Generalized coherent states for arbitrary quantum systems

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Abstract. Recent developments [3] in the construction of coherent states for an arbitrary quantum system with Hamiltonian H are presented here. These CS families $\{\mid J,\gamma\rangle\}$ are required to solve the unity, to be temporally stable, $e^{-iHt/\hbar}\mid J,\gamma\rangle = \mid J,\gamma+\omega t\,\rangle$, and the J functional dependence of the energy mean value $\langle J,\gamma\mid H\mid J,\gamma\rangle = E(J)$ is assigned possibly along lines which are classically meaningful. Further extensions are finally proposed.

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1. A review of standard CS properties

Since (standard) Coherent States were invented first by Schrödinger and rediscovered by Klauder [1], Glauber, Sudarshan, many generalizations have been proposed. Each of them retain some of the properties of the standard ones. Let us recall the familiar expression of the latter as infinite superposition of harmonic oscillator eigenstates,

$$\mid z\rangle = e^{-|z|^2/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}\mid n\rangle, \tag{1.1}$$

and let us give a brief review of those properties which on their whole own give these states a strong status of uniqueness.

P$_0$ *The map* $\mathbb{C}\ni z\rightarrow\mid z\rangle\in \mathrm{L}^2(\mathbb{R})$ *is continuous.*

P$_1$ $\mid z\rangle$ *is annihilation eigenvector:* $a\mid z\rangle = z\mid z\rangle$.

P$_2$ *The CS family resolves the unity:* $\frac{1}{\pi}\int_{\mathbb{C}}\mid z\rangle\langle z\mid\, d^2z = \mathrm{I}$.

P$_3$ *The CS saturate Heisenberg inequality:* $\Delta Q\,\Delta P = 1/2$.

P$_4$ *The CS family is temporally stable:* $e^{-i\mathscr{H}t}\mid z\rangle = \mid e^{-i\omega t}z\rangle$, *where* $\mathscr{H}$ *is the harmonic oscillator Hamiltonian. The latter is shifted such that* $\mathscr{H}\mid n\rangle = \omega n\mid n\rangle$.

P$_5$ *The mean value (or "lower symbol") of the Hamiltonian mimics the classical relation energy-action:* $\check{\mathscr{H}}(z)\equiv\langle z\mid\mathscr{H}\mid z\rangle = \omega|z|^2$.

P$_6$ *The CS family is the orbit of the ground state under the action of the Weyl-Heisenberg displacement operator:* $\mid z\rangle = e^{za^\dagger-\bar{z}a}\mid 0\rangle\equiv D(z)\mid 0\rangle$.

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P$_7$ *The Weyl-Heisenberg covariance follows from the above:*

$$\mathscr{U}(s,\zeta) \mid z\rangle = e^{i(s+\Im(\zeta\bar{z}))} \mid z+\zeta\rangle.$$

P$_8$ *The coherent states provide a straightforward quantization scheme:*

Classical state $z \rightarrow \mid z\rangle\langle z \mid$ *quantum state.*

Thus, these properties cover a wide spectrum, starting from a "wave-packet" side (Equation (1.1) and Properties P_3 and P_4) through an algebraic side (P_1), a group representation side (P_6 and P_7), a functional analysis side (P_2) to end with the ubiquitous problematic of the relationship between classical and quantum worlds (P_5 and P_8) [4], [5]. The generalisations we are going to describe here take place on the wave-packet side. They are in the continuation of recent papers [2, 3], and preserve most of the properties listed in the above, if not all of them, at the price of some adaptation, of course. For the group representation and functional analysis aspects, we refer to [6] with references therein.

2. Coherent states for discrete spectrum dynamics

Let us start like in [3] with a minimal set of requirements. Suppose we deal with a quantum system described by a positive Hamiltonian $\mathscr{H}$ with purely discrete nondegenerate spectrum,

$$\mathscr{H} \mid n\rangle = \omega e_n \mid n\rangle, \quad n \geq 0, \tag{2.1}$$

where the strictly increasing sequence $0 = e_0 < e_1 < e_2 < \cdots < e_n < \cdots$ is made of pure numbers ($\hbar = 1$) and can be finite or infinite. Let us denote by H the Hilbertian span of the orthonormal set $\{\mid n\rangle\}_{n\geq 0}$. We then consider the abstract **normalized** wave packet in H:

$$\mid J,\gamma\rangle_\rho = \frac{1}{\sqrt{\mathscr{N}_\rho(J)}} \sum_{n\geq 0} \frac{J^{n/2}}{\sqrt{\rho_n}} e^{-ie_n\gamma} \mid n\rangle, \tag{2.2}$$

where $\rho_0 = 1$, $J \geq 0$ and $-\infty < \gamma < \infty$. It is clear that we have here a formal generalization of (1.1) where $z = \sqrt{J}e^{-in\gamma}$, $\rho_n = n!$, and $\mathscr{N}_\rho(J) = e^{|z|^2}$. Let us now examine the validity of some of the standard CS properties in the present case.

