Tensor products of Frobenius manifolds and moduli spaces of higher spin curves

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Abstract. We review progress on the generalized Witten conjecture and some of its major ingredients. This conjecture states that certain intersection numbers on the moduli space of higher spin curves assemble into the logarithm of the τ function of a semiclassical limit of the r-th Gelfand-Dickey (or KdV_r) hierarchy. Additionally, we prove that tensor products of the Frobenius manifolds associated to such hierarchies admit a geometric interpretation in terms of moduli spaces of higher spin structures. We also elaborate upon the analogy with Gromov-Witten invariants of a smooth, projective variety.

Keywords: cohomological field theory, Gromov-Witten invariants, moduli spaces, integrable hierarchies, KdV, Frobenius manifolds

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In memory of Moshé Flato

0. Introduction

In recent years, there has been a great deal of interaction between mathematics and quantum field theory. One such area has been in the context of topological gravity coupled to topological matter.

The notion of a cohomological field theory (CohFT), due to Kontsevich and Manin [22] is an axiomatization of the expected factorization properties of the correlators appearing in such a theory, cf. [26, 27, 28, 3]. The Gromov-Witten invariants of a smooth, projective variety V provide a rigorous construction of

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a CohFT, one for each V, with a state space $H^\bullet(V)$ and a (Poincaré) metric η. (All cohomology groups in this paper will be assumed to be over $\mathbb{C}$.) The genus zero correlators endow $H^\bullet(V)$ with the structure of a (formal) Frobenius manifold which is a deformation of the cup product, and which yields the structure of quantum cohomology. Furthermore, for two smooth, projective varieties V and V', the quantum cohomology of $V \times V'$ is related to $H^\bullet(V) \otimes H^\bullet(V')$ through a deformation of the Künneth formula. The Gromov-Witten invariants are known to be symplectic invariants of V and are therefore of great mathematical interest.

When V is a point, Witten conjectured [28] and Kontsevich proved [21] that intersection numbers of tautological cohomological classes on the moduli space of stable curves assemble into a τ function of the KdV hierarchy. However, the KdV hierarchy is only the first of a series of integrable hierarchies, one for each integer $r \geq 2$, called the KdV$_r$ or the r-th Gelfand-Dickey hierarchy, where the usual KdV hierarchy is KdV$_2$. Witten stated a generalization [26] of this conjecture where he suggested a construction of moduli spaces and cohomology classes on them whose intersection numbers assemble into a τ function of the KdV$_r$ hierarchy. At the time that his conjecture was formulated, however, the relevant compact moduli spaces had not yet been constructed.

In [14, 15], the moduli space of stable r-spin curves of genus g, $\overline{\mathscr{M}}^{1/r}_{g,n}$, was constructed. In [16], it was proved that certain intersection numbers on $\overline{\mathscr{M}}^{1/r}_{0,n}$ assemble into the genus zero part of the τ function of the KdV$_r$ hierarchy. In addition, there is a (genus zero part of a) CohFT associated to KdV$_r$ whose associated Frobenius manifold has a potential that is polynomial in its flat coordinates. The relevant potential function in genus zero satisfies the string equation, dilaton equation, and topological recursion relations. These proofs are purely algebro-geometric. Axioms have also been formulated for a *virtual class* in higher genus. The existence of a class meeting these axioms gives a CohFT in all genera. For the case of $g = 0$ and all $r > 1$ as well as for the case of $r = 2$ and all $g \geq 0$, this class was constructed in [16], but an algebro-geometric construction of this class for all r and g has yet to be completed.

From a physical perspective, it should not be surprising that the theory of Gromov-Witten invariants and the KdV$_r$ theory should share many features (see the table on next page). Gromov-Witten invariants correspond to a theory of topological gravity with a matter sector arising from the topological sigma model with target variety V. However, one may also consider other choices of matter sector. Witten was led to his KdV$_r$ conjecture by studying the theory of topological gravity coupled to a matter sector arising from a coset model. Even mirror symmetry has an analog in this theory; namely, the Frobenius manifold given by the CohFT associated to KdV$_r$ is isomorphic to that arising from the A_{r-1} singularity, cf. [4, 5, 24, 25].

In this paper, we elaborate upon the analogy between the theory of Gromov-Witten invariants and the KdV$_r$ theory (see the table) using the results in [16].

We construct a geometric "A-model" realization (extending the results in [16]) of the tensor product of the Frobenius manifolds associated to KdV_r in terms of the moduli space of multiple spin structures on stable curves.

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1. Comparison of CohFTs related to stable maps and higher spin curves

This table illustrates some of the similarities and differences between stable maps and higher spin curves. Details and notation will be explained in subsequent sections.

	Stable maps	**Higher spin curves**
Key moduli spaces	Stable maps: $\overline{\mathscr{M}}_{g,n}(V)$	r-spin curves: $\overline{\mathscr{M}}_{g,n}^{1/r}$
State space	$H^\bullet(V)$	$\mathscr{H}^{(r)}$ with basis $\{e_0,\ldots,e_{r-2}\}$
Metric	$\eta(\gamma',\gamma'')=\int_V \gamma'\cup\gamma''$	$\eta(e_{m'},e_{m''})=\delta_{m'+m'',r-2}$
Flat identity	$\mathbf{1}$	e_0
Grading	Integral	Fractional
Fundamental class	Virtual fundamental class $[\overline{\mathscr{M}}_{g,n}(V)]^{\mathrm{virt}}$	Orbifold fundamental class $[\overline{\mathscr{M}}_{g,n}^{1/r}]$
"Gromov-Witten" classes	True Gromov-Witten classes: $\mathrm{ev}^*(\boldsymbol{\gamma}):=\mathrm{ev}_1^*\gamma_1\cup\ldots\cup\mathrm{ev}_n^*\gamma_n$	Virtual class $c^{1/r}(\mathbf{m})$: constructed for $r=2$ and all g as well as for $g=0$ and all r.
Gravitational correlators	$\left\langle\tau_{a_1}(\gamma_1)\ldots\tau_{a_n}(\gamma_n)\right\rangle_g$ $=\int_{[\overline{\mathscr{M}}_{g,n}(V)]^{\mathrm{virt}}}\mathrm{ev}^*(\boldsymbol{\gamma})\prod_{i=1}^n\psi_i^{a_i}$	$\left\langle\tau_{a_1}(e_{m_1})\ldots\tau_{a_n}(e_{m_n})\right\rangle_g$ $=\int_{[\overline{\mathscr{M}}_{g,n}^{1/r,\mathbf{m}}]}c^{1/r}\prod_{i=1}^n\psi_i^{a_i}$
Free Energy	Large phase space potential: $\langle\exp(\mathbf{t}\cdot\boldsymbol{\tau})\rangle$	Logarithm of τ function: $\langle\exp(\mathbf{t}\cdot\boldsymbol{\tau})\rangle$
Virasoro algebra action	The Virasoro Conjecture: proved for V a point. Proved for $g=0$ and in some cases for $g=1$.	W_r-algebra conjecture: proved for $g=0$ as well as for $r=2$
Integrable system	KdV_2 when V is a point. Known in $g=0$ and in some cases in $g=1$.	KdV_r hierarchy
(Generalized) Witten conjectures	Proved for V a point.	Proved for $g=0$ as well as for $r=2$.
Cohomological field theory	$\Lambda_{g,n}=\mathrm{st}_*(\mathrm{ev}^*(\boldsymbol{\gamma})\cap[\overline{\mathscr{M}}_{g,n}]^{\mathrm{virt}})$	$\Lambda_{g,n}=p_*c^{1/r}(\mathbf{m})$
WDVV equation	Holds	Holds
String equation	Holds	Holds
TRR	Holds	Holds
Tensor product of CohFTs	Realized by the product $V_1\times V_2$ (Quantum Künneth theorem in $g=0$)	Realized by the fibered product $\overline{\mathscr{M}}_{g,n}^{1/r,1/s}$
Isomorphisms	Mirror symmetry	Versal deformation of A_{r-1} singularity

Remarks 1.1

1. From a geometric point of view the role of the virtual fundamental class of stable maps is played more by the virtual class $c^{1/r}$ than by the orbifold fundamental class. Indeed, most of the real geometric subtlety of the r-spin theory is contained in the construction of $c^{1/r}$, whereas the smoothness of the moduli spaces $\overline{\mathscr{M}}^{1/r}_{g,n}$ implies that the orbifold class is a simple modification of the standard fundamental class.

2. For general V, the generalized Witten conjecture is not precisely formulated, although it is expected that for each V the exponential of the free energy is a τ function for some integrable system. In $g = 0$ and for some cases in $g = 1$ this integrable system is described in [6, 7].

