = -2i \int_{z\in\Gamma^+, |z|=1} \frac{e^{-\frac{y}{|s|}}}{1+z^2 e^{-\frac{2y}{|s|}}} \exp\left(\mid s \mid (X+c)\frac{z^2 e^{-\frac{2y}{|s|}} - 1}{z^2 e^{-\frac{2y}{|s|}} + 1}\right) z^{2n+1} dz.$$

In this integral, we write

$$a = e^{-\frac{2y}{|s|}}, \quad u = \frac{z^2 a - 1}{z^2 a + 1}, \quad \text{and} \quad z = e^{it}. \tag{30}$$

It then becomes

$$J_y^+(2n+1;X) = -ie^{\frac{y}{|s|}(2n+1)} \int_0^{\pi} (1+u)^n (1-u)^{-(n+1)} e^{(|s|(X+c)u)} du. \tag{31}$$

We can write $u = u_1 + iu_2$. From (30), we get $u_1^2 + u_2^2 + 2\coth\left(\frac{2y}{|s|}\right)u_1 + 1 = 0$. u is thus a point in the above circumference Υ_y. We see also that the mappings $t \in [0;\pi] \xrightarrow{\tilde{y}} u \in \Upsilon_y$ are bijections. The point $u = -1$ is interior to any circle whose circumference is Υ_y for every $y > 0$, but the point $u = +1$ is not. Conversely, the point $u = +1$ is interior to any circle whose circumference is Υ_y, $\forall\ y < 0$, but the point $u = -1$ is not. We may compute (31) for $y > 0$, and then for $y < 0$, by means of the Cauchy formula applied to the contours Υ_y, $y > 0$, and Υ_y, $y < 0$.

We get, for $y > 0$,

$$J_{y>0}^+(-2(N_0 + \frac{1}{2});X) = 2\pi e^{-\frac{2y}{|s|}(N_0+\frac{1}{2})}(-1)^{N_0} e^{-|s|(X+c)} L_{N_0}^0(2 \mid s \mid (X+c)),$$

where $L_{N_0}^0(Y) = \frac{1}{N_0!} e^Y \left(\frac{d}{dY}\right)^{N_0}[e^{-Y}Y^{N_0}]$ are the Laguerre polynomials [21]. We can now write expression (29) as

$$I_{y>0}(\lambda;X) = 2\pi e^{cy} \sum_{N_0 \geq 0} \delta_{(\frac{2}{|s|}(N_0+\frac{1}{2})-c)}(\lambda)\pi_{N_0}(\mid s \mid (X+c))e^{-\frac{2y}{|s|}(N_0+\frac{1}{2})}, \tag{32}$$

where $\pi_{N_0}(\mid s \mid (X+c)) = 2(-1)^{N_0} e^{-|s|(X+c)} L_{N_0}^0(2 \mid s \mid (X+c)$, and the series converges in $Z'(\mathbb{R})\hat{\otimes}\mathscr{D}'(\mathbb{R})$.

Similarly, for $y < 0$, we obtain

$$I_{y<0}(\lambda;X) = 2\pi e^{cy} \sum_{N_0 \geq 0} \delta_{(-\frac{2}{|s|}(N_0+\frac{1}{2})-c)}(\lambda)\pi'_{N_0}(- \mid s \mid (X+c))e^{-\frac{2|y|}{|s|}(N_0+\frac{1}{2})}, \tag{33}$$

where $\pi'_{N_0}(- \mid s \mid (X+c)) = 2(-1)^{N_0} e^{|s|(X+c)} L_{N_0}^0(-2 \mid s \mid (X+c))$, and the series converges in $Z^{(2)\prime}(\mathbb{R})\hat{\otimes}\mathscr{D}'(\mathbb{R})$.