Eigenvector property. Property P_1 formally holds with the lowering operator defined by

$$a_\rho \mid n\rangle = \sqrt{\rho_n/\rho_{n-1}} \mid n-1\rangle,\ e^{-i\mathscr{H}t} a_\rho e^{i\mathscr{H}t} \mid J,\gamma\rangle_\rho = \sqrt{J} \mid J,\gamma\rangle_\rho, \tag{2.3}$$

for $n \geq 1$, and $a_\rho \mid 0\rangle = 0$.

Temporal stability. Property P_4 holds by construction:

$$e^{-i\mathscr{H}t} \mid J,\gamma\rangle_\rho = \mid J,\gamma+\omega t\rangle_\rho. \tag{2.4}$$

Normalization. This property imposes the power series

$$\mathscr{N}_\rho(J) = \sum_{n\geq 0} \frac{J^n}{\rho_n} \tag{2.5}$$

to be non trivial, which means that the convergence radius

$$R = \overline{\lim}_{n\to\infty} \sqrt[n]{\rho_n} > J \geq 0$$

is not null. Continuity property P_0 immediately follows.

Resolution of the unity. This is the crucial point because we have to solve a classical moment problem. Indeed, $\rho_n > 0$ is assumed to be a probability distribution moment

$$\rho_n = \int_0^R J^n w_\rho(J)\, dJ,\; w_\rho(J) \geq 0. \tag{2.6}$$

Then Property P_2 holds in the following sense:

$$\begin{aligned} \mathrm{I} &= \int \mid J,\gamma\rangle_{\rho\,\rho}\langle J,\gamma \mid \, d\mu_\rho(J,\gamma) \\ &\equiv \lim_{\Gamma\to\infty} \frac{1}{2\Gamma}\int_{-\Gamma}^{\Gamma} d\gamma \int_0^R \mathscr{N}_\rho(J) w_\rho(J)\, dJ \mid J,\gamma\rangle_{\rho\,\rho}\langle J,\gamma\mid . \end{aligned} \tag{2.7}$$

It often happens that the determination of the distribution w_ρ, with support in $[0,R]$, is quite intractable, while another one, say $w_{\rho'}$, with same support, and giving rise to the moments

$$\rho'_n = \int_0^R J^n w_{\rho'}(J)\, dJ,\; w_{\rho'}(J) \geq 0, \tag{2.8}$$

may be also relevant to our purposes. Of course ρ'_n should be equivalent to ρ_n in the sense that both series $\mathscr{N}_\rho(J)$ and $\mathscr{N}_{\rho'}(J)$ have same convergence radius. So let us see what happens when we replace w_ρ by $w_{\rho'}$ in the $d\mu_\rho$ of (2.7). We now get the resolution of a positive diagonal operator $\Delta \equiv \delta(N)$, $\Delta \mid n\rangle = \delta_n \mid n\rangle$,

$$\begin{aligned} \Delta &= \int \mid J,\gamma\rangle_{\rho\,\rho}\langle J,\gamma\mid \, d\mu'(J,\gamma) \\ &\iff \mathrm{I} = \int (\Delta)^{-1/2} \mid J,\gamma\rangle_{\rho\,\rho}\langle J,\gamma\mid (\Delta)^{-1/2}\, d\mu'(J,\gamma), \end{aligned} \tag{2.9}$$

where $d\mu'(J,\gamma) = \mathscr{N}_\rho(J) w_{\rho'}(J)\, dJ\, d\Gamma$, and $\delta_n = \rho'_n/\rho_n$. This operator should be bounded with bounded inverse so that $\mathscr{N}_\rho(J)$ and $\mathscr{N}_{\rho'}(J)$ do have same convergence radius. The triple $(\Delta, \{\mid J,\gamma\rangle_\rho\}, d\mu')$ is called a *frame* [6].