3. The W_r-algebra contains the Virasoro algebra, and so the corresponding conjecture is a generalization of the Virasoro Conjecture.

2. The KdV$_r$ hierarchy and its semiclassical limit

2.1. The KdV$_r$ hierarchy

In this section we introduce the generalized KdV hierarchies (also called the Gelfand-Dickey hierarchies) and some of their solutions which will be used in the subsequent sections.

Fix an integer $r \geq 2$ and consider differential operators of order r in $D = \frac{i}{\sqrt{r}}\frac{\partial}{\partial x}$ (where the factor $\frac{i}{\sqrt{r}}$ is added for convenience). A monic differential operator of order r, after conjugation by the operator of multiplication by an appropriate function, can be written in the form

$$L = D^r - \sum_{m=0}^{r-2} u_m(x) D^m, \tag{1}$$

where u_m are formal functional variables. Denote by $\mathscr{D}$ the affine space of the operators of type (1).

For every operator $L \in \mathscr{D}$ there exists a unique pseudo-differential operator

$$L^{1/r} = D + \sum_{m>0} w_m D^{-m},$$

such that $(L^{1/r})^r = L$. All coefficients w_m of $L^{1/r}$ are differential polynomials in $u_0, u_1, \ldots, u_{r-2}$.

A pseudodifferential operator $Q = \sum_{m \geq -n} v_m D^{-m}$ can be decomposed as $Q = Q_+ + Q_-$, where $Q_+ = \sum_{m=-n}^{0} v_m D^{-m}$ is the differential part of Q.

For every $k \geq 0$, the operators $Q = (L^{1/r})^k$ and $L = (L^{1/r})^r$ commute yielding $[Q_+, L] = -[Q_-, L]$ and, therefore, $[Q_+, L]$ is a differential operator of order not exceeding $r-2$. Thus the expression $\frac{\partial L}{\partial t} = [Q_+, L]$ gives a meaningful system of differential equations for unknown functions $u_0, \ldots, u_{r-2}$. It can be shown that for different values of k these equations are consistent (i.e., the corresponding flows on $\mathscr{D}$ commute) which allows one to define the KdV$_r$ (or the r-th Gelfand-Dickey) hierarchy as the following infinite family of differential equations on $\mathscr{D}$:

$$i\frac{\partial L}{\partial t_n^m} = \frac{k_{n,m}}{\sqrt{r}}\left[(L^{n+\frac{m+1}{r}})_+, L\right], \tag{2}$$

where the constants

$$k_{n,m} = \frac{(-1)^n r^{n+1}}{(m+1)(r+m+1)\ldots(nr+m+1)}$$

have been inserted for convenience.

In terms of the unknown functions u_k the equations (2) take the form

$$\frac{\partial u_k}{\partial t_n^m} = S_{k,n}^m,$$

where $S_{k,n}^m$ are polynomials in u_j and their derivatives with respect to x.

There are several simple special cases.

1. When $m+1$ is a multiple of r, the commutator in (2) vanishes and the corresponding equation just means that the functions u_k do not depend on t_n^m.
2. In the case $m = n = 0$ the corresponding equation is $\frac{\partial u_k}{\partial t_0^0} = \frac{\partial u_k}{\partial x}$ and therefore we can identify t_0^0 with x.
3. Finally, when $r = 2$ the KdV$_r$ hierarchy becomes the ordinary KdV hierarchy.

2.2. Potential

In the notation of the previous subsection, we introduce the functions

$$v_n = -\frac{r}{n+1}\mathrm{res}(L^{1/r})^{n+1}, \tag{3}$$

where the residue of a pseudodifferential operator is defined as the coefficient of D^{-1}. The functions v_k can be expressed in terms of u_j by a triangular system of differential polynomials. This means that u_j can be expressed in terms of v_n in a similar way, and we can consider $v_0, v_1, \ldots, v_{r-2}$ as a new system of coordinates in $\mathscr{D}$.

We call a function $\widetilde{\Phi}(\mathbf{t})$ in formal variables t_n^m, $n, m \geq 0$, a *potential* of the KdV$_r$ hierarchy if, first, $\widetilde{\Phi}(\mathbf{0}) = 0$ and second, the functions

$$v_m(\mathbf{t}) = \frac{\partial \widetilde{\Phi}(\mathbf{t})}{\partial t_0^0 \partial t_0^m}$$

satisfy the equations (2) with $x = t_0^0$ and u_j is related to v_m via (3) and, finally, $\widetilde{\Phi}(\mathbf{t})$ satisfies the so-called *string equation*

$$\frac{\partial\widetilde{\Phi}(\mathbf{t})}{\partial t_0^0} = \frac{1}{2}\sum_{m,n=0}^{r-2} \eta_{mn} t_0^m t_0^n + \sum_{k=0}^{\infty}\sum_{m=0}^{r-2} t_{k+1}^m \frac{\partial\widetilde{\Phi}(\mathbf{t})}{\partial t_k^m}, \tag{4}$$

where $\eta_{mn} = \delta_{m+n,r-2}$.

It can be shown that the potential $\widetilde{\Phi}(\mathbf{t})$ is uniquely determined by these conditions (cf. [28]).

2.3. Semiclassical approximation

The hierarchy KdV_r (2) has a semiclassical (or dispersionless) limit which is defined in terms of the formalism of Section 2.1 as follows. For a differential operator $L \in \mathscr{D}$ given by (1) denote by $\widetilde{L} = p^r - \sum_{m=0}^{r-2} u_m(x) p^m$ the polynomial in a formal variable p obtained by replacing D with p. The commutator $[L,Q]$ of differential operators will be replaced in (2) by the Poisson bracket

$$\{\widetilde{L},\widetilde{Q}\} = \frac{\partial\widetilde{L}}{\partial p}\frac{\partial\widetilde{Q}}{\partial x} - \frac{\partial\widetilde{Q}}{\partial p}\frac{\partial\widetilde{L}}{\partial x}.$$

The semiclassical limit of the KdV_r *hierarchy*, KdV_r^s is the system of equations

$$\frac{\partial\widetilde{L}}{\partial t_n^m} = \frac{k_{m,n}}{r}\left\{\widetilde{L^{n+\frac{m+1}{r}}},\widetilde{L}\right\} \tag{5}$$

in unknown functions $u_0,\dots,u_{r-2}$.

The corresponding potential function $\widetilde{\Phi}_0(\mathbf{t})$ is defined as the unique function satisfying the string equation (4), the condition $\widetilde{\Phi}_0(\mathbf{0}) = 0$, and such that the functions $u_0,\dots,u_{r-2}$ given by (3) and

$$v_m(\mathbf{t}) = \frac{\partial\widetilde{\Phi}_0(\mathbf{t})}{\partial t_0^0 \partial t_0^m}$$

satisfy the equations of the hierarchy (5).

3. The generalized Witten conjecture

In [28], Witten conjectured that the potential $\widetilde{\Phi}(\mathbf{t})$ of the KdV hierarchy is equal to a generating function for the intersection numbers of certain tautological classes on the moduli space of stable curves $\overline{\mathscr{M}}_{g,n}$. This conjecture arose from the study of physical realizations of pure topological gravity and provided an unexpected link between integrable systems and the geometry of these moduli spaces. This conjecture was proved by Kontsevich [21].

To be more precise, let $\overline{\mathscr{M}}_{g,n} := \{[\Sigma, x_1, \ldots, x_n]\}$ be the moduli space of stable curves of genus g with n marked points. That is, Σ is a conformal equivalence class of genus g Riemann surfaces, with (at worst) nodal singularities and $x_1, \ldots, x_n$ are marked points, distinct from each other and the nodes. These data are subject to the stability condition that they must have no infinitesimal automorphisms. This moduli space is a $(3g-3+n)$-dimensional, compact, complex orbifold and is a compactification of the moduli space of genus g Riemann surfaces with n marked points.

