Anticipative Chen-Souriau cohomology and Hochschild cohomology

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Abstract. We define random forms over the based loop space of a manifold respectively to the set of semi-martingales and we show that the stochastic cohomology is equal to the Hochschild cohomology.

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1. Introduction

Let M be a compact manifold. Let $L_x(M)$ be the based loop space of M. There are two stochastic cohomology theories of $L_x(M)$.

1) In the first one, we suppose that the manifold is Riemannian. We endow the loop space $L_x(M)$ with the Brownian bridge measure. We consider on a stochastic loop (which is only continuous), a Hilbert tangent space, which was introduced in a preliminary form by Bismut [5] and under a final form by Jones and Léandre [18]. This allows us to define Sobolev spaces of functionals [20, 21] and Sobolev spaces of forms on the Brownian bridge. The stochastic exterior derivative acts continuously on the intersection of all Sobolev spaces of forms. This allows to define some Sobolev stochastic cohomology groups of the based loop space $L_x(M)$ [22, 23, 24]. Since there is no Sobolev imbedding theorem in infinite dimension, the space of smooth forms (those which belong to all of the Sobolev spaces) is constituted of forms which are only almost surely defined. It is the purpose of the Malliavin calculus to do analysis over objects which are almost surely defined [34]. If we consider smooth loops, the Hochschild complex constitutes an algebraic model of the loop space: If the manifold is simply connected, the de Rham cohomology groups of $L_x(M)$ are equal to the Hochschild cohomology groups [1, 11, 16]. It is the purpose of the works of Léandre [23, 24] to show that this classical result of algebraic topology remains true for the Sobolev cohomology of the loop space. References [23] and [24] furnish a stochastic generalization of the classical result of Adams [1] and Chen [11], for the Sobolev cohomology of the loop space. Therefore, the Sobolev cohomology is equal to the deterministic de Rham cohomology of the smooth loop space.

2) The second stochastic cohomology theory deals with objects which are only almost surely defined too, but without to give a (Sobolev) number to a random

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form. The main idea goes back to the considerations of Chen [11], Souriau [36], and Iglesias [17] in the deterministic case. These authors consider a family of plots valued in the loop space (or another complicated space), which satisfy a system of axioms: This family constitutes a diffeology. A form over the loop space is seen through a plot: The problem to define an exterior derivative becomes trivial, because the exterior derivative is seen by using this system of plots. A stochastic generalization was performed in [25, 26, 27, 28]. There are two different stochastic diffeologies of the loop space: The first one uses the general theory of semi-martingales on a manifold which belong to all of the H^p [14]. The second one uses stochastic integrals with respect to a flat Brownian motion. The main interest of the second one is that stochastic integrals are in fact Hölderian (and not only continuous), and that there are partitions of unity over the Hölderian loop space [2, 3]. We work for the second stochastic diffeology over the Hölderian loop space, and we can apply the traditional tools of de Rham cohomology theory, which are strongly based upon the existence of partitions of unity. This allows to show that the stochastic cohomology of Chen-Souriau with respect of the diffeology involved with stochastic integrals is equal to the deterministic cohomology of the Hölderian loop space [26, 28]. We give in Section 2 an example of a form which is a form with respect of the diffeology involved with stochastic integrals and not with the general diffeology of semi-martingales which are in all of the H^p. (See [14].)

The purpose of this work is to compute the stochastic cohomology, when we consider a set of general semi-martingales as diffeology: We work over the continuous loop space, where there is no partition of unity in general. The interest to consider more general processes is the following: In algebraic topology, the natural object is not a manifold but a $C-W$ complex. People in algebraic topology consider the loop space of a $C-W$ complex which is a notion which is more general than the loop space of a manifold. A beginning of theory of random processes over an Euclidean complex was performed by Brin and Kifer in [8]. In these random processes the Walsh martingale plays an important role, which is not a stochastic integral. So the eventual generalization of stochastic homology of a $C-W$ complex needs a more general class of random processes than stochastic integrals. With respect to this larger class of diffeology, in [27] it was already shown that the stochastic cohomology groups are invariant under differentiable homotopies. The stochastic cohomology in this sense gives an invariant under differentiable homotopies, although we are not able to compute explicitely these cohomology groups; namely, the traditional tools for computing the de Rham cohomology are inefficients.

But, if the manifold is simply connected, we know that the cohomology of the based smooth loop space is equal to the Hochschild cohomology. If we can show that the stochastic cohomology with respect to the diffeology involved with general semi-martingales is equal to the Hochschild cohomology, we will then be able to show as a consequence that the stochastic cohomology is equal to the cohomology of the smooth loop space.

The fact that the Hochschild cohomology for a simply connected manifold is equal to the cohomology of the smooth based loop space uses a spectral sequence because the based path space is a fibration over the manifold with fiber the loop space. For the Sobolev cohomology, this last property remains true by using Driver's flow [15, 13, 24]. The work in [23] uses the quasi-invariance formulas of Albeverio-Hoegh-Krohn in order to show this property for the loop space of a homogeneous manifold.

In order to get this fibration property for this stochastic diffeology, we have to impose that the semi-martingales depend, in some sense, on the future, which give a little bit richer diffeology than the diffeology of [25].