3. Statistical and semi-classical aspects

At this point we should notice the remarkable interplay between two statistical aspects of the states (2.2):

Probability distribution $w_\rho(J)$
We could say this $w_\rho(J)$ pertains to the classical (statistical) side of the states $\mid J,\gamma\rangle_\rho$ if we take the freedom to give the measure $d\mu_\rho(J,\gamma)$ in (2.7) a classical phase space interpretation, which would be perfectly allowable in the standard case. As a matter of fact, let us associate to a quantum observable A its (non-unique) components $\hat{A}(J,\gamma)$ as seen from the continuous "coherent frame" $\{\mid J,\gamma\rangle_{\rho\rho}\langle J,\gamma\mid\}$:

$$A = \int \hat{A}(J,\gamma) \mid J,\gamma\rangle_{\rho\rho}\langle J,\gamma \mid \, d\mu_\rho(J,\gamma). \tag{3.1}$$

The function $\hat{A}(J,\gamma)$, which exists for a large class of operators A, is also called upper (see Lieb in [5]) or contravariant (Berezin in [7]) symbol. In particular, for an observable $f(N)$ diagonal with respect to $\{\mid n\rangle\}$, we have

$$f(N) = \int \hat{f}(J) \mid J,\gamma\rangle_{\rho\rho}\langle J,\gamma \mid \, d\mu_\rho(J,\gamma),$$
$$\text{which implies } f(n) = \frac{1}{\rho_n}\int_0^R \hat{f}(J) J^n w_\rho(J)\, dJ. \tag{3.2}$$

Wave-packet weighting probability
Alternatively, the moment ρ_n is part of the purely quantal weighting probability

$$|\langle n \mid J,\gamma\rangle_\rho|^2 \equiv |c_n|^2 = \frac{1}{\mathcal{N}_\rho(J)}\frac{J^n}{\rho_n}. \tag{3.3}$$

Let us now associate to the observable A its lower (or covariant) symbol which is nothing but its mean value in the state $\mid J,\gamma\rangle_\rho$:

$$\check{A}(J,\gamma) = {}_\rho\langle J,\gamma \mid A \mid J,\gamma\rangle_\rho. \tag{3.4}$$

In particular, we have for the diagonal observable $f(N)$:

$$\check{f}(J) = \sum_{n\geq 0} |c_n|^2 f(n) \equiv \langle f(n)\rangle. \tag{3.5}$$

Therefore, the weighting probability determines lower symbols of diagonal observables. It also determines the semi-classical revival structure (Averbukh and Perelman [8]) of the coherent state $\mid J,\gamma\rangle_\rho$. Let us briefly recall what we mean by this temporal concept of revival and refer to the comprehensive review of Bluhm, Kostelecký, and Porter [9] for more details. Suppose the expansion (2.2) is strongly weighted around a mean value $\check{N} = \langle n\rangle$ for the number operator N. For instance, in the case of the harmonic oscillator (1.1), the distribution $|c_n|^2 = e^{-|z|^2}|z|^2/n!$ is Poissonian and is weighted around $\check{N} = |z|^2$ with spread $\sigma \approx |z| = \sqrt{\check{N}}$. In the general case (2.2), let us choose $\bar{n} \in \mathbb{N}$ as being the closest integer to $\check{N}$ for a given J. With the hypothesis that the spread $\sigma \approx \Delta N$ is small compared with $\check{N} \approx \bar{n}$, what we should expect in general for large J (semi-classical behavior!), we expand the energy $E_n = \omega e_n$ in a Taylor series in n around the centrally excited value $\bar{n}$:

$$E_n \simeq E_{\bar{n}} + E'_{\bar{n}}(n-\bar{n}) + \frac{1}{2}E''_{\bar{n}}(n-\bar{n})^2 + \frac{1}{6}E'''_{\bar{n}}(n-\bar{n})^3 + \cdots , \tag{3.6}$$

where each prime on $E_{\bar{n}}$ denotes a derivative. These derivatives define distinct time scales: the classical period $T_{\rm cl} = 2\pi/|E'_{\bar{n}}|$, the revival time $t_{\rm rev} = 2\pi/\frac{1}{2}|E''_{\bar{n}}|$, the super-revival time $t_{\rm sr} = 2\pi/\frac{1}{6}|E'''_{\bar{n}}|$, and so on. Reporting this expansion in the evolution factor $e^{-i\gamma E_n/\omega}$ of (2.2) allows us to understand the possible occurrence of a quasiperiodic revival structure of these coherent states at high J, which would be well illustrated by the behavior in function of the evolution parameter γ of the absolute square of the so-called autocorrelation function of the coherent states:

$$|_\rho\langle J,0 \mid J,\gamma\rangle_\rho|^2 = |_\rho\langle J,0 \mid e^{-i\mathscr{H}\gamma/\omega} \mid J,0\rangle_\rho|^2. \tag{3.7}$$

It is clear from (3.6) that the revival features will be more or less apparent according to the value of the deviation $(n-\bar{n})$ relatively to n. A quantitative estimate is afforded by the so-called Mandel parameter $Q(J)$ defined as follows.

$$Q(J) = \frac{(\Delta N)^2}{\check{N}} - 1 = J\frac{d}{dJ}\log\frac{d}{dJ}\log\mathscr{N}_\rho(J). \tag{3.8}$$

This parameter allows one to compare the weighting distribution $|c_n|^2$ with the Poisson distribution: The latter is

- Poissonian if $Q = 0$,
- super-Poissonian if $Q > 0$,
- sub-Poissonian if $Q < 0$.