This space $\overline{\mathscr{M}}_{g,n}$ is equipped with tautological holomorphic line bundles $\mathscr{L}_{(g,n),i} \to \overline{\mathscr{M}}_{g,n}$ where $i = 1, \ldots, n$, such that the fiber of $\mathscr{L}_{(g,n),i}$ at the point $[\Sigma; x_1, \ldots, x_n]$ is the cotangent space $T^*_{x_i}\Sigma$. Define $\psi_i \in H^2(\overline{\mathscr{M}}_{g,n})$ to be $c_1(\mathscr{L}_{(g,n),i})$ for all $i = 1, \ldots, n$. Let

$$\left\langle \tau_{a_1} \ldots \tau_{a_n} \right\rangle_g := \int_{\overline{\mathscr{M}}_{g,n}} \psi_1^{a_1} \ldots \psi_n^{a_n}$$

for all nonnegative integers $a_1, \ldots, a_n \geq 0$. Let $\mathbf{t} := (t_0, t_1, \ldots)$ be formal variables and consider the function $\Phi_g(\mathbf{t}) \in \mathbb{C}[[\mathbf{t}]]$ defined by

$$\Phi_g(\mathbf{t}) := \langle \exp(\mathbf{t} \cdot \boldsymbol{\tau}) \rangle_g = \sum_{n \geq 0} \frac{t_{a_1} \cdots t_{a_n}}{n!} \left\langle \tau_{a_1} \cdots \tau_{a_n} \right\rangle_g \tag{6}$$

where $\mathbf{t} \cdot \boldsymbol{\tau} = \sum_{a=0}^{\infty} t_a \tau_a$, the expression $\langle \cdots \rangle_g$ is linear over $\mathbb{C}[[\mathbf{t}]]$, and the exponential function is a formal power series. The summation convention for repeated indices has also been used. Let $\Phi(\mathbf{t}) = \sum_{g \geq 0} \Phi_g(\mathbf{t})$.

Theorem 3.1. **(Kontsevich [21])** *The function* $\Phi(\mathbf{t})$ *defined in (6) is equal to the function* $\widetilde{\Phi}(\mathbf{t})$ *for* KdV_2.

Kontsevich's proof is fascinating, but it does not explain why such a theorem *should* hold. Kontsevich realizes the function $\widetilde{\Phi}(\mathbf{t})$ as a large N limit of an integral over the space of Hermitian $N \times N$ matrices, a so-called Hermitian matrix model. The terms in the resulting perturbative expansion are indexed by ribbon graphs which, in turn, can be related to a cell decomposition (constructed using Strebel differentials) of a closely related moduli space.

A natural question to ask is whether there is a moduli space and tautological classes on it, whose generating function assembles into the potential $\widetilde{\Phi}$ of the KdV_r hierarchy for general r. Witten [26, 27] stated a conjecture for this case as well. He expected the relevant theory to be topological gravity coupled to a topological matter sector corresponding to a class of topological minimal models which he realized in terms of an $\mathfrak{su}(2)_{r-2}/\mathfrak{u}(1)$ coset model. He suggested a construction of the relevant moduli space and tautological classes. However, the rigorous construction of the appropriate compact moduli space had not yet been completed at the time that the conjecture was formulated.

Recently, Jarvis [14, 15] constructed the moduli space, $\overline{\mathscr{M}}^{1/r}_{g,n}$, called *the moduli space of r-spin curves of genus g with n marked points*. This space is the disjoint union

$$\overline{\mathscr{M}}^{1/r}_{g,n} = \sqcup_{\mathbf{m}} \overline{\mathscr{M}}^{1/r,\mathbf{m}}_{g,n}$$

where the union is over n-tuples of nonnegative integers $\mathbf{m} = (m_1, \dots, m_n)$ where $0 \le m_i \le r-1$ for all $i = 1, \dots n$. (In fact, the moduli space $\overline{\mathscr{M}}^{1/r}_{g,n}$ is defined for general integral $\mathbf{m}$ but we will only consider those pieces relevant to the generalized Witten conjecture.)

The spaces $\overline{\mathscr{M}}^{1/r}_{g,n}$ are equipped with tautological classes $\psi_i \in H^2(\overline{\mathscr{M}}^{1/r}_{g,n})$ where $i = 1, \dots, n$. Each ψ_i is the Chern class of a tautological line bundle over $\overline{\mathscr{M}}^{1/r}_{g,n}$. Furthermore, Witten [26] sketched an analytic construction of a tautological class $c^{1/r}$ in $H^D(\overline{\mathscr{M}}^{1/r}_{0,n})$ where $D = \frac{1}{r}(2 - r + \sum_{i=1}^n m_i)$. The correlators

$$\left\langle \tau_{a_1}(e_{m_1}) \dots \tau_{a_n}(e_{m_n}) \right\rangle_g := r \int_{\overline{\mathscr{M}}^{1/r,\mathbf{m}}_{g,n}} c^{1/r} \psi_1^{a_1} \dots \psi_n^{a_n}$$

assemble into a generating function Φ, which is conjectured to coincide with the potential $\widetilde{\Phi}$ of KdV_r. The integral above is to be understood in the orbifold sense.[1]

In genus zero a rigorous algebro-geometric construction of this class was given in [16] (see also Section 4.3), and the following result holds.

Theorem 3.2. **(The Generalized Witten Conjecture in Genus Zero [16])** *Consider the generating function*

$$\Phi_0(\mathbf{t}) := \langle \exp(\mathbf{t} \cdot \boldsymbol{\tau}) \rangle_0 := \sum_{n \ge 3} \frac{t^{m_1}_{a_1} \cdots t^{m_n}_{a_n}}{n!} \left\langle \tau_{a_1}(e_{m_1}) \dots \tau_{a_n}(e_{m_n}) \right\rangle_0$$

in $\mathbb{Q}[[\mathbf{t}]]$ where $\mathbf{t}$ is the set of variables t^m_a for $a \ge 0$ and $m = 0, \dots, r-1$, $\mathbf{t} \cdot \boldsymbol{\tau} := \sum_{a,m} \tau_a(e_m) t^m_a$, and $a_i \ge 0$ and $0 \le m_i \le r-1$. The function $\Phi_0(\mathbf{t})$ which is in fact independent of t^{r-1}_a is equal to $\widetilde{\Phi}_0(\mathbf{t})$, the (semiclassical) potential of the KdV^s_r *hierarchy.*

The precise algebro-geometric formulation of this conjecture is, as yet, incomplete in higher genera since the class $c^{1/r}$ has not yet been constructed algebro-geometrically

More detailed descriptions of the above objects will be given in subsequent sections.

[1] The factor of r can be interpreted as coming from the string coupling constant.

4. The moduli space of curves with multiple spin structures

In this section we review some definitions and facts from [16] about spin curves, and then generalize these to define curves with multiple spin structures.

4.1. Definitions

Definition 4.1 *Let* $(X, p_1, \ldots, p_n)$ *be a nodal n-pointed algebraic curve, and let* $\mathscr{K}$ *be a rank-one, torsion-free sheaf on* X. *A* d*-th root of* $\mathscr{K}$ *of type* $\mathbf{m} = (m_1, \ldots, m_n)$ *is a pair* $(\mathscr{E}, b)$ *of a rank-one, torsion-free sheaf* $\mathscr{E}$*, and an* $\mathscr{O}_X$*-module homomorphism*

$$b : \mathscr{E}^{\otimes d} \longrightarrow \mathscr{K} \otimes \mathscr{O}_X(-\sum m_i p_i)$$

with the following properties:

- $d \cdot \deg \mathscr{E} = \deg \mathscr{K} - \sum m_i$;
- *b is an isomorphism on the locus of* X *where* $\mathscr{E}$ *is locally free;*
- *for every point* $p \in X$ *where* $\mathscr{E}$ *is not free, the length of the cokernel of b at p is* $d-1$.

For any d-th root $(\mathscr{E}, b)$ of type $\mathbf{m}$, and for any $\mathbf{m}'$ congruent to $\mathbf{m} \mod d$, we can construct a unique d-th root $(\mathscr{E}', b')$ of type $\mathbf{m}'$ simply by taking $\mathscr{E}' = \mathscr{E} \otimes \mathscr{O}(1/d \sum (m_i - m_i') p_i)$. Consequently, the moduli of curves with d-th roots of a bundle $\mathscr{K}$ of type $\mathbf{m}$ is canonically isomorphic to the moduli of curves with d-th roots of type $\mathbf{m}'$. Therefore, unless otherwise stated, we will always assume the type $\mathbf{m}$ of a d-th root has the property that $0 \le m_i < d$ for all i.

Unfortunately, the moduli space of curves with d-th roots of a fixed sheaf $\mathscr{K}$ is not smooth when d is not prime, and so we must consider not just roots of a bundle, but rather coherent nets of roots [14]. This additional structure suffices to make the moduli space of curves with a coherent net of roots smooth.

Definition 4.2 *Let* $\mathscr{K}$ *be a rank-one, torsion-free sheaf on a nodal n-pointed curve* $(X, p_1, \ldots, p_n)$. *A* coherent net of r-th roots *of* $\mathscr{K}$ *of type* $\mathbf{m} = (m_1, \ldots, m_n)$ *consists of the following data:*

- *For every divisor d of r, a rank-one torsion-free sheaf* $\mathscr{E}_d$ *on* X*;*
- *For every pair of divisors* d', d *of r, such that* d' *divides d, an* $\mathscr{O}_X$*-module homomorphism*
$$c_{d,d'} : \mathscr{E}_d^{\otimes d/d'} \longrightarrow \mathscr{E}_{d'}.$$

These data are subject to the following restrictions.