This leads us to give a diffeological calculus with anticipating conditions. With respect to this system of anticipative plots, we get a stochastic cohomology theory which is equal to the Hochschild cohomology, therefore to the cohomology of the smooth based loop space. For this we suppose that the manifold M is simply connected.

Let us motivate the study of stochastic cohomology: Equivariant cohomology of the loop space is related to the finite-dimensional index theory [6, 7, 16]. There are series of forms over the loop space, for instance the Bismut-Chern character, and it is interesting to know if they converge in some convenient functional spaces. Moreover, the infinite-dimensional index theory is related to a theory of bundles over the loop space [19, 33, 12, 10, 37]. The simplest case of such a bundle is constituted by line bundles over the loop space [9], which is given by its curvature, if the loop space is simply connected. It becomes therefore interesting in order to speak of Hilbert spaces of sections of bundles to consider stochastic cohomology groups [29, 30, 26, 28]. In a particular case, we can consider a Kac-Moody group: Over the underlying loop group we can construct the Brownian bridge measure or another measure. For instance, we can perform a change of time over the loop group such that the typical loops are no longer Hölderian. We can define the Hilbert space of functionals over the Kac-Moody group, but in the curvature of the circle bundle of the Kac-Moody group, some stochastic integrals appear. This shows us that in the transition functional some stochastic integrals appear, where we have to deform the typical loop in the group into a constant loop. By this deformation, we get laws which are not quasi-invariant with respect of the original mesure for the non-Hölderian loops in the group. If we make a change of time over the loop group, the theory of measures over the Kac-Moody group implies an understanding of stochastic forms (which are almost surely defined) over the set of semi-martingales (see [31]).

2. Statement of the conjecture

Let M be a compact Riemannian manifold of dimension d. We embed it into the Euclidean space $R^{d'}$.

Let (Ω, F_t, P) be a filtered probability space with a filtration F_t satisfying the usual conditions. A semi-martingale on M can be seen as a semi-martingale on $R^{d'}$ constrained to be on M.

On semi-martingales from $[0,1]$ on $R^{d'}$, starting from a given deterministic point on $R^{d'}$ and which are in all of the H^p (see [14]), we construct a convenient topology: If $X_t = M_t + V_t$ is a semi-martingale where M_t is a local martingale and V_t an adapted continuous finite variational process, we put

$$\|X_\bullet\|_p^p = E[< M, M >_1^{p/2}] + E[(\int_0^1 |dV_s|)^p]. \quad (1)$$

We consider over the set of semi-martingales the Fréchet topology endowed with the family of norms $\|\cdot\|_p$. We call $S_{\infty-}$ the Fréchet space which is obtained from this procedure.

Definition 1 *A stochastic plot $\psi = (U, \phi_i, Z_i, \Omega_i)$ of dimension n from an open subset U of R^n is given by the following data:*

i) A countable partition of Ω in measurable sets Ω_i.

ii) A smooth map $\phi_i : U \times R^m \to S_{\infty-}$ for any m. This implies that $\phi_i(u,v)_\bullet$ has almost surely a smooth version in (u,v) defined for the uniform topology over the set of paths in $R^{d'}$.

iii) A family of random variables $Z_i(u)$ from Ω into R^m which depend smoothly on u for $u \in U$ in all of the L^p. We choose a smooth version of this family by the Kolmogorov lemma [35].

iv) $\phi_i(u, Z_i(\omega)(u))(\omega)_\bullet$ belongs almost surely to $L_x(M)$ if $\omega \in \Omega_i$ for the considered smooth version in (u,v) of $\phi_i(u,v)(\omega)_\bullet$.

The system of stochastic plots ψ constitutes a diffeology of $L_x(M)$. We identify two stochastic plots $(U, \phi_i^1, Z_i^1, \Omega_i^1)$ and $(U, \phi_i^2, Z_i^2, \Omega_i^2)$ if $\phi_i^1(\cdot, Z_i^1(\cdot))_\bullet = \phi_j^2(\cdot, Z_j^2(\cdot))_\bullet$ almost surely in $\Omega_i^1 \cap \Omega_j^2$ as smooth maps from U into the based loop space endowed with the uniform topology.

Definition 2 *A stochastic form σ_{st} over $L_x(M)$ is given by the following data:*

i) To each stochastic plot $\psi = (U, \phi_i, Z_i, \Omega_i)$, we associate almost surely a random smooth form over U $\psi^* \sigma_{st}$, for almost all ω belonging to each Ω_i.