We are now aware of the basic importance of the distribution $w_\rho(J)$ in regard to these statistical considerations.

4. Imposing Hamiltonian lower symbol

So far the only observable we have at our disposal is the Hamiltonian. Suppose there exists some presupposed functional form

$$\check{\mathscr{H}}(J) = {}_\rho\langle J,\gamma \mid \mathscr{H} \mid J,\gamma\rangle_\rho = \omega \sum_{n\geq 0} |c_n|^2 e_n = \omega\langle e_n\rangle, \tag{4.1}$$

for the lower symbol of the Hamiltonian. We could for instance consider as a prerequisite the classical energy-action dependence for systems with one degree of freedom,

$$J = \frac{1}{2\pi}\oint p(E = \check{\mathscr{H}})dq, \tag{4.2}$$

and this way deserves to be carefully explored in view of studying the (semi-) classical behavior of the corresponding quantum system. Suppose that $\check{\mathscr{H}}(J)$ has a series expansion in the neighborhood of $J = 0$.

$$\check{\mathscr{H}}(J) = \omega J^{\nu} \sum_{n\geq 0} \alpha_n J^n. \tag{4.3}$$

Then, by simple identification with the expansion

$$\mathscr{N}_{\rho}(J)\check{\mathscr{H}}(J) = \omega \sum_{n\geq 0} \frac{e_n}{\rho_n} J^n, \tag{4.4}$$

we obtain

$$\nu = 0, \quad \alpha_0 = 0, \quad \frac{e_n}{\rho_n} = \sum_{m=0}^{n} \alpha_{n-m} \frac{1}{\rho_m}, \quad \rho_0 = 1. \tag{4.5}$$

The solution ρ_n to Equation (4.5) can be written in terms of the Jackson "factorial" [10]

$$[e_n]! \stackrel{def}{=} e_1 e_2 \cdots e_n, \; [e_n]! = 1, \tag{4.6}$$

as follows:

$$\rho_n = \frac{[e_n]!}{p_n}, \quad \text{with} \tag{4.7}$$

$$p_n = \sum_{m=0}^{n} \frac{[e_{n-1}]!}{[e_m]!} \alpha_{n-m}, \quad p_0 = 1. \tag{4.8}$$

For instance,

$$\rho_1 = \frac{e_1}{\alpha_1}, \quad \rho_2 = \frac{e_1 e_2}{e_1 \alpha_2 + \alpha_1^2}, \quad \rho_3 = \frac{e_1 e_2 e_3}{e_1 \alpha_2 + (e_1 + e_2)\alpha_1 \alpha_2 + \alpha_1^3} \cdots$$

Let us give a few explicit examples.

- *The linear case*: $\check{\mathscr{H}}(J) = \omega J$. It is the simplest one, of course, and gives rise to the "Action-Angle" coherent states introduced in [3]. Here the ρ_n's are exactly the Jackson factorials:

$$\rho_n = [e_n]!. \tag{4.9}$$

- *The single power case*: $\check{\mathscr{H}}(J) = \omega J^q$, $q = 2, 3, \cdots$. Then the corresponding CS do not generate the whole Hilbert space H:

$$|\, J, \gamma \rangle_{\rho} = \frac{1}{\sqrt{\mathscr{N}_{\rho}(J)}} \sum_{n\geq 0} \frac{J^{nq/2}}{\sqrt{e_q e_{2q} \cdots e_{nq}}} e^{-ie_{nq}\gamma} \,|\, nq \rangle. \tag{4.10}$$

- *The geometric case*: $\check{\mathscr{H}}(J) = \omega/(1-J)$. Then the denominator p_n in (4.7) is given by

$$p_n = \sum_{k=1}^{n-1} \sum_{1\leq i_1<i_2<\cdots<i_k\leq n-1} e_{i_1} e_{i_2} \cdots e_{i_k}. \tag{4.11}$$

Let us end this section by noting that the reverse to the above approach is also of great interest, at least from a more "experimental" point of view. Suppose the coherent states $\mid J,\gamma\rangle_\rho$ are prepared in such a way that the weighting distribution $|c_n|^2 = \frac{1}{\mathscr{N}_\rho(J)} \frac{J^n}{\rho_n}$ (*e.g.* a Gaussian-like distribution) is determined by the experimental conditions. Then the coefficients of the lower symbol series (4.3) will be given by inverting Equation (4.5): $\alpha_0 = 0$, $\alpha_n = \sum_{m=0}^{n} c_m(\rho_1,\cdots,\rho_n)e_m$.