1. *$\mathscr{E}_1 = \mathscr{K}$ and $c_{1,1} = \mathbf{1}$.*

2. *For each divisor d of r and each divisor d' of d, let $\mathbf{m}'' = (m''_1, \ldots, m''_n)$ be such that m''_i is the unique non-negative integer less than d/d', and congruent to m_i $\bmod d/d'$. Then the homomorphism $c_{d,d'}$ makes $(\mathscr{E}_d, c_{d,d'})$ into a d/d' root of $\mathscr{E}_{d'}$ of type $\mathbf{m}''$.*

3. *The homomorphisms $\{c_{d,d'}\}$ are compatible. That is, the diagram*

$$\begin{tikzcd}
(\mathscr{E}_d^{\otimes d/d'})^{\otimes d'/d''} \arrow[r, "(c_{d,d'})^{\otimes d'/d''}"] \arrow[dr, "c_{d,d''}"'] & \mathscr{E}_{d'}^{\otimes d'/d''} \arrow[d, "c_{d',d''}"] \\
 & \mathscr{E}_{d''}
\end{tikzcd}$$

commutes for every $d''|d'|d|r$.

If r is prime, then a coherent net of r-th roots is simply an r-th root of $\mathscr{K}$. Moreover, if $\mathscr{E}_d$ is locally free of type $\mathbf{m}$, then for $d'|d$, if $\mathbf{m} \equiv \mathbf{m}' \bmod d'$ and $0 \le m'_i < d'$, then the sheaf $\mathscr{E}_{d'}$ is isomorphic to $\mathscr{E}_d^{\otimes d/d'} \otimes \mathscr{O}(\frac{1}{d'}\sum(m_i - m'_i)p_i)$ and $c_{d,d'}$ is uniquely determined up to an automorphism of $\mathscr{E}_{d'}$.

Definition 4.3 *An n-*pointed *r-*spin curve of type $\mathbf{m} = (m_1, \ldots, m_n)$ *is an n-pointed nodal curve $(X, p_1, \ldots, p_n)$ with a coherent net of r-th roots of ω_X of type $\mathbf{m}$, where ω_X is the canonical (dualizing) sheaf of X. An r-spin curve is called* smooth*, if X is smooth, and it is called* stable*, if X is stable.*

Example 4.4 Smooth 2-spin curves of type $\mathbf{m} = \mathbf{0}$ correspond to classical spin curves (a curve and a theta-characteristic) *together with an explicit isomorphism* $\mathscr{E}_2^{\otimes 2} \xrightarrow{\sim} \omega$.

Definition 4.5 *An isomorphism of r-spin curves*

$$(X, p_1, \ldots, p_n, \{\mathscr{E}_d, c_{d,d'}\}) \xrightarrow{\sim} (X', p'_1, \ldots, p'_n, \{\mathscr{E}'_d, c'_{d,d'}\})$$

of the same type $\mathbf{m}$ is an isomorphism of pointed curves

$$\tau : (X, p_1, \ldots, p_n) \xrightarrow{\sim} (X', p'_1, \ldots, p'_n)$$

and a family of isomorphisms $\{\beta_d : \tau^*\mathscr{E}'_d \xrightarrow{\sim} \mathscr{E}_d\}$, with β_1 being the canonical isomorphism $\tau^*\omega_{X'}(-\sum_i m_i p'_i) \xrightarrow{\sim} \omega_X(-\sum m_i p_i)$, and such that the β_d are compatible with all the maps $c_{d,d'}$ and $\tau^* c'_{d,d'}$.

Every r-spin structure on a smooth curve X is determined up to isomorphism by a choice of a line bundle $\mathscr{E}_r$, such that $\mathscr{E}_r^{\otimes r} \cong \omega_X(-\sum m_i p_i)$. In particular, if X has no automorphisms, then the set of isomorphism classes of r-spin structures of a fixed type $\mathbf{m}$ is a principal $\mathrm{Jac}\,_r X$-bundle, where $\mathrm{Jac}\,_r X$ is the group of r-torsion points of the Jacobian of X, thus there are r^{2g} such isomorphism classes.

Example 4.6 If $g = 1$ and $\mathbf{m} = \mathbf{0}$, then $\omega_X \cong \mathscr{O}_X$ and a smooth r-spin curve is just an r-torsion point of the elliptic curve X, again with an explicit isomorphism of $\mathscr{E}_r^{\otimes r} \xrightarrow{\sim} \mathscr{O}_X$. In particular, the stack of stable r-spin curves of genus one and type $\mathbf{0}$ forms a gerbe over the disjoint union of modular curves $\coprod_{d|r} X_1(d)$.

Definition 4.7 *The moduli space of stable n-pointed, r-spin curves of genus g and type $\mathbf{m}$ is denoted $\overline{\mathscr{M}}^{1/r,\mathbf{m}}_{g,n}$. The disjoint union $\coprod_{\substack{\mathbf{m}\\ 0\le m_i<r}} \overline{\mathscr{M}}^{1/r,\mathbf{m}}_{g,n}$ is denoted $\overline{\mathscr{M}}^{1/r}_{g,n}$.*

Remark 4.8 As mentioned above, no geometric information is lost by restricting $\mathbf{m}$ to the range $0 \le m_i \le r-1$, since when $\mathbf{m} \equiv \mathbf{m}' \mod r$ every r-spin curve of type $\mathbf{m}$ naturally gives an r-spin curve of type $\mathbf{m}'$ simply by

$$\mathscr{E}_d \mapsto \mathscr{E}_d \otimes \mathscr{O}(\sum \frac{m_i - m'_i}{d} p_i),$$

and thus $\overline{\mathscr{M}}^{1/r,\mathbf{m}}_{g,n}$ is canonically isomorphic to $\overline{\mathscr{M}}^{1/r,\mathbf{m}'}_{g,n}$.

More generally, if $\mathbf{r} = (r^1, \ldots, r^k)$ is a k-tuple of integers $r^i \ge 2$, and if $\bar{\mathbf{m}} := (\mathbf{m}^1, \ldots, \mathbf{m}^k)$, $\mathbf{m}^j := (m^j_1, \ldots, m^j_n)$, for nonnegative integers m^j_i we define $\mathbf{r}$-spin curves of type $\bar{\mathbf{m}}$ as follows:

Definition 4.9 *An n-pointed $\mathbf{r}$-spin curve of type $\bar{\mathbf{m}}$ is an n-pointed, nodal curve $(X, p_1, \ldots, p_n)$ with a k-tuple of $\mathbf{r}$-spin structures of type $\bar{\mathbf{m}}$, that is, a k-tuple whose i-th term is a coherent net $\mathscr{N}_i := \{\mathscr{E}_{d^i}, c_{d^i,d^{i'}}\}$ of r^i-th roots of ω_X of type $\mathbf{m}^i = (m^i_1, \ldots, m^i_n)$. Here, of course, the d^i run over all positive integers dividing r^i, and the $d^{i'}$ run over all positive divisors of d^i.*

Definition 4.10 *We denote by $\overline{\mathscr{M}}^{\mathbf{1/r},\bar{\mathbf{m}}}_{g,n}$ the moduli space of genus g, n-pointed, stable $\mathbf{r}$-spin curves of type $\bar{\mathbf{m}}$ and we denote by $\overline{\mathscr{M}}^{\mathbf{1/r}}_{g,n}$ the disjoint union*

$$\overline{\mathscr{M}}^{\mathbf{1/r}}_{g,n} := \coprod_{\substack{\bar{\mathbf{m}}=(\mathbf{m}^1,\ldots,\mathbf{m}^k)\\ \mathbf{m}^i=(m^i_1,\ldots,m^i_n)\\ 0\le m^i_j<r^i}} \overline{\mathscr{M}}^{\mathbf{1/r},\bar{\mathbf{m}}}_{g,n} .$$

4.2. Basic properties of the moduli space $\overline{\mathscr{M}}^{\mathbf{1/r},\bar{\mathbf{m}}}_{g,n}$

These spaces are endowed with a number of canonical morphisms, including the projections

$$q_i : \overline{\mathscr{M}}^{\mathbf{1/r},\bar{\mathbf{m}}}_{g,n} \longrightarrow \overline{\mathscr{M}}^{1/r^i,\mathbf{m}^i}_{g,n}$$

and

$$p_i : \overline{\mathscr{M}}^{1/r^i,\mathbf{m}^1}_{g,n} \longrightarrow \overline{\mathscr{M}}_{g,n}.$$

It is clear that

$$\overline{\mathscr{M}}^{\mathbf{1/r},\bar{\mathbf{m}}}_{g,n} = \prod_{i=1}^{k} \overline{\mathscr{M}}^{1/r^i,\mathbf{m}^i}_{g,n} \tag{7}$$

where $\prod_i$ denotes the fibered product over $\overline{\mathscr{M}}_{g,n}$ with respect to the maps p_i.