Moreover, the system of random finite-dimensional forms $\psi^* \sigma_{st}$ check the following consistency properties:

ii) Let $j : U_1 \to U_2$ be a smooth deterministic map. Let $\psi^2 = (U_2, \phi_i^2, Z_i^2, \Omega_i)$ be a stochastic plot. Let $\psi^1 = (U_1, \phi_i^1, Z_i^1, \Omega_i)$ be the composite plot:

$$\phi_i^1(u_1, v) = \phi_i^2(j \circ u_1, v), \tag{2}$$

and $Z_i^1(u_1) = Z_i^2(j \circ u_1)$ almost surely as random smooth maps. Then almost surely, $\psi^{1*}\sigma_{st} = j^*\psi^{2*}\sigma_{st}$.

iii) Let $\psi^1 = (U, \phi_i^1, Z_i^1, \Omega_i^1)$ and $\psi^2 = (U, \phi_j^2, Z_j^2, \Omega_j^2)$ be two stochastic plots. Suppose that there exist i, j, and a measurable transformation $F : \Omega_i^1 \to \Omega_j^2$ defined over a set of measure strictly larger than 0 such that almost surely:

$$\phi_i^1(\cdot, Z_i^1(\omega)(\cdot))(\omega)_\bullet = \phi_j^2(\cdot, Z_j^2(F\omega)(\cdot))(F\omega). \tag{3}$$

Then almost surely

$$\psi^{1*}\sigma_{st}(\omega) = \psi^{2*}\sigma_{st}(F(\omega)). \tag{4}$$

Remark We will not make precise the technicalities which arise in condition iii), because we take smooth versions of $\phi_1^1(u, Z_i^1(\omega)(u))(\omega)$, $\phi_j^2(\cdot, Z_j^2(\omega)(u))(\omega)$, $\psi^{1*}\sigma_{st}(\omega)$, and $\psi^{2*}\sigma_{st}(\omega)$.

Let us give an example of stochastic forms over $L_x(M)$ with respect to this stochastic diffeology. Let us introduce

$$\tilde{\omega} = \omega_1 \otimes \cdots \otimes \omega_n \tag{5}$$

an element of $\Omega_\bullet(M)^{\otimes n}$, where $\Omega_\bullet(M)$ is the set of smooth forms of degree larger than 1 over the manifold M. Let us consider the Chen form

$$\sigma_{st}(\tilde{\omega}) = \int_{0<s_1<\cdots<s_n<1} \omega_1(d\gamma_{s_1}, \cdot) \wedge \cdots \wedge \omega_n(d\gamma_{s_n}, \cdot). \tag{6}$$

We can extend the forms ω_i into the forms $\omega_{i,e}$ on the Euclidean space where the manifold is imbedded. We can consider the stochastic form on $U \times R^m$ defined for a plot ψ by

$$\sigma_{U\times R^m}(v)(\tilde{\omega}) = \int\limits_{0<s_1<\cdots<s_n<1} \omega_{1,e}(d\phi_i(u,v)_{s_1}, \partial\phi_i(u,v)_{s_1}) \wedge \cdots \wedge \omega_{n,e}(d\phi_i(u,v)_{s_n}, \partial\phi_i(u,v)_{s_n}). \tag{7}$$

We take the derivative of $\phi_i(u,v)_s$ in the direction of $u \in U$ and $v \in R^m$. We get by this procedure a random form over $U \times R^m$. Let us explain a little more this formula. Suppose that $\omega_{i,e}$ is of degree r_i, such that $\sigma_{U\times R^m}(v)(\tilde{\omega})$ is of degree $N = \sum(r_i - 1)$. We consider vector fields over $U \times R^m$, $X_1, \ldots, X_N$, we apply $\omega_{i,e}(d\phi_i(u,v)_{s_i}, \cdot)$ to the $(r_i - 1)$-uple $\partial_{X_j}\phi_i(u,v)_{s_i}$ where j runs from $r_1 - 1 + r_2 - 1 + \cdots r_{i-1}$ to $r_1 - 1 + r_2 - 1 + \cdots r_i - 1$, and we perform the stochastic integral over the simplex. This gives by the Kolmogorov lemma [35] a random smooth multilinear expression over $U \times R^m$, which is partially antisymmetric. It remains to antisymmetrize the expression because we have wedge products in $\omega_{i,e}(d\phi_i(u,v)_{s_i},\cdot)$. We have the random map from U into $U \times R^m$ denoted by $\tilde{Z}_i$: $u \to (u, Z_i(u))$. We get a random smooth form over $U \times R^m$. On Ω_i, we put

$$\psi^*\sigma_{st} = \tilde{Z}_i^{\,*}\sigma_{U\times R^m}(\tilde{\omega}). \tag{8}$$

$\psi^*\sigma_{st}$ checks clearly the hypothesis of compatibility of Definition 2 ii) because the Stratonovitch integrals are purely local. We denote by $\Sigma(\tilde{\omega})$ the stochastic Chen iterated integral σ_{st}, and the map stochastic Chen iterated integral is denote by Σ.

Let us define the Hochschild boundary:

$$\begin{aligned} b(\tilde{\omega}) = &-\sum(-1)^{\varepsilon i-1}\omega_1\otimes\cdots\otimes d\omega_i\otimes\cdots\otimes\omega_n \\ &-\sum(-1)^{\varepsilon_i}\omega_1\otimes\cdots\otimes\omega_i\wedge\omega_{i+1}\otimes\cdots\otimes\omega_n, \end{aligned} \tag{9}$$

where $\varepsilon_i = \sum_{j<i} deg\omega_j - 1$.