From (28), (32), (33), and the continuity of the Fourier transform, we deduce the following

Proposition 8

$$\delta_+ \mathscr{U}(t;X) = \lim_{y\downarrow 0}\left(\sum_{N_0} e^{i\left(\frac{2}{|s|}(N_0+\frac{1}{2})-c\right)t} \pi_{N_0}(\mid s \mid (X+c))e^{-\frac{2y}{|s|}(N_0+\frac{1}{2})}\right). \tag{34}$$

This limit being taken in the space $\mathscr{D}'(\mathbb{R})\hat{\otimes}\mathscr{D}'(\mathbb{R})$.

$$\delta_- \mathscr{U}(t;X) = \lim_{y\uparrow 0}\left(\sum_{N_0} e^{i\left(-\frac{2}{|s|}(N_0+\frac{1}{2})-c\right)t} \pi'_{N_0}(- \mid s \mid (X+c))e^{-\frac{2|y|}{|s|}(N_0+\frac{1}{2})}\right). \tag{35}$$

This limit being taken in the space $\mathscr{D}^{(2)\prime}(\mathbb{R})\hat{\otimes}\mathscr{D}'(\mathbb{R})$.

Let us write

$$u = -e^{i\frac{2}{|s|}(t+iy)}, \text{ for } y > 0, \quad \text{and} \quad u = -e^{-i\frac{2}{|s|}(t+iy)}, \text{ for } y < 0. \tag{36}$$

Proposition 9 *The series*

$$i)\ \mathscr{U}(t+iy;X) = \sum_{N_0} e^{i\left(\frac{2}{|s|}(N_0+\frac{1}{2})-c\right)(t+iy)} \pi_{N_0}(|\, s\,|\,(X+c)), \quad y>0,$$

$$ii)\ \mathscr{U}(t+iy;X) = \sum_{N_0} e^{i\left(-\frac{2}{|s|}(N_0+\frac{1}{2})-c\right)(t+iy)} \pi'_{N_0}(-\,|\, s\,|\,(X+c)), \quad y<0,$$

are absolutely convergent $\forall (u;(X+c)) \in \{|\, u\,| < 1\} \times \mathbb{C}$.

Proof. Let us consider the generating function of the Laguerre polynomials [21]:

$$(1-u)^{-1} e^{(-Y\frac{u}{1-u})} = \sum_{N_0} u^{N_0} L^0_{N_0}(Y).$$

This series is *absolutely convergent* in $\{|\, u\,| < 1\} \times \mathbb{C}$. By inserting the values (36) in this series, we get the series *i*) and *ii*), respectively.

As a consequence of that proposition we have

Proposition 10 *Let* $X \in \mathbb{R}$ *be fixed. At any point* $t \neq |\, s\,|\,(2n+1)\frac{\pi}{2}$, $\delta_{\pm}\mathscr{U}(t;X)$ *are holomorphic functions, and the series* $\sum_{N_0} e^{i(\frac{2}{|s|}(N_0+\frac{1}{2})-c)t} \pi_{N_0}(|\, s\,|\,(X+c))$ *and* $\sum_{N_0} e^{i\left(-\frac{2}{|s|}(N_0+\frac{1}{2})-c\right)t} \pi'_{N_0}(-\,|\, s\,|\,(X+c))$ *are Abel summable to* $\delta_{+}\mathscr{U}(t;X)$ *and* $\delta_{-}\mathscr{U}(t;X)$*, respectively. In particular, the equalities* $1 = \sum_{N_0} \pi_{N_0}(|\, s\,|\,(X+c))$ *and* $1 = \sum_{N_0} \pi'_{N_0}(-\,|\, s\,|\,(X+c))$ *are verified in the Abel sense.*

Proposition 11 *The following equalities hold:*

$$X * \pi_{N_0}(|\, s\,|\,(X+c)) = \big(\frac{2}{|\, s\,|}(N_0+\frac{1}{2}) - c\big)\pi_{N_0}(|\, s\,|\,(X+c)), \tag{37}$$

$$X * \pi'_{N_0}(-\,|\, s\,|\,(X+c)) = \big(-\frac{2}{|\, s\,|}(N_0+\frac{1}{2}) - c\big)\pi'_{N_0}(-\,|\, s\,|\,(X+c)), \tag{38}$$

$$\pi_{N_0}(|\, s\,|\,(X+c)) * \pi_{M_0}(|\, s\,|\,(X+c)) = \delta_{N_0 M_0}\pi_{N_0}(|\, s\,|\,(X+c)),$$

$$\pi'_{N_0}(-\,|\, s\,|\,(X+c)) * \pi'_{M_0}(-\,|\, s\,|\,(X+c)) = \delta_{N_0 M_0}\pi'_{N_0}(-\,|\, s\,|\,(X+c)).$$

Proof. We get the first two equalities from the second-order differential operator $X*$ and the properties of the Laguerre polynomials. The third equality is in [5] and we obtain the fourth in a similar way.