In [14] it is shown that $\overline{\mathscr{M}}^{1/r,\mathbf{m}}_{g,n}$ is a smooth Deligne-Mumford stack (orbifold), finite over $\overline{\mathscr{M}}_{g,n}$, with a projective coarse moduli space. Consequently, $\overline{\mathscr{M}}^{\mathbf{1/r},\bar{\mathbf{m}}}_{g,n}$ has the same properties.

There is one other canonical morphism associated to these spaces; namely, when d divides r, the morphism

$$[r/d] : \overline{\mathscr{M}}^{1/r,\mathbf{m}}_{g,n} \longrightarrow \overline{\mathscr{M}}^{1/d,\mathbf{m}''}_{g,n} \text{ and } [r/d] : \overline{\mathscr{M}}^{1/r}_{g,n} \longrightarrow \overline{\mathscr{M}}^{1/d}_{g,n},$$

which forgets all of the roots and homomorphisms in the net of r-th roots except those associated to divisors of d. Here, as above, $\mathbf{m}''$ is congruent to $\mathbf{m} \mod d$ and has $0 \leq m''_i < d$. In the case that the underlying curve is smooth, this is equivalent to replacing the line bundle $\mathscr{E}_r$ by its r/d-th tensor power (and tensoring with $\mathscr{O}(1/d\sum(m_i - m''_i)p_i)$).

Remark 4.11 It is important to keep in mind that the space $\overline{\mathscr{M}}^{1/rs}_{g,n}$ is not isomorphic to $\overline{\mathscr{M}}^{(1/r,1/s)}_{g,n} = \overline{\mathscr{M}}^{1/r}_{g,n} \times_{\overline{\mathscr{M}}_{g,n}} \overline{\mathscr{M}}^{1/s}_{g,n}$, even when r and s are relatively prime. However, on the open locus of smooth spin curves we do have $\mathscr{M}^{1/rs}_{g,n} \cong \mathscr{M}^{(1/r,1/s)}_{g,n}$, whenever r and s are relatively prime [14].

Throughout this paper we will denote the universal curve by $\pi : \mathscr{C}^{\mathbf{1/r}}_{g,n} \longrightarrow \overline{\mathscr{M}}^{\mathbf{1/r}}_{g,n}$. As in the case of the moduli space of stable curves, the universal curve possesses canonical sections $\sigma_i : \overline{\mathscr{M}}^{\mathbf{1/r}}_{g,n} \longrightarrow \mathscr{C}^{\mathbf{1/r}}_{g,n}$ for $i = 1, \ldots, n$.

4.3. Cohomology classes

As in the case of stable maps and stable curves we have some tautological cohomology classes. Most important for our purposes are the classes

$$\psi_i := c_1(\mathscr{L}_{(g,n),i}) = c_1(\sigma_i^*(\omega_\pi)),$$

the first Chern class of the restriction of the canonical (relative dualizing) sheaf ω_π to the image of the i-th section. Of course, ψ_i on $\overline{\mathscr{M}}^{\mathbf{1/r},\bar{\mathbf{m}}}_{g,n}$ is just the pullback $p^*\psi_i$

of the usual class ψ_i on $\overline{\mathscr{M}}_{g,n}$ because $\mathscr{L}_{(g,n),i} = \sigma_i^*(\omega_\pi) = p^*\sigma_i^*(\omega_{(g,n)})$, where $\omega_{(g,n)}$ is the relative dualizing sheaf of the universal curve over $\overline{\mathscr{M}}_{g,n}$.

In addition to the tautological classes, we will also need a class $c^{1/r}$ which plays the role of Gromov-Witten classes $(ev_1^*(\gamma_1)\cup\cdots\cup ev_n^*(\gamma_n))\cap([\overline{\mathscr{M}}_{g,n}(V,\beta)]^{virt})$ in the case of stable maps.

The class $c^{1/r}$ will be called the *virtual class*, and in the case of $g=0$ it has a simple description as the Euler class of the pushforward of the r-th root bundle from the universal curve to $\overline{\mathscr{M}}^{1/r}_{g,n}$.

Definition 4.12 *If $g=0$, let $\mathscr{E}^i_{r^i}$ be the r^i-th root of ω_π associated to the i-th net $\mathscr{N}_i$ of the universal $\mathbf{r}$-spin structure $(\mathscr{N}_1,\ldots,\mathscr{N}_k)$ on the universal curve $\pi: \mathscr{C}^{\mathbf{1/r}}_{0,n} \longrightarrow \overline{\mathscr{M}}^{\mathbf{1/r}}_{0,n}$. Let $c^{\mathbf{1/r}}_{0,n}$ be the Euler class (top Chern class) of the pushforward of $\mathscr{E}^1_{r^1}\oplus\cdots\oplus\mathscr{E}^k_{r^k}$*

$$c^{\mathbf{1/r}}_{0,n} = e(-R\pi_*(\mathscr{E}^1_{r^1}\oplus\cdots\oplus\mathscr{E}^k_{r^k})).$$

Here $R\pi_*\mathscr{F}$ is the K-theoretic pushforward $\pi_*\mathscr{F} - R^1\pi_*\mathscr{F}$. This class is well defined because in genus zero $\mathscr{E}^i_{r^i}$ is *convex* [16], that is $\pi_*\mathscr{E}^i_{r^i}=0$, and so $R^1\pi_*\mathscr{E}^i_{r^i}$ is a vector bundle of rank $D_i = \chi(\mathscr{E}^i_{r^i}) = ((2-r^i)+\sum_j m^i_j)/r^i$. Thus $R\pi_*(\mathscr{E}^1_{r^1}\oplus\cdots\oplus\mathscr{E}^k_{r^k})$ is also a vector bundle and has a top Chern class.

Let us consider the case where $k=1$ and $r\geq 2$. The class $c^{1/r}$ on $\overline{\mathscr{M}}^{1/r}_{0,n}$ defined above is precisely the class described in Theorem 3.2 (the generalized Witten conjecture).

Lemma 4.13 *When the moduli space $\overline{\mathscr{M}}^{\mathbf{1/r}}_{g,n}$ of curves with multiple spin structures is viewed as a fibered product, as in (7), then the virtual class $c^{\mathbf{1/r}}$ is the product of the pullbacks of the virtual classes from each of the factors:*

$$c^{\mathbf{1/r}} = q_1^*c^{1/r^1}\cup q_2^*c^{1/r^2}\cup\cdots\cup q_n^*c^{1/r^n}.$$

Proof. Since the maps $p_i: \overline{\mathscr{M}}^{1/r^i}_{g,n} \longrightarrow \overline{\mathscr{M}}_{g,n}$ are flat, since $\overline{\mathscr{M}}^{\mathbf{1/r}}_{g,n}$ is a fibered product, and since $\mathscr{E}^i_{r^i}$ is the pullback of the universal r^i-th root on $\mathscr{C}^{1/r^i}_{0,n}$ via q_i, we have

$$\begin{aligned} c^{\mathbf{1/r}} &= e(-(R\pi_*(\mathscr{E}^1_{r^1}\oplus\cdots\oplus\mathscr{E}^k_{r^k}))) \\ &= e(-(q_1^*R\pi_*\mathscr{E}^1_{r^1}\oplus\cdots\oplus q_k^*R\pi_*\mathscr{E}^k_{r^k})) \\ &= q_1^*c^{1/r^1}\cup q_2^*c^{1/r^2}\cup\ldots q_n^*c^{1/r^n}. \end{aligned}$$

■

Remark 4.14 In [16] it is shown that in genus zero the virtual class $c^{1/r}$ has a number of properties resembling those of the Gromov-Witten classes. One additional property that is unique to this construction, and which plays an important role, is the fact that $c^{1/r}$ vanishes on $\overline{\mathscr{M}}^{\mathbf{1/r,m}}_{g,n}$ whenever any one of the m^i_j is equal to $r^i - 1$. Also it vanishes in some other special cases.

5. Cohomological field theories and Frobenius manifolds

5.1. Operads and cohomological field theories

The objects defined in the previous section allow a construction of a $g = 0$ cohomological field theory (CohFT) whose potential function coincides with the potential function of KdV$_r$ and gives rise to an associated $(r-1)$ dimensional Frobenius manifold. It is precisely this property, together with the topological recursion relations (TRR) which was used to prove the generalized Witten conjecture in genus zero. It is also the framework in which the notion of tensor product most readily appears. These constructions fit into the framework of stable graphs and operads which we will now describe.