We can define the exterior derivative of a stochastic form σ_{st} by putting for a stochastic plot ψ:

$$\psi^* d\sigma_{st} = d\psi^*\sigma_{st}. \tag{10}$$

By the same proof of [25, p. 22], we get:

$$d\sigma_{st}(\tilde{\omega}) = \sigma_{st}(b\tilde{\omega}). \tag{11}$$

The map Σ: $\tilde{\omega} \to \sigma_{st}(\tilde{\omega})$ has a kernel [16]. It is defined as follows. Let us consider a smooth function $f : M \to R$. Let $S_i(f)$ be the following operation: Between ω_i and ω_{i+1}, we intercalate f in the tensor product which constitutes $\tilde{\omega}$. We get the expression $\omega_1\otimes\omega_2\otimes\cdots\otimes\omega_i\otimes f\otimes\omega_{i+1}\otimes\cdots\otimes\omega_n$. The kernel of the map stochastic Chen iterated integrals Σ is constituted of finite linear combinations $[b, S_i(f)]\tilde{\omega}$, b being the Hochschild boundary.

$H_\bullet H_\bullet(R, \Omega_\bullet(M), R)$ is the cohomology group for the Hochschild boundary of the Hochschild complex, when we consider a **finite** sum of elementary tensor products $\tilde{\omega}$, with different length of tensor products, modulo the kernel of the map stochastic Chen iterated integrals.

$H^*_{st}(L_x(M))$ is the cohomology group for the stochastic exterior derivative defined on stochastic differential forms by (10).

The main result of this work is the following:

Conjecture*Let us suppose M simply connected. The map stochastic Chen iterated integrals induces an isomorphism between* $H_\bullet H_\bullet(R, \Omega_\bullet(M), R)$ *and* $H^*_{st}(L_x(M))$.

As we shall see, the proof of this conjecture presents a difficulty. In the next part, we will prove a statement which is very close to this conjecture.

Remark We give an example in a simple case of a 0-form (that is a functional) for the diffeology associated to stochastic integral which has no chance to be extended to a form over the set of semi-martingale which belong to all of the H^p. We work over R and we consider as diffeology the set of $\phi(u)_t = \int_0^t A_s(u)\delta B_s$ where B_s is a flat one-dimensional Brownian motion and $A_s(u)$ depends smoothly in all of the L^p from u belonging to the parameter space U which is an open subset of R^n. σ_{st} is defined as follows: $\phi^*\sigma_{st} = \int_0^1 A_s(u)ds$ and constitutes an example of a functional for our diffeology which seems difficult to extend to more general semi-martingales in R.

3. Proof of the approximated theorem of the conjecture

We recall that the based path space $P_x(M)$ is the set of continuous maps $\gamma_\bullet$ from $[0,1]$ into M such that $\gamma_0 = x$. We define a stochastic plot of the path space.

Definition 3 *A stochastic plot $\psi = (U, \phi_i, Z_i(u), \Omega_i)$ for an open subset U of R^n of the based path space is given by the following data:*

i) A countable partition of Ω in measurable sets Ω_i.

ii) A smooth map $\phi_i : U \times R^m \to S_{\infty-}$. This implies that $\phi_i(u,v)_\bullet$ has almost surely a smooth version in (u,v) defined for the uniform topology over the set of paths in $R^{d'}$.

iii) A family of random variables $Z_i(u)$ from Ω into R^m depending smoothly on $u \in U$ in L^p. The Kolmogorov lemma implies that there exists a smooth version of $Z_i(u)$ [35].

iv) $\phi_i(u, Z_i(\omega)(u))(\omega)_\bullet$ belongs almost surely to $P_x(M)$ if $\omega \in \Omega_i$ for the considered smooth version in (u,v) of $\phi_i(u,v)(\omega)$.

We consider the same identification of two stochastic plots with values in $P_x(M)$ as the equivalence relation stated after Definition 1. We have an analogous definition of a stochastic form over $P_x(M)$ along the lines of Definition 2.

We remark that a stochastic plot over $L_x(M)$ defines a stochastic plot over $P_x(M)$. Therefore, by restriction, a stochastic form over $P_x(M)$ defines a stochastic form over $L_x(M)$.

The exterior derivative of a stochastic form over $P_x(M)$ is defined as in (10).

One of the main ingredient of the proof of the main theorem is the following, which reflects that the based path space is contractible:

Theorem 1 *$H^*_{st}(P_x(M)) = R$.*

Proof. Let us extend the Levi-Cività connection over M, which we suppose endowed with a Riemannian metric, to a connection over $R^{d'}$. We get the notion of parallel transport over a stochastic plot $\phi_i(u,v)_\bullet$: We note it $\tau_i(u,v)_\bullet$. The main property of $\tau_i(u,v)_\bullet$ is the following: If $\omega \in \Omega_i$, $\tau_i(u, Z_i(\omega)(u))(\omega)_\bullet$ defines a map from $T_x(M)$ onto $T_{\phi_i(u,Z_i(\omega)(u))(\omega)_\bullet}(M)$, which is a rotation, because we have extended the Levi-Cevità connection from M to $R^{d'}$ in a metric connection, and because on Ω_i, $\phi_i(u,Z_i(\omega)(u))(\omega)_\bullet$ is an element of $P_x(M)$. We note by $\xi_i(u,v)_\bullet$ the semi-martingale in $R^{d'}$ defined in the sense of Stratonovitch by:

$$d\xi_i(u,v)(\omega)_s = \tau_i^{-1}(u,v)_s d\phi_i(u,v)_s \tag{12}$$

and which starts at 0. If $\omega \in \Omega_i$, $\xi_i(u,Z_i(\omega))(\omega)_\bullet$ is a path starting at 0 in $T_x(M)$. We consider $\varepsilon \in [0,1]$ and the path $\varepsilon\xi_i(u,v)(\omega)_s = \xi_i(\varepsilon,u,v)(\omega)_s$. We introduce the Stratonovitch differential equation:

$$d\phi_i(\varepsilon,u,v)(\omega)_s = \tau_i(\varepsilon,u,v)_s d\xi_i(\varepsilon,u,v)(\omega)_s \tag{13}$$

starting at x. $\tau_i(\varepsilon,u,v)_s$ is the parallel transport over the random path which is realized by the semi-martingale $\phi_i(\varepsilon,u,v)(\omega)_\bullet$. Moreover, we can suppose that the Christoffel symbols of the extended connection on $R^{d'}$ are bounded, so that $\phi_i(\varepsilon,u,v)$ depends smoothly on (ε,u,v) in $S_{\infty-}$. Moreover, if $\omega \in \Omega_i$, $\phi_i(\varepsilon,u,Z_i(\omega)(u))(\omega)_\bullet$ is almost surely in $P_x(M)$. On the other hand, $\phi_i(0,u,Z_i(\omega)(u))(\omega)_s = x$ and $\phi_i(1,u,Z_i(\omega)(u))(\omega)_\bullet = \phi_i(u,Z_i(\omega)(u))(\omega)_\bullet$. In others words, we have deformed the plot ψ into the constant plot by remaining in the considered set of plots. We get a plot ψ^{ext} given by $(U\times[0,1],\phi_i(\varepsilon,u,v),Z_i,\Omega_i)$.

Let us consider a closed stochastic form σ_{st} of degree strictly larger than 0. $\psi^{ext,*}\sigma_{st}$ gives a random closed form on $U\times[0,1]$. We consider the retraction operation from $U\times[0,1]$ to U given by $g_\lambda(u,\varepsilon) = (u,f_\lambda(\varepsilon))$, $\lambda\in[0,1]$, for a convenient smooth function f_λ from $[0,1]$ into 0 which is equal to 0 at $\lambda = 0$. Since $\psi^{ext,*}\sigma_{st}$ is closed and since a form over a point vanishes if its degree is strictly larger than 0, we get by the Cartan formula:

$$\psi^{ext,*}\sigma_{st} = d\int_0^1 g_\lambda^* i_{\frac{\partial g_\lambda}{\partial\lambda}} \psi^{ext,*}\sigma_{st} d\lambda. \tag{14}$$

This implies by condition ii) of Definition 2 that

$$\psi^{*}\sigma_{st} = d_u\int_0^1 g_\lambda^* i_{\frac{\partial g_\lambda}{\partial\lambda}} \psi^{ext,*}\sigma_{st} d\lambda. \tag{15}$$

Therefore $\sigma_{st} = d\tilde{\sigma}_{st}$ if the degree of σ_{st} is strictly larger than 0. If this degree vanishes, we have by the same type of argument

$$\psi^*\sigma_{st} = C, \tag{16}$$

where C is almost surely constant by condition iii) of Definition 2, and corresponds to the value of σ_{st} associated to the constant stochastic plot. ■

Let us recall what is the Hochschild complex associated to the based path space. Let

$$\tilde{\omega} = \omega_1\otimes\cdots\otimes\omega_n \tag{17}$$

be an element of the algebraic tensor product $\Omega_\bullet(M)^{\otimes n-1} \otimes \Omega(M)$, where $\Omega(M)$ denotes the set of smooth forms over M without any restriction upon the degree. Let us consider the Chen form:

$$\sigma_{st}(\tilde{\omega}) = \int_{0<s_1<\cdots<s_{n-1}<1} \omega_1(d\gamma_{s_1},\cdot) \wedge \cdots \wedge \omega_{n-1}(d\gamma_{s_{n-1}},\cdot) \wedge \omega_n. \tag{18}$$

We can extend the forms ω_i to forms $\omega_{i,e}$ on the Euclidean space where the manifold is imbedded. Let ψ be a stochastic plot over $P_x(M)$. Instead of (7), we consider the iterated integral between 1 and $n-1$ and we consider the wedge product by $\omega_{n,e}(\partial\phi_i(u,v)_1)$. $\sigma_{st}(\tilde{\omega})$ is a Chen form over the based path space of degree $\sum_0^{n-1}(\deg(\omega_i) - 1) + \deg(\omega_n)$. Moreover, $d\sigma_{st}(\tilde{\omega}) = \sigma_{st}(b\tilde{\omega})$ where b is given by the same formula as in (9).

The kernel of the operation stochastic Chen iterated integral is given just after Equation (11). We get a complex, called the algebraic Hochschild complex, which operates over **finite sums**, and we denote its cohomology groups by $H_\bullet H_\bullet(R,\Omega_\bullet(M),\Omega(M))$.

Theorem 2 $H_\bullet H_\bullet(R,\Omega_\bullet(M),\Omega(M)) = H^*_{st}(P_x(M))$.