5. "Action-Angle" coherent states

Let us now concentrate on the linear case $\check{\mathscr{H}}(J) = \omega J$, which amounts to require Property P_5. We shall adopt here the simplified notations:

$$\mid J,\gamma\rangle = \frac{1}{\sqrt{\mathrm{E}(J)}} \sum_{n\geq 0} \frac{J^{n/2}}{\sqrt{[e_n]!}} e^{-ie_n\gamma} \mid n\rangle, \tag{5.1}$$

where $\mathrm{E}(J)$ is the Jackson generalized exponential

$$\mathrm{E}(J) = \sum_{n\geq 0} \frac{J^n}{[e_n]!}. \tag{5.2}$$

The algebraic structure exemplified by Property P_1 and (2.3) may now display nontrivial interesting features. Indeed, it is clear from the definition of lowering and raising operators

$$a \mid n\rangle = \sqrt{e_n} \mid n-1\rangle, \quad a^\dagger \mid n\rangle = \sqrt{e_{n+1}} \mid n+1\rangle, \tag{5.3}$$

that the Hamiltonian factorizes as

$$\mathscr{H} = \omega a^\dagger a \equiv \omega e(N), \quad e(N) \mid n\rangle = e_n \mid n\rangle. \tag{5.4}$$

Now given any lowering operator a, $a \mid 0\rangle = 0$, of the above type, and any diagonal (real) operator $f(N)$, $f(N) \mid n\rangle = f_n \mid n\rangle$, the following commutation rules hold true:

$$af(N) = f(N+1)a, \quad f(N)a^\dagger = a^\dagger f(N+1), \tag{5.5}$$

$$[a, f(N)] = f'(N)a = af'(N-1), \quad [a^\dagger, f(N)] = -a^\dagger f'(N) = -f'(N-1)a^\dagger, \tag{5.6}$$

where $f'(N) \equiv f(N+1) - f(N)$ is the finite difference derivative of $f(N)$. Of course, the sense of the above *any*'s should be mellowed in view of operator domain considerations. In the present case, we also have

$$[a, a^\dagger] = e'(N). \tag{5.7}$$

So we can think about the existence of an interesting "dynamical" or "spectrum generating" Lie algebra generated by $\{a, a^\dagger, e'(N), \cdots\}$, where the dots mean further generators coming out from iterating the above commutators [11]. This algebra is generically infinite. However, non trivial structures, like quadratic algebra, can occur provided that the operator function $e(N)$ is solution of some required finite difference equation. Work in this very rich classification scheme is in progress (Lafortune, Gazeau, and Winternitz [12]).

6. Two examples of action-angle coherent states

6.1. Coulomb-like spectrum

Our first example concerns a Coulomb-like discrete spectrum:

$$e_n = (1 - \frac{1}{(n+1)^2}), \text{ for which } \rho_n = [e_n]! = \frac{1}{2}\frac{n+2}{n+1}. \tag{6.1}$$

So the convergence radius of the generalized exponential is $R = 1$. This series can be given the closed form:

$$\mathrm{E}(J) = \frac{2}{J}(\frac{1}{1-J} + \frac{\log(1-J)}{J}). \tag{6.2}$$

The distribution probability $w(J)$ is explicitly given by:

$$\rho_n = \int_0^1 J^n w(J)\, dJ \ \text{ where } \ w(J) = \frac{1}{J}(1 + \delta(J - 1^-)). \tag{6.3}$$

On the other hand, the weighting distribution is given by

$$|c_n|^2 = \frac{n+1}{n+2}\frac{(1-J)J^{n+2}}{J + (1-J)\log(1-J)}. \tag{6.4}$$

This distribution is super-Poissonian. This is easily inferred from the estimate on the parameter [13]:

$$\sigma \equiv \frac{e_{n+1}/e_n}{(n+1)/n} < 1 \ \Rightarrow\ Q < 0. \tag{6.5}$$

At the moment we do not have any interesting result about the algebraic structure, let alone the defining finite difference equation for $e(N)$

$$e(N) = \exp\left(\log N\right)''.$$

6.2. Infinite square well and Pöschl-Teller potentials

The paper of Ref.[14] is devoted to the infinite square well and Pöschl-Teller potential example. So we just give here a brief account of this nice illustration of many features of the action-angle coherent states. We consider the Hamiltonian

$$\mathscr{H} = \frac{P^2}{2m} + V(Q) - \frac{\hbar^2}{8ma^2}(\lambda+\kappa)^2, \tag{6.6}$$

for a particle on the line and submitted to the potential

$$V(x) = \begin{cases} 0 & 0 < x < \pi a \\ \infty & x \leq 0,\ x \geq \pi a \end{cases}, \tag{6.7}$$