Consider a connected graph Γ with n tails (or legs) indexed by numbers $1,\dots,n$, such that each vertex v is labeled by a nonnegative integer $g(v)$ (called the *genus of* v). The *genus* $g(\Gamma)$ of Γ is defined as $g(\Gamma) = \sum_{v\in V(\Gamma)} g(v) + b_1(\Gamma)$, where $V(\Gamma)$ is the set of vertices of Γ and $b_1(\Gamma)$ is the first Betti number (number of loops) of Γ. Such a graph Γ is called a *stable graph* of genus g with n tails if for every $v \in V(\Gamma)$ with $g(v) = 0$ its valence (denoted by $n(v)$) satisfies $n(v) \geq 3$ and if $g(v) = 1$ then $n(v) \geq 1$. In particular, genus zero stable graphs are trees whose vertices are all labeled by 0.

Each stable curve $[\Sigma, x_1, \dots, x_n] \in \overline{\mathscr{M}}_{g,n}$ is associated to a stable graph of genus g with n tails in the following way.

To each irreducible component, associate a vertex, and draw as many half-edges emerging from this vertex as there are *special points* (either marked points or nodes) on that component. Label vertex v by an integer $g(v)$ equal to the genus of that irreducible component. Connect any two half edges if they are associated to the same node. Finally, label the remaining half-edges (tails) by an integer from $1,\dots,n$ corresponding to the associated marked point.

Let Γ be a stable graph of genus g with n tails. The stratification of $\overline{\mathscr{M}}_{g,n}$ yields *gluing maps*

$$\rho_\Gamma : \prod_{v\in V(\Gamma)} \overline{\mathscr{M}}_{g(v),n(v)} \to \overline{\mathscr{M}}_{g,n}.$$

This map is obtained by composing the maps $\chi : \prod_{v\in V(\Gamma)} \overline{\mathscr{M}}_{g(v),n(v)} \to \overline{\mathscr{M}}_\Gamma$ and $i : \overline{\mathscr{M}}_\Gamma \to \overline{\mathscr{M}}_{g,n}$. The space $\overline{\mathscr{M}}_\Gamma$ is the closure of the locus of points in $\overline{\mathscr{M}}_{g,n}$ whose associated stable graph is Γ. The map χ is the quotient map corresponding to the action of $\mathrm{Aut}(\Gamma)$, the automorphism group of Γ, and i is the inclusion map. The symmetric group $S_{n(v)}$ acts on the space $\overline{\mathscr{M}}_{g(v),n(v)}$ by permuting the marked points. Similarly, S_n acts on $\overline{\mathscr{M}}_{g,n}$. Clearly, the map ρ_Γ is equivariant under these actions.

The collection $\overline{\mathscr{M}} := \{\overline{\mathscr{M}}_{g,n}\}$, together with the symmetric group actions and the maps ρ_Γ, is the model example of a *modular operad* [11]. Restricting to genus zero stable graphs, one obtains a *cyclic operad*, a refinement of the notion of an operad. The modular operad structure on $\overline{\mathscr{M}}$ induces a modular operad structure on the homology $H_\bullet(\overline{\mathscr{M}}) := \{H_\bullet(\overline{\mathscr{M}}_{g,n})\}$. From this perspective, a cohomological field theory (CohFT) is simply a vector space $\mathscr{H}$ with a metric η, which is an algebra over $H_\bullet(\overline{\mathscr{M}})$. The algebra maps $H_\bullet(\overline{\mathscr{M}}_{g,n}) \to T^n\mathscr{H}^*$ (where $T^n\mathscr{H}^*$ denotes the n-fold tensor product of $\mathscr{H}^*$) are the correlators of the theory. However, in the context of algebraic geometry, it turns out to be more useful to use the dual definition of a cohomological field theory, which is given in terms of cohomology.

Definition 5.1 *A* cohomological field theory (CohFT) of rank d *(denoted by* $(\mathscr{H},\eta,\Lambda)$ *or just* $(\mathscr{H},\eta)$*) is a d-dimensional vector space $\mathscr{H}$ with a metric η and a collection of forms $\Lambda := \{\Lambda_{g,n}\}$ where*

$$\Lambda_{g,n} \in H^\bullet(\overline{\mathscr{M}}_{g,n}) \otimes T^n\mathscr{H}^* = Hom(T^n\mathscr{H}, H^\bullet(\overline{\mathscr{M}}_{g,n})) \tag{8}$$

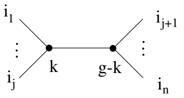
are defined for stable pairs (g,n) and satisfy the following axioms **C1–C3**. *Let $\{e_0,\dots,e_{d-1}\}$ be a fixed basis of $\mathscr{H}$, and let $\eta^{\mu\nu}$ be the inverse of the matrix of the metric η in this basis. We use the summation convention in the following.*

C1. *The form $\Lambda_{g,n}$ is invariant under the action of the symmetric group S_n.*

C2. *Let*

$$\rho_{\Gamma_{\text{tree}}} : \overline{\mathscr{M}}_{k,j+1} \times \overline{\mathscr{M}}_{g-k,n-j+1} \longrightarrow \overline{\mathscr{M}}_{g,n} \tag{9}$$

be the gluing map corresponding to the stable graph

$\Gamma_{\text{tree}} =$ 

then the forms $\Lambda_{g,n}$ satisfy the composition property

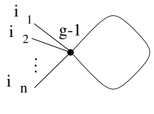
$$\begin{aligned} &\rho^*_{\Gamma_{\text{tree}}}\Lambda_{g,n}(\gamma_1,\gamma_2\dots,\gamma_n) \;= \\ &\Lambda_{k,j+1}(\gamma_{i_1},\dots,\gamma_{i_j},e_\mu)\eta^{\mu\nu} \;\otimes\; \Lambda_{g-k,n-j+1}(e_\nu,\gamma_{i_{j+1}},\dots,\gamma_{i_n}) \end{aligned} \tag{10}$$

for all $\gamma_i \in \mathscr{H}$.

C3. *Let*

$$\rho_{\Gamma_{\text{loop}}} : \overline{\mathscr{M}}_{g-1,n+2} \longrightarrow \overline{\mathscr{M}}_{g,n} \tag{11}$$

be the gluing map corresponding to the stable graph

$\Gamma_{\text{loop}} =$  (12)

then

$$\rho^*_{\Gamma_{\text{loop}}} \Lambda_{g,n}(\gamma_1, \gamma_2, \ldots, \gamma_n) = \Lambda_{g-1,n+2}(\gamma_1, \gamma_2, \ldots, \gamma_n, e_\mu, e_\nu)\, \eta^{\mu\nu}. \tag{13}$$

The pair $(\mathscr{H}, \eta)$ *is called the* state space *of the CohFT. The maps* $H_\bullet(\overline{\mathscr{M}}_{g,n}) \to T^n\mathscr{H}^*$ *given by* $[c] \mapsto \int_{[c]} \Lambda_{g,n}$ *are called the* correlators of the CohFT.

An element $e_0 \in \mathscr{H}$ *is called* a flat identity *of the CohFT, if in addition, the following equations hold.*

C4a. *For all* γ_i *in* $\mathscr{H}$ *we have*

$$\Lambda_{g,n+1}(\gamma_1, \ldots, \gamma_n, e_0) = \pi^* \Lambda_{g,n}(\gamma_1, \ldots, \gamma_n), \tag{14}$$

where $\pi : \overline{\mathscr{M}}_{g,n+1} \to \overline{\mathscr{M}}_{g,n}$ *is the universal curve on* $\overline{\mathscr{M}}_{g,n}$ *and*

C4b.

$$\int_{\overline{\mathscr{M}}_{0,3}} \Lambda_{0,3}(\gamma_1, \gamma_2, e_0) = \eta(\gamma_1, \gamma_2). \tag{15}$$

A CohFT with flat identity is denoted by $(\mathscr{H}, \eta, \Lambda, e_0)$.

A genus g CohFT *on the state space* $(\mathscr{H}, \eta)$ *is the collection of forms* $\{\Lambda_{\tilde{g},n}\}_{\tilde{g} \leq g}$ *that satisfy only those of the equations (10), (13), (14), and (15), where* $\tilde{g} \leq g$.

Remark 5.2 The state space $\mathscr{H}$ of a CohFT can, in general, be $\mathbb{Z}_2$-graded, but for simplicity we will assume that $\mathscr{H}$ contains only even elements as is the case for KdV_r.

Let Γ be a genus g stable graph with n tails, then, since the map ρ_Γ can be constructed from gluing morphisms (9) and (11), the forms $\Lambda_{g,n}$ satisfy the restriction property

$$\rho^*_\Gamma \Lambda_{g,n} = \eta^{-1}_\Gamma \Big(\bigotimes_{v \in V(\Gamma)} \Lambda_{g(v),n(v)} \Big) \tag{16}$$

where

$$\eta^{-1}_\Gamma : \bigotimes_{v \in V(\Gamma)} T^{n(v)}\, \mathscr{H}^* \to T^n\, \mathscr{H}^*$$

contracts the factors $T^n\mathscr{H}^*$ by means of the inverse of the metric η and successive application of equations (10) and (13) .