Proof. We refer to [16] or [22, Theorem II.8] for the details of the proof. It is based upon a contraction from $\Omega_\bullet(M)^{\otimes n} \otimes \Omega(M)$ into $\Omega_\bullet(M)^{n+1} \otimes \Omega(M)$ which to $\omega_1 \otimes \cdots \otimes \omega_{n+1}$ associates $\omega_1 \otimes \cdots \otimes \omega_{n+1} \otimes 1$ if the degree of ω_{n+1} is strictly larger than 0, and which associates 0 otherwise. ∎

Let O be an open subset of M: $P_O(M)$ denotes the set of based paths $\gamma_\bullet$ such that $\gamma_0 = x$ and such that $\gamma_1 \in O$. We can give the definition of a stochastic plot with values in $P_O(M)$. In Definition 3, we replace the condition iv) by the condition that $\phi_i(u,Z_i(\omega)(u))(\omega)_\bullet$ belongs almost surely to $P_O(M)$ if $\omega \in \Omega_i$. Associated to this system of stochastic plots with values in $P_O(M)$, we can define a system of stochastic forms $\Lambda_{st}(P_O(M))$ over $P_O(M)$. There is another way, which was used in [26] to define a stochastic form over $P_O(M)$, by starting from the full set of stochastic plots over the total based path space $P_x(M)$.

Definition 4 *A stochastic form $\sigma_{st} \in \Lambda'_{st}(P_O(M))$ is given by the following data: Let $\psi = (U,\phi_i,Z_i,\Omega_i)$ be a stochastic plot over $P_x(M)$. To ψ we associate $U(\omega)$ the random open subset of U such that, if $\omega \in \Omega_i$, $\phi_i(u,Z_i(\omega)(u))(\omega)_\bullet$ belongs to $P_O(M)$. We associate a random smooth form $\psi^*\sigma_{st}$ over $U(\omega)$, which checks the analoguous consistency relations ii) and iii) of Definition 2.*

Remark In Definition 1, the parameter set U is deterministic, while in Definition 2, the parameter set $U(\omega) \subseteq U$ is random.

Proposition 1 *$\Lambda'_{st}(P_O(M))$ is included in $\Lambda_{st}(P_O(M))$.*

Proof. The statement is clear, because a stochastic plot with values in $P_O(M)$ is still a stochastic plot with values in $P_x(M)$. Let us show what is the difficulty in order to show the converse inclusion. Let σ_{st} be an element of $\Lambda_{st}(P_O(M))$, and let $\psi = (U, \phi_i, Z_i(u), \Omega_i)$ be a stochastic plot over $P_x(M)$. R^n has a countable basis of neighborhoods. Let U_m, $m \in Z$ be such a basis. We construct a stochastic plot $\psi_m = (U_m, \varphi_i, Z_i, \Omega'_i)$ with values in $P_O(M)$: We consider the subset of Ω_i, denoted by Ω'_i, such that if $u \in U_m$, $\phi_i(u, Z_i(\omega)(u))(\omega)_\bullet$ belongs to $P_O(M)$. If ω does not belong to the union of the Ω'_i, we choose any deterministic curve with bounded variation belonging to $P_O(M)$. We deduce a random form over U_m. The problem now is that $U(\omega)$ is an union of U_m and that the set of subsets of Z is not countable. ■

The main property of the set $\Lambda'_{st}(P_O(M))$ is described as follows.

Proposition 2 *Let O_i be a finite open cover of M and g_i be a partition of unity subordinated to it. Let σ^i_{st} belong to $\Lambda'_{st}(P_{O_i}(M))$. Then $\sum g_i(\gamma_1)\sigma^i_{st}$ defines an element of $\Lambda_{st}(P_x(M))$.*

Proof. It is clear, because if we consider a plot $\psi = (U, \phi_i, Z_i, \Omega_i)$ with values in $P_{O_i}(M)$ over $U(\cdot)$, then $\psi^* g_i(\gamma_1)\sigma^i_{st}$ determines a smooth form over U. ■

We will slightly modify the notion of a stochastic form over $L_x(M)$: We will generalize it to the space of paths joining x to y, denoted by $L_{x,y}(M)$.

Definition 5 *A stochastic form σ_{st} over $L_{x,y}(M)$ is given by the following data:*

To each stochastic plot $\psi = (U, \phi_i, Z_i(u), \Omega_i)$ over $P_x(M)$, we associate almost surely a random smooth form $\psi^*\sigma_{st}$ for almost all ω belonging to each Ω_i over the interior $U(\omega)$ of the subset of U such that $\phi_i(u, Z_i(\omega)(u))(\omega_\bullet)_\bullet \in L_{x,y}(M)$.

Moreover, the set of random finite-dimensional forms $\psi^*\sigma_{st}$ satisfy over $U(\omega)$ the analoguous consistency relations ii) and iii) of Definition 3.

The set of stochastic forms in this sense is denoted by $\Lambda'_{st}(L_{x,y}(M))$.

Remark In the Definition 2 ii), the parameter set U is deterministic. In Definition 5, it is random.

We can define the stochastic exterior derivative of a stochastic form in the sense of Definition 5 as in (10). We get stochastic cohomology groups denoted by $H^*(\Lambda'_{st}(L_{x,y}(M))$.