for $\lambda = 1 = \kappa$ (*infinite square well*),

$$V(x) = \begin{cases} \frac{\hbar^2}{8ma^2}\left(\frac{\lambda(\lambda-1)}{\cos^2 x/2a} + \frac{\kappa(\kappa-1)}{\sin^2 x/2a}\right) & 0 < x < \pi a \\ \infty & x \leq 0,\ x \geq \pi a \end{cases}, \tag{6.8}$$

for $\lambda > 1,\ \kappa > 1$ (*Pöschl-Teller potential*). Note that we have here reintroduced the Planck constant into the physical quantities expressions. In a certain sense, the Pöschl-Teller potential [15] interpolates between the harmonic oscillator and the infinite square well. We shall here treat the former and the latter on the same footing, since the infinite well takes place at the limit $\lambda = 1 = \kappa$. Eigenvalues and eigenvectors are given by:

$$\mathscr{H} \mid n\rangle = \hbar\omega e_n \mid n\rangle, \quad e_n = n(n+\lambda+\kappa), \quad \omega = \frac{\hbar}{2ma^2}, \tag{6.9}$$

$$\begin{aligned}\langle x \mid n\rangle = & k_n(\cos x/2a)^{\lambda}(\sin x/2a)^{\kappa}\,{}_2\times \\ & \mathrm{F}_1(-n, n+\lambda+\kappa; \kappa+1/2; \sin^2 x/2a).\end{aligned} \tag{6.10}$$

- *Algebraic side.* We put $e(N) = N(N+\lambda+\kappa)$. The Lie algebra generated by $\{a, a^\dagger, e'(N), \cdots\}$ is finite and <u>is</u> $su(1,1)$, as it actually holds for any algebra built from an $e(N)$ quadratic in N. Indeed,

  $$[a,a^\dagger] = e'(N) = 2N+1+\lambda+\kappa,\ [a,e'(N)] = 2a,\ [a^\dagger, e'(N)] = -2a^\dagger. \tag{6.11}$$

  Note that $e''(N) = 2$, $e'''(N) = 0$. Therefore, we can identify the Hilbertian span of the $\mid n\rangle$'s via the values assumed by the $su(1,1)$ Casimir operator.

  $$\mathscr{C} = -2e'(N)(2e'(N)-1) + e(N) \equiv \eta(\eta-1)\mathrm{I} \ \Rightarrow\ \eta = \frac{\lambda+\kappa+1}{2} \geq \frac{3}{2}. \tag{6.12}$$

  This range of values falls into the discrete series representation interval $1/2 < \eta < \infty$ for $su(1,1)$ or, equivalently, for the universal covering of $SU(1,1)$. Note that $\eta = 3/2$ corresponds to the infinite well.

- *Coherent states.* They read in the present case:

$$| J,\gamma\rangle = \frac{1}{\sqrt{\mathrm{E}(J)}} \sum_{n\geq 0} \frac{J^{n/2}}{\sqrt{[e_n]!}} e^{-ie_n\gamma} \,| n\rangle, \tag{6.13}$$

$$\rho_n = [e_n]! = \frac{n!\Gamma(n+\nu+1)}{\Gamma(\nu+1)}, \; \nu \equiv \lambda+\kappa \geq 2, \tag{6.14}$$

$$\mathrm{E}(J) = \Gamma(\nu+1)\frac{\mathrm{I}_\nu(2\sqrt{J})}{J^{\nu/2}}, \; R=\infty, \tag{6.15}$$

where I_ν is a modified Bessel function . Note that the $\nu=2$ case corresponds to the infinite well. At this point, we should remark that, for $\gamma=0$ and for complex values of J, these coherent states are nothing else but the Barut-Girardello states [16]. However, the construction of the latter was motivated by purely mathematical considerations, more precisely by the aim to extend to the algebra $su(1,1)$ the eigenvector property P_1.

- *Resolution of the identity.*

$$\mathrm{I} = \lim_{\Gamma\to\infty} \frac{1}{2\Gamma}\int_{-\Gamma}^{\Gamma} d\gamma \int_0^\infty E(J)w(J)\,dJ \;| J,\gamma\rangle\langle J,\gamma\,|,$$

$$\text{with } \rho_n = \int_0^\infty J^n w(J)\,dJ, \; w(J) = \frac{2J^{\nu/2}}{\Gamma(\nu+1)}\mathrm{K}_\nu(2\sqrt{J}), \tag{6.16}$$

where K_ν is a modified Bessel function.