Definition 5.3 *The* potential function *of the CohFT* $(\mathscr{H}, \eta, \Lambda)$ *is a formal series* $\Phi \in \mathbb{C}[[\mathscr{H}]]$ *given by*

$$\Phi(\mathbf{x}) := \sum_{g=0}^{\infty} \Phi_g(\mathbf{x}), \tag{17}$$

where

$$\Phi_g(\mathbf{x}) := \sum_n \frac{1}{n!} \int_{\overline{\mathscr{M}}_{g,n}} \langle \Lambda_{g,n}, \mathbf{x}^{\otimes n} \rangle .$$

Here $\langle \cdots \rangle$ *denotes evaluation, the sum over n is understood to be over the stable range, and* $\mathbf{x} = \sum_\alpha x^\alpha e_\alpha$*, where* $\{ e_\alpha \}$ *is a basis of* $\mathscr{H}$*.*[2]

All of the information of a genus zero CohFT is encoded in its potential.

Theorem 5.4. **([22, 24])** *An element* Φ_0 *in* $\mathbb{C}[[\mathscr{H}]]$ *is the potential of a rank d, genus zero CohFT* $(\mathscr{H}, \eta)$ *if and only if it contains only terms which are of cubic and higher order in the coordinates* $x^0, \dots, x^{d-1}$ *(corresponding to a basis* $\{ e_0, \dots e_{d-1} \}$ *of* $\mathscr{H}$*) and it satisfies the* associativity, or WDVV *(Witten-Dijkgraaf-Verlinde*[2]*) equation*

$$\partial_a \partial_b \partial_e \Phi_0 \, \eta^{ef} \, \partial_f \partial_c \partial_d \Phi_0 = \partial_b \partial_c \partial_e \Phi_0 \, \eta^{ef} \, \partial_f \partial_a \partial_d \Phi_0,$$

where η^{ab} *is the inverse of the matrix of* η *in the basis* $\{e_a\}$*,* ∂_a *is derivative with respect to* x^a*, and the summation convention has been used.*

Conversely, a genus zero CohFT structure on $(\mathscr{H}, \eta)$ *is uniquely determined by its potential* Φ_0*, which must satisfy the WDVV equation.*

A formal power series Φ which satisfies the WDVV equations determines a *formal Frobenius manifold*, so the theorem shows that a genus zero CohFT with flat identity is equivalent to endowing the state space $(\mathscr{H}, \eta)$ with the structure of a formal Frobenius manifold [5, 13, 24]. The theorem follows from the work of Keel [20] who proved that $H^\bullet(\overline{\mathscr{M}}_{0,n})$ is generated by boundary classes, and that all relations between boundary divisors arise from lifting the basic codimension one relation on $\overline{\mathscr{M}}_{0,4}$.

5.2. Gromov-Witten invariants

The best known examples of CohFTs come from the Gromov-Witten classes of a smooth, projective variety V, where the state space $\mathscr{H}$ is $H^\bullet(V)$, and η is the Poincaré pairing. Let $\overline{\mathscr{M}}_{g,n}(V)$ be the moduli space of stable, genus g maps into V with n marked points, a compactification of the moduli space of holomorphic maps $f : \Sigma \to V$ from a genus g Riemann surface Σ with n marked points $(x_1, \dots, x_n)$ into V. There are canonical evaluation maps $\mathrm{ev}_i : \overline{\mathscr{M}}_{g,n}(V) \to V$ corresponding to evaluating f at x_i. There is also the stabilization morphism $\mathrm{st} : \overline{\mathscr{M}}_{g,n}(V) \to \overline{\mathscr{M}}_{g,n}$ which "forgets" f. Finally, the space $\overline{\mathscr{M}}_{g,n}(V)$ has a virtual fundamental class $[\overline{\mathscr{M}}_{g,n}(V)]^{\mathrm{virt}}$ in the Chow group of $\overline{\mathscr{M}}_{g,n}(V)$. In this situation, $(H^\bullet(V), \eta, \Lambda, \mathbf{1})$ is a CohFT with flat identity, where $\mathbf{1}$ is the usual unit element and

$$\Lambda_{g,n}(\gamma_1, \dots, \gamma_n) := \mathrm{st}_*(\mathrm{ev}_1^* \gamma_1 \cup \ldots \cup \mathrm{ev}_n^* \gamma_n \cap [\overline{\mathscr{M}}_{g,n}(V)]^{\mathrm{virt}})$$

[2] The *string coupling constant* factor of λ^{2g-2} in front of $\Phi_g(\mathbf{x})$ has been suppressed to avoid notational clutter.

for $\gamma_i \in H^\bullet(V)$. In the case of $g = 0$, the correlators endow $H^\bullet(V)$ with a multiplication, which is a deformation of the cup product known as the quantum cohomology of V.

The *large phase space potential* $\Phi(\mathbf{t})$ is the sum of formal power series $\Phi_g(\mathbf{t})$ for $g \geq 0$ where

$$\Phi(\mathbf{t})_g := \langle \exp(\mathbf{t}\cdot\boldsymbol{\tau}) \rangle_g = \sum \frac{t_{a_1}^{m_1}\cdots t_{a_n}^{m_n}}{n!} \left\langle \tau_{a_1}(e_{m_1})\cdots\tau_{a_n}(e_{m_n})\right\rangle_g$$

for a given basis $\{e_m\}$ of $H^\bullet(V)$ and

$$\left\langle \tau_{a_1}(e_{m_1})\cdots\tau_{a_n}(e_{m_n})\right\rangle_g = \int_{[\overline{\mathscr{M}}_{g,n}(V)]^{\mathrm{virt}}} \mathrm{ev}_1^* e_{m_1}\psi_1^{a_1}\ldots \mathrm{ev}_n^* e_{m_n}\psi_n^{a_n}.$$

Setting t_a^m to zero and $x^m := t_0^m$, one recovers the potential $\Phi(\mathbf{x})$ of the CohFT associated to Λ which is the generating function for the Gromov-Witten invariants of V.[3]

If V is a point then $\overline{\mathscr{M}}_{g,n}(V)$ reduces to $\overline{\mathscr{M}}_{g,n}$ and one has Kontsevich's theorem, which identifies Φ with the potential of the KdV_2 hierarchy. For general V a similar relation is expected, but the corresponding integrable system has only been described in $g = 0$ [5], and in some cases, in $g = 1$ [7]. However, there is a related conjecture by Eguchi, Hori, Xiong, and S. Katz [8], which states that the exponential of $\Phi(\mathbf{t})$ satisfies a highest weight condition for an action of the Virasoro algebra. The evidence for this conjecture is growing [9, 10, 12].

5.3. The KdV$_r$ Frobenius manifold

Consider an integer $r \geq 2$. Let $(\mathscr{H}^{(r)}, \eta)$ be an $(r-1)$-dimensional vector space $\mathscr{H}^{(r)}$ with basis $\{e_0, \ldots e_{r-2}\}$ and a metric $\eta(e_{m'}, e_{m''}) = \delta_{m'+m'', r-2}$ such that $m', m'' = 0, \ldots, r-2$.