Proposition 3 *For any $y \in M$, $H^*(\Lambda'_{st}(L_{x,y}(M))$ is equal to $H^*(\Lambda'_{st}(L_x(M))$.*

Proof. Let $g_t(z)$ be a smooth family of diffeomorphisms of M extended to a smooth family in $t \in [0,1]$ of diffeomorphisms of $R^{d'}$ with bounded derivatives of all orders as well as their inverse. We suppose that $g_0(x) = x$ and $g_1(y) = x$. Let $\psi = (U, \phi_i, Z_i(u), \Omega_i)$ be a stochastic plot over $P_x(M)$ with values in $L_{x,y}(M)$ over $U(\cdot)$. We associate to it the stochastic plot $g_\bullet\psi = (U, g_\bullet \circ \psi, Z_i(u), \Omega_i)$. This gives a bijection between stochastic plots with values in $L_{x,y}(M)$ over $U(\cdot)$ and stochastic plots with values in $L_x(M)$ over $U(\cdot)$. Let σ_{st} be an element of $\Lambda'_{st}(L_x(M))$. We associate to $g^*\sigma_{st}$ an element of $\Lambda'_{st}(L_{x,y}(M))$ defined as follows: Let ψ be a stochastic plot with values in $L_{x,y}(M)$ over $U(\cdot)$. We have $\psi^*(g^*\sigma_{st}) = (g\psi)^*\sigma_{st}$. This definition has a sense because $g\psi$ is a stochastic plot with values in $L_x(M)$ over $U(\cdot)$. Clearly, g^* induces a bijection from $\Lambda'(L_x(M))$ to $\Lambda'_{st}(L_{x,y}(M))$ which is compatible with the stochastic exterior derivative because:

$$\psi^*(d(g^*\sigma_{st})) = d\psi^*(g^*\sigma_{st}) = d(g\psi)^*\sigma_{st} = (g\psi)^*(d\sigma_{st}) = \psi^*(g^*d\sigma_{st}). \quad (19)$$

Therefore the proposition holds. ■

Remark This isomorphism property is uniquely based upon the Itô formula and is much more simpler to prove than the isomorphism formula for stochastic cohomology groups in the sense of Nualart-Pardoux [23, 24] based upon the Albeverio-Hoegh-Krohn quasi-invariance formulas or upon Driver's flow, which brings into heavy technicalities. This remark also relevant for the following proposition.

Proposition 4 *Let O be a small convex neighborhood of y in M. Then the group $H^*(\Lambda'_{st}(L_{x,y}(M)))$ is equal to $H^*(\Lambda'_{st}(P_O(M)))$.*

Proof. Let $\lambda \in [0,1]$ and $t \in [0,1]$. We consider a smooth family of maps parameterized by λ and t from $R^{d'} \times R^{d'}$ into $R^{d'}$ with bounded derivatives of all orders such that:

1) $g_{\lambda,0}(x,z') = x$.

2) $z \to g_{\lambda,t}(z,z')$ is a diffeomorphism of M with bounded derivatives of all orders as well as its inverse. This diffeomorphism is equal to Id_M at $t = 0$.

3) $z \to g_{\lambda,t}(z,y)$ is equal to Id_M.

4) $z \to g_{1,t}(z,z')$ is equal to Id_M.

5) $z \to g_{\cdot,1}(z,z)$ realises a retraction from O onto y.

Let $\psi = (U,\phi_i,Z_i(u),\Omega_i)$ be a plot in $P_x(M)$ and $s \to g_{\lambda,s}(\psi_s,\psi_1)$ be its associated plot. We get another plot in $P_x(M)$, denoted $g_{\lambda,\bullet}(\psi)$. If ψ is in $P_O(M)$ over $U(\cdot) \subseteq U$, $g_{\lambda,\bullet}(\psi)$ is still in $P_O(M)$ over the same $U(\cdot)$, if $\lambda \neq 0$. We deduce a map $g^*_{\lambda,\bullet}$ from $\Lambda'_{st}(P_O(M))$ into $\Lambda'_{st}(P_O(M))$. If $\lambda = 0, g_{0,\bullet}(\psi)$ is a plot with values in $L_{x,y}(M)$ over $U(\cdot)$. We deduce a map $g^*_{0,\bullet}$ from $\Lambda'_{st}(L_{x,y}(M))$ into $\Lambda'_{st}(P_O(M))$ as in the proof of Proposition 3, which preserves the exterior stochastic differential. We deduce a map $g^*_{0,\bullet}$ from $H^*(\Lambda'_{st}(L_{x,y}(M))$ into $H^*(\Lambda'_{st}(P_O(M))$. Let i^* the inclusion map from $\Lambda'_{st}(P_O(M))$ into $\Lambda'_{st}(L_{x,y}(M))$, which has a sense by the definition of

$\Lambda'_{st}(L_{x,y}(M))$. This extends to a map of the cohomology group $H^*(\Lambda'_{st}(P_O(M)))$ into $H^*(\Lambda'_{st}(L_{x,y}(M)))$.

By 3), $i^*g^*_{0,\bullet} = I_d$ in cohomology, because it is the identity at the level of forms.