- *Infinite square well.* Let us end this section by some considerations on the specific case of the infinite well [14], since it is the simplest one and should not qualitatively differ from the Pöschl-Teller cases. The weighting probability $|c_n|^2$ is sub-Poissonian and reads

$$|c_n|^2 = \frac{1}{n!(n+2)!}\frac{J^{n+1}}{\mathrm{I}_2(2\sqrt{J})}. \tag{6.17}$$

The revival structure is encoded by the expansion

$$E_n = \frac{\hbar^2}{2ma^2}n(n+2) = \frac{\hbar^2}{2ma^2}[\bar{n}(\bar{n}+2)+2(\bar{n}+1)(n-\bar{n})+(n-\bar{n})^2], \tag{6.18}$$

and so the two characteristic times are the classical period $T_{\mathrm{cl}} = 2\pi ma^2/\hbar(\bar{n}+1)$ and the revival time $t_{\mathrm{rev}} = 4\pi ma^2/\hbar = 2(\bar{n}+1)T_{\mathrm{cl}}$. There is no super-revival time here. We refer to [14] for a systematic study of the temporal behavior of the probability density, $|\langle x \mid J,\gamma\rangle|^2$ with $\gamma=\omega t$, of the square modulus of the autocorrelation, $|\langle J,0 \mid J,\gamma\rangle|^2$, of the mean position and mean "momentum" and their respective dispersions, and so on.

7. Coherent states for continuum dynamics

We now turn to the continuum case. Suppose the quantum system has a Hamiltonian with positive non degenerate purely continuous spectrum,

$$\mathscr{H} \mid \mathscr{E} \rangle = \mathscr{E} \mid \mathscr{E} \rangle, \quad 0 \leq \mathscr{E} < \bar{\mathscr{E}}, \tag{7.1}$$

where $\mid \mathscr{E} \rangle$ denotes some eigendistribution. Mimicking the discrete case, we define the **normalized** coherent states as the continuous superposition

$$\mid J,\gamma\rangle_\rho = \frac{1}{\sqrt{\mathscr{N}_\rho(J)}} \int_0^{\bar{\mathscr{E}}} \frac{J^{\mathscr{E}/2}}{\sqrt{\rho(\mathscr{E})}} e^{-i\gamma\mathscr{E}} \mid \mathscr{E} \rangle \, d\mathscr{E}. \tag{7.2}$$

The normalization condition reads

$$\mathscr{N}_\rho(J) = \int_0^{\bar{\mathscr{E}}} \frac{J^{\mathscr{E}}}{\rho(\mathscr{E})} \, d\mathscr{E}, \tag{7.3}$$

which has to be finite for, say, $0 \leq J < R$. Equation (7.3) gives the ratio $J^{\mathscr{E}}/(\mathscr{N}_\rho(J)\rho(\mathscr{E}))$ a weighting distribution meaning.

These coherent states clearly have Properties P_0 (continuity) and P_1 (temporal stability). For having the resolution of the unity, we first suppose the existence of a nonnegative density $w_\rho(J)$ such that

$$\int_0^R J^{\mathscr{E}} w_\rho(J) \, dJ = \rho(\mathscr{E}). \tag{7.4}$$

Then we have,

$$\mathrm{I} = \int \mid J,\gamma\rangle_{\rho\rho}\langle J,\gamma \mid \, d\mu_\rho(J,\gamma) \equiv \int_{-\infty}^{\infty} d\gamma \int_0^R \mathscr{N}_\rho(J) w_\rho(J) \, dJ \, \mid J,\gamma\rangle_{\rho\rho}\langle J,\gamma \mid . \tag{7.5}$$

Note the simplification in the γ integration afforded by the continuous spectrum. Like in the discrete case, there is an interplay between the functional dependence of the lower symbol and the weighting distribution:

$$\check{\mathscr{H}}(J) = {}_\rho\langle J,\gamma \mid \mathscr{H} \mid J,\gamma\rangle_\rho = \frac{1}{\mathscr{N}_\rho(J)} \int_0^{\bar{\mathscr{E}}} \frac{\mathscr{E} J^{\mathscr{E}}}{\rho(\mathscr{E})} \, d\mathscr{E} = J \frac{\partial}{\partial J} \log \mathscr{N}_\rho(J) \tag{7.6}$$

$$\textit{i.e.}, \quad \mathscr{N}_\rho(J) = \mathscr{N}_\rho(J_0) \exp\left(\int_{J_0}^{J} \check{\mathscr{H}}(x) \frac{dx}{x} \right). \tag{7.7}$$

8. Coherent states for discrete *and* continuum dynamics

Finally, we briefly present the more general case of an Hamiltonian having both discrete and continuous spectra:

$$\mathscr{H} = \mathscr{H}_D \oplus \mathscr{H}_C, \quad 0 \leq \mathscr{H}_D \leq \Omega,\ \Omega < \mathscr{H}_C : \tag{8.1}$$