Theorem 5.5. **([16])** *For each integer $r \geq 2$, the pair $(\mathscr{H}^{(r)}, \eta)$ is a Frobenius manifold (and thus defines a $g = 0$ CohFT) with flat identity e_0, Euler vector field*

$$E := \sum_{m=0}^{r-2} \left(\frac{m}{r} - 1\right) x^m \frac{\partial}{\partial x^m}$$

and

$$\Lambda_{0,n}(e_{m_1}, \ldots, e_{m_n}) = r p_* c^{1/r}$$

where $p : \overline{\mathscr{M}}_{g,n}^{1/r,\mathbf{m}} \to \overline{\mathscr{M}}_{g,n}$, $\mathbf{m} := (m_1, \ldots, m_n)$, and $0 \leq m_i \leq r-2$. The corresponding potential function $\Phi_0(\mathbf{x})$ (of the Frobenius manifold) is given by the formula

$$\Phi_0(\mathbf{x}) := \sum_{n\geq 3} \frac{1}{n!} x^{m_1}\ldots x^{m_n}\, r \int_{\overline{\mathscr{M}}_{0,n}^{1/r,\mathbf{m}}} c^{1/r},$$

[3] To avoid notational clutter, factors of q^β where $\beta \in H_2(V,\mathbb{Z})$ in the Novikov ring have been suppressed.

where $\mathbf{x} := (x^0, \ldots, x^{r-2})$ *are coordinates corresponding to the basis* $\{e_0, \ldots, e_{r-2}\}$ *and* $\mathbf{m} := (m_1, \ldots, m_n)$. *Summation over* $m_i = 0, \ldots, r-2$ *is assumed, and the integral is understood in the sense of orbifolds.*

Furthermore, the potential $\Phi_0(\mathbf{x})$ *is equal to the large phase space potential* $\Phi_0(\mathbf{t})$ *from Theorem 3.2 when* t_a^m *is set to zero for* $a > 0$ *and* $x^m := t_0^m$ *for all* $m = 0 \ldots, r-2$. *The potential* $\Phi_0(\mathbf{t})$ *satisfies the* $g = 0$ *topological recursion relations (TRR)*

$$\frac{\partial^3 \Phi_0}{\partial t^{m_1}_{a_1+1} \partial t^{m_2}_{a_2} t^{m_3}_{a_3}} = \sum_{m_+, m_-} \frac{\partial^2 \Phi_0}{\partial t^{m_1}_{a_1} \partial t^{m_+}_0} \eta^{m_+, m_-} \frac{\partial^3 \Phi_0}{\partial t^{m_-}_0 \partial t^{m_2}_{a_2} \partial t^{m_3}_{a_3}}.$$

The potential also satisfies the string equation, equation (4), where the symbol $\widetilde{\Phi}(\mathbf{t})$ *is replaced by* $\Phi_0(\mathbf{t})$.

This result was conjectured by Witten [26, 27] before the proper moduli spaces had even been constructed. The proof of the $g = 0$ CohFT property uses intersection theory on $\overline{\mathscr{M}}^{1/r}_{0,n}$ which is rather involved because of its orbifold (or stacky) nature. The proof of the topological recursion relations follows from the properties of the classes involved when they are restricted to boundary strata and from a presentation of the ψ classes in terms of boundary classes. The string equation follows from the lifting properties of the classes under the map of "forgetting" a point labeled by $m = 0$ and from the grading. Finally, one obtains the structure of a Frobenius manifold on $(\mathscr{H}^{(r)}, \eta)$ because $\Phi_0(\mathbf{x})$ is a polynomial. We refer the interested reader to [16] for details.

Remark 5.6 Strictly speaking, the state space of the KdV$_r$ theory should be $(\tilde{\mathscr{H}}^{(r)}, \tilde{\eta})$, where $\tilde{\mathscr{H}}^{(r)}$ is an r-dimensional vector space with basis $\{e_0, \ldots, e_{r-1}\}$, and with a metric in this basis given by $\tilde{\eta}_{m_1,m_2} := 1$ if $m_1 + m_2 \equiv (r-2) \mod r$ and 0 otherwise. However, there is an obvious orthogonal decomposition of vector spaces $\tilde{\mathscr{H}}^{(r)} = \mathscr{H}^{(r)} \oplus \mathscr{H}'$ where $\mathscr{H}'$ is the one dimensional vector space with basis $\{e_{r-1}\}$, and because $c^{1/r}$ vanishes on $\overline{\mathscr{M}}^{1/r,\mathbf{m}}_{g,n}$ whenever one of the m_i is $r-1$. Therefore, the decomposition $\tilde{\mathscr{H}}^{(r)} = \mathscr{H}^{(r)} \oplus \mathscr{H}'$ is a direct sum of CohFTs where $\mathscr{H}'$ is the trivial, one-dimensional CohFT. For this reason, we can and will restrict ourselves to the state space $(\mathscr{H}^{(r)}, \eta)$.

What is interesting about this approach is that it offers the possibility of generalization to higher genera. In [16], axioms were formulated for the virtual class $c^{1/r}$ in order to obtain a CohFT in all genera drawing upon an analogy with Gromov-Witten invariants. These axioms should be viewed as an analog of those of Behrend-Manin [2] for the virtual fundamental class in the case of the moduli space of stable maps.

Finally, it is also worth observing that the Frobenius structure here *cannot* arise as the quantum cohomology of a smooth, projective variety because it would correspond to fractional dimensional cohomology classes on the target variety.

6. The tensor product of Frobenius manifolds

The category of cohomological field theories has a natural tensor product [23] described as follows.

Definition 6.1 *Let $(\mathscr{H}', \eta', \Lambda')$ and $(\mathscr{H}'', \eta'', \Lambda'')$ be CohFTs. Their tensor product is $(\mathscr{H}' \otimes \mathscr{H}'', \eta' \otimes \eta'', \Lambda)$ where*

$$\Lambda_{g,n}(v_1' \otimes v_1'', \ldots, v_n' \otimes v_n'') := \Lambda'_{g,n}(v_1', \ldots v_n') \cup \Lambda''_{g,n}(v_1'', \ldots, v_n'')$$

for all v' in $\mathscr{H}'$ and v'' in $\mathscr{H}''$.

This is nothing more than the fact that the diagonal map $\overline{\mathscr{M}}_{g,n} \to \overline{\mathscr{M}}_{g,n} \times \overline{\mathscr{M}}_{g,n}$ is a coproduct with respect to the composition maps of the modular operad $\{H_\bullet(\overline{\mathscr{M}}_{g,n})\}$. We will only discuss the situation where $g = 0$ and the potential functions are polynomial in the flat coordinates.

The tensor product operation, when written in terms of the underlying potential functions, is highly nontrivial [18, 19], even in genus zero. In the case of Gromov-Witten invariants, Behrend [1] proved that the CohFT arising from $\overline{\mathscr{M}}_{g,n}(V' \times V'')$ is the tensor product of that arising from $\overline{\mathscr{M}}_{g,n}(V')$ and $\overline{\mathscr{M}}_{g,n}(V'')$. When restricted to genus zero, one can view this result as a deformation of the Künneth theorem.

There exists a notion of the tensor product of (nonformal) Frobenius manifolds [18]. Because the potential functions of KdV_r are polynomial, their tensor product is also a (nonformal) Frobenius manifold. There is, however, an unanswered question. Is there a natural moduli space whose intersection numbers assemble into a generating function related to the tensor products of the Frobenius manifolds associated to KdV_{r^i} for $i = 1 \ldots k$? These moduli spaces would be analogs of $\overline{\mathscr{M}}_{0,n}(V_1 \times \cdots \times V_k)$ in the theory of Gromov-Witten invariants. The answer turns out to be *yes*, as explained in the following theorem. This theorem is an immediate consequence of Lemma 4.13.

Theorem 6.2 *Let $(\mathscr{H}^{(r^i)}, \eta^{(i)})$ denote the Frobenius manifold associated to the r^i-th Gelfand-Dickey hierarchy, KdV_{r^i}, for $r^i \geq 2$ and $i = 1 \ldots, k$. Let $(\mathscr{H}, \eta)$ denote the tensor product of these Frobenius manifolds. The tensor product Frobenius structure arises from $\Lambda := \{\Lambda_{0,n}\}$ where the forms $\Lambda_{0,n} \in H^\bullet(\overline{\mathscr{M}}_{g,n}) \otimes T^n \mathscr{H}^*$, are defined by*

$$\Lambda_{0,n}(e_{m_1^1} \otimes \ldots \otimes e_{m_1^k}, \ldots, e_{m_n^1} \otimes \ldots \otimes e_{m_n^k}) = (\prod_{i=1}^k r^i) p_* c^{\mathbf{1/r}},$$

and where $p : \overline{\mathscr{M}}_{g,n}^{\mathbf{1/r}, \bar{\mathbf{m}}} \to \overline{\mathscr{M}}_{g,n}$, $\mathbf{r} := (r^1, \ldots, r^k)$, $\bar{\mathbf{m}} := (\mathbf{m}^1, \ldots, \mathbf{m}^k)$, $m_j^i = 0, \ldots, r^i - 2$ for all $i = 1, \ldots, k$ and $j = 1, \ldots, n$.

One may regard the above result as a geometric realization (a kind of "A-model") for the tensor product of these Frobenius manifolds. Isomorphic Frobenius manifolds can be constructed from versal deformations of A_{r^i-1} singularities in [24, 25] (a Landau-Ginzburg model). The latter construction can be generalized to yield a Frobenius manifold associated to any singularity of type A-D-E, or even more generally, to Coxeter groups. It is known that as Frobenius manifolds, the tensor product of the Frobenius manifolds associated to A_2 (KdV$_3$) and A_3 (KdV$_4$) is isomorphic to the Frobenius manifold associated to E_6, while the tensor product of the Frobenius manifolds associated to A_2 (KdV$_3$) and A_4 (KdV$_5$) is isomorphic to the Frobenius manifold associated to E_8 [4, 6].

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