Let us show that $g^*_{0,\bullet}i^* = Id$ in cohomology. The proof closely follows the proof of Lemma II.7 of [26], Lemma 4.9 of [28], Theorem III.3 of [27], and Theorem 1. We consider the plot ψ^{ext} indexed by $U \times [0,1]$, $g_{\lambda,\bullet}(\psi)$. Suppose that ψ takes its values in $P_O(M)$ over $U(\cdot) \subseteq U$. The main remark is the following: ψ^{ext} still takes its values in $P_O(M)$ over $U(\cdot) \times [0,1]$. Suppose that σ_{st} over $P_O(M)$ is closed. We deduce a random finite-dimensional form over $U(\cdot) \times [0,1]$ $(\psi^{ext})^*\sigma_{st}$. We have a retract from $U \times [0,1]$ into U: $(u,\lambda) \to (u, f_\varepsilon(\lambda))$. We apply the Cartan formula as in (2.16) of [26] or (14) of Theorem 3, λ and ε being interchanged. By (2.17) of [26], we deduce that $\sigma_{st} = g^*_{0,\bullet}i^*\sigma_{st} + d\tilde{\sigma}_{st}$. This shows the result. ■

We can now state the main result of this section.

Theorem 3 *Let us suppose that M is simply connected. Then the cohomology group $H^*(\Lambda'_{st}(L_x(M)))$ is equal to $H_\bullet H_\bullet(R, \Omega_\bullet(M), R)$.*

Proof. Let O_i be a cover by small open convex subsets of M. Let $O_{i_0,\ldots,i_r} = \cap O_{i_j}$. We consider the set $\Lambda'_{st}(P_{O_{i_0,\ldots,i_r}}(M)$ endowed with 2 gradings:

1) r, the length of the intersection minus 1.

2) The degree of the stochastic form.

There is a bicomplex associated to this bigrading. It is the Cech complex δ in the length of the intersection, and, in the degree of the stochastic form, it is the stochastic exterior derivative.

There are two spectral sequences which converge to the total cohomology of the bicomplex. But by Proposition 2, $\tau_{st} = \sum g_i(\gamma_1)\sigma_{st,i,i_0,\ldots,i_{r-1}}$, where g_i is a smooth partition of unity associated to O_i, is a stochastic form such that $\delta\tau_{st} = \sigma_{st}$. Therefore the two spectral sequences converge to the stochastic cohomology groups of the path space.

The remaining of the proof is exactly the same as the remaining of the proof of this statement for Sobolev cohomology groups of the loop space (see [23, 24, 16]).

We consider a small contractible open subset of M: We see that the cohomology group $H_\bullet H_{\bullet O}(R, \Omega_\bullet(M), \Omega^O(M))$ is equal to the cohomology group $H_\bullet H_\bullet(R, \Omega_\bullet(M), R)$ where in $H_\bullet H_{\bullet O}(R, \Omega_\bullet(M), \Omega^O(M))$ we consider the last form ω_n in (18) which is a form over O instead of M. (See [23, Lemma II.6].) Over the set of Chen forms with support in $O_{i_0,\ldots,i_r}$ included in M, there is a bigrading: The length of the intersection $O_{i_0,\ldots,i_r}$ and the degree associated to the Hochschild boundary which corresponds to the degree of the associated form over the path space. We get a spectral sequence where the Hochschild boundary (or the stochastic exterior derivative) is the vertical complex and the horizontal complex is the

Cech complex. The map stochastic Chen iterated integral preserves these 2 spectral sequences.

The first term $E_1^{p,q}$ of the spectral sequence is the space of the Cech complex with values in the presheaf $H^q(\Lambda'_{st}(P_O(M)))$ which is locally constant because $H^*(\Lambda'_{st}(P_O(M)))$ is equal to $H^*(\Lambda'_{st}(L_x(M)))$ for a small open contractible set O. But it is constant since M is simply connected. Therefore $E_2^{p,q} = H^p(M) \otimes H^q(\Lambda'_{st}(L_x(M))$.

It is the same for the spectral sequence associated to the Hochschild complex. Therefore $E_2^{p,q} = H^p(M) \otimes H_\bullet H_\bullet^q(R, \Omega_\bullet(M), R)$.

The map iterated stochastic Chen integral induces a map of spectral sequences. It induces an isomorphism by Theorem 3 on E_∞. If it induces an isomorphism on $E_2^{p,0}$, it will induce by the Zeeman comparison theorem [32] an isomorphism on $E_2^{0,q}$. These facts will show the theorem.

But $H^0(\Lambda'_{st}(L_x(M))) = R$. Namely, $H^0(\Lambda_{st}(P_x(M))) = R$ (see Theorem 2). We consider the spectral sequence associated to the Cech coboundary δ as horizontal complex and to the stochastic exterior derivative d as vertical complex. $E_2^{p,q} = H^p(M) \otimes H^q(\Lambda'_{st}(L_x(M)))$, because M is simply connected. Since we have a first quadrant spectral sequence, then $E_l^{0,0} = E_\infty^{0,0}$ and $E_\infty^{0,0} = H_{st}^0(P_x(M)) = R$. (See [4] for a proof in the cae of smooth loop space.) The same argument holds for $H_\bullet H_\bullet^0(R, \Omega_\bullet(M), R)$. ■

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