References

1. Avez, A. and Lichnerowicz, A.: Dérivations et premier groupe de cohomologie pour des algèbres de Lie attachées à une variété symplectique, *C. R. Acad. Sci. Paris Sér. A-B* **275** (1972), 113–118.
2. Basart, H., Flato, M., Lichnerowicz, A., and Sternheimer, S.: Deformation theory applied to quantization and statistical mechanics, *Lett. Math. Phys.* **8** (1984), 483–494.
3. Basart, H. and Lichnerowicz, A.: Conformal symplectic geometry, deformations, rigidity and geometrical (KMS) conditions, *Lett. Math. Phys.* **10** (1985), 167–177.
4. Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A., and Sternheimer, D.: Deformation theory and quantization I. Deformations of symplectic structures, *Ann. Physics* **111** (1978), 61–110.
5. Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A., and Sternheimer, D.: Deformation theory and quantization II. Physical applications, *Ann. Physics* **111** (1978), 111–151.
6. Bayen, F. and Maillard, J.-M.: Star exponentials of the elements of the inhomogeneous symplectic Lie algebra, *Lett. Math. Phys.* **6** (1982), 481–497.
7. Cahen, M., Flato, M., Gutt, S., and Sternheimer, D.: Do different deformations lead to the same spectrum? *J. Geom. Phys.* **2** (1985), 35–48.
8. Gelfand, I. M. and Shilov, G. E.: *Generalized functions*, Vol. 1 and 2, Academic Press, New York-London, 1964 and 1968.
9. Kammerer, J.-B.: Analysis of the Moyal product in a flat space, *J. Math. Phys.* **27** (1986), 529–535.
10. Komatsu, H.: Hyperfunctions and linear partial differential equations, in: textitHyperfunctions and pseudo-differential equations, (Katata, 1971), Lecture Notes in Math., Vol. 287, Springer, Berlin, 1973, pp. 180–191.
11. Komatsu, H.: Ultradistributions I, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **20**, (1973), 25–105; part II, **24** (1977), 607–628; part III, **29** (1982), 653–717.
12. Kaneko, A.: *Introduction to hyperfunctions,* Kluwer Academic Publishers, Dordrecht, 1988.
13. Kawai, K.: On the theory of Fourier hyperfunctions and its applications to partial differential equations with constant coefficients, *J. Fac. Sci. Univ. Tokyo, Sect. IA Math.* **17** (1970), 467–517.
14. Maillard, J.-M.: On the twisted convolution product and the Weyl transformation of tempered distributions, *J. Geom. Phys.* **3** (1986), 230–261.
15. Martineau, A.: Distributions et valeurs au bord des fonctions holomorphes, *Theory of distributions (Lisbon, 1964)*, Inst. Gulbenkian Ciencia, Lisbon, 1964, pp. 193–326.
16. Martineau, A.: Sur les fonctionnelles analytiques et la transformation de Fourier-Borel, *J. Analyse Math.* **11** (1963), 1–164.
17. Meise, R.: Representation of distributions and ultradistributions by holomorphic functions, in: *Functional analysis: surveys and recent results (Paderborn, 1976)*, North-Holland Mathematical Studies 27, North-Holland, Amsterdam, 1977, pp. 189–208.
18. Moreno, C.: Produits $\star$ et analyse spectrale, in: *Journées Relativistes*, Grenoble, 1981.
19. Moreno, C. and Pereira da Silva, J. A.: The theory of $\star$-products, preprint Universidade de Coimbra, Portugal 1980.
20. Omori, H., Maeda, Y., Miyazaki, N., and Yoshioka, A.: Singular systems of exponential functions, to be published in the proceedings of the workshop on noncommutative differential geometry and its applications to physics, Shonan Kokusaimura, May 1999. *Math. Phys. Studies*, Kluwer Acad. Publ., Dordrecht, 2000.
21. Sansone, G.: *Orthogonal functions,* Interscience Publishers, Inc., New York, 1959.
22. Schwartz, L.: Théorie des distributions à valeurs vectorielles I, *Ann. Inst. Fourier. Grenoble* **7** (1957), 1–141; part II, **8** (1958), 1–209.
23. Slater, L. J.: *Confluent hypergeometric functions*, Cambridge University Press, New York, 1960.