We introduce for this situation the following superposition of respective coherent states:

$$\mid J,\gamma;K,\delta;\eta,\phi\rangle_\rho = f(K,\delta-\eta) \mid J,\gamma\rangle_\rho + e^{-i\phi} g(J) \mid K,\delta\rangle_\rho, \tag{8.2}$$

where $\mid J,\gamma\rangle_\rho$ (resp. $\mid K,\delta\rangle_\rho$) takes place for the discrete (resp. continuous) spectrum. Continuity and temporal stability follow from this expression:

$$e^{-i\mathscr{H}t} \mid J,\gamma;K,\delta;\eta,\phi\rangle_\rho = \mid J,\gamma+\omega t;K,\delta+\omega t;\eta+\omega t,\phi+\Omega t\rangle_\rho. \tag{8.3}$$

Establishing the resolution of the unity,

$$\mathrm{I} = \int \mid J,\gamma;K,\delta;\eta,\phi\rangle_{\rho\,\rho}\langle J,\gamma;K,\delta;\eta,\phi \mid d\lambda, \tag{8.4}$$

$$d\lambda\,(J,\gamma;K,\delta;\eta,\phi) \equiv d\mu_D(J,\gamma)\,d\mu_C(K,\delta)\,h(\eta)\,d\eta\,d\phi/2\pi, \tag{8.5}$$

is more elaborate and explains the presence in (8.2) of the functions f, g, and the parameters η and ϕ. The ranges of values assumed by the latter are respectively $(-\infty,\infty)$ and $[0,2\pi)$. The conditions the three functions f, g, h, have to satisfy are the following:

$$\int |f(K,\delta-\eta)|^2 \mid J,\gamma\rangle_{\rho\,\rho}\langle J,\gamma \mid d\lambda = \mathrm{I}_D, \tag{8.6}$$

$$\int |g(J)|^2 \mid K,\delta\rangle_{\rho\,\rho}\langle K,\delta \mid d\lambda = \mathrm{I}_C, \tag{8.7}$$

$$\int_0^{2\pi} d\phi e^{i\phi} \int g(J)^\star f(K,\delta-\eta) \mid J,\gamma\rangle_{\rho\,\rho}\langle K,\delta \mid d\lambda = 0, \tag{8.8}$$

$$\text{with} \quad \int_{-\infty}^{\infty} h(\eta)\,d\eta = 1, \ \int |f(K,\delta)|^2\,d\mu_C = 1, \ \int |g(J)|^2\,d\mu_D = 1. \tag{8.9}$$

Again here, a predefinite functional form assigned to the lower symbol $\check{\mathscr{H}}(J,K,\delta,\eta,\phi) \equiv |f(K,\delta-\eta)|^2 \check{\mathscr{H}}_D + |g(J)|^2 \check{\mathscr{H}}_C$ implies determining constraints for linear combination coefficients and normalization factors, i.e., for the weighting distributions present in the construction.

9. Concluding general expression

Through the successive generalizations of coherent states presented in this paper, we have been becoming gradually aware of the existence of some unifying formalism which underlies all various constructions. So let us conclude by proposing the following general form. Let a quantum system be with one degree of freedom and with Hamiltonian $\mathscr{H}$. We suppose that the spectrum of $\mathscr{H}$, Sp($\mathscr{H}$), is bounded from below (as it should be!), and we choose its lower bound equal to zero. Let σ be a **complex** measure with support contained in Sp($\mathscr{H}$). We then consider the linear superposition of Hamiltonian eigenstates or eigendistributions $\mid \lambda \rangle$, $\mathscr{H} \mid \lambda \rangle = \lambda \mid \lambda \rangle$, in a possibly abusive notation:

$$\int \mid \lambda \rangle d\sigma(\lambda) \equiv \mid \sigma \rangle, \tag{9.1}$$

with the following requirements on σ.

Temporal stability:
The complex nature of the measure σ should assure the temporal stability:

$$e^{-i\mathscr{H}t} \mid \sigma \rangle = \mid e^{-i\lambda t} \sigma \rangle. \tag{9.2}$$

Normalization:

$$1 = \langle \sigma \mid \sigma \rangle = \int |\frac{d\sigma}{d\lambda}|^2 d\lambda, \tag{9.3}$$

with again a possibly abusive notation for the "density" $d\sigma/d\lambda$ of the measure σ.

Operator resolution:

$$A = \int \mid \sigma \rangle \langle \sigma \mid \, d\mu(\sigma), \tag{9.4}$$

where A is some positive bounded operator, *e.g.* $A = P\Delta P$ where P is a projector and Δ defines a frame.

Lower symbol identity:

$$\langle \sigma \mid \mathscr{H} \mid \sigma \rangle = \check{\mathscr{H}}(\sigma). \tag{9.5}$$

The extension of the formalism presented in this paper to systems with several degrees of freedom is possible, and rather straightforward for completely integrable systems.

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