Gauged q-fields

R. J. Finkelstein
Department of Physics and Astronomy
University of California, Los Angeles, CA 90095-1547 USA

Abstract. The straightforward description of q-deformed systems leads to transition amplitudes that are not numerically valued. To give physical meaning to these expressions without introducing *ad hoc* remedies, one may exploit an "internal" Fock space already defined by the q-algebra. This internal space may be interpreted in terms of internal degrees of freedom of the deformed system or alternatively in terms of non-locality. It is shown that the q-deformation may give stringy characteristics to a Yang-Mills theory.

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1. Introduction

Corresponding to the well known systems of quantum mechanics such as the harmonic oscillator or the hydrogen atom there are q-systems obtained by going over to quantum groups. The q-systems have more degrees of freedom than the system from which they are derived. When the standard field theories are similarly deformed the new degrees of freedom may be interpreted as expressions of non-locality exhibited by extended particles. Here we explore one aspect of this non-locality as it might appear in q-Yang-Mills theories.

2. The q-Yang-Mills theory

Let $\psi(x)$ be the basis of the fundamental representation of a gauge group. Then if $\psi'(x) = T(x)\psi(x)$, the covariant derivative ∇_μ transforms as

$$\nabla'_\mu = T\nabla_\mu T^{-1} \tag{2.1}$$

so that $\nabla'_\mu \psi' = T(\nabla_\mu \psi)$.

The corresponding gauge field A_μ is defined by $A_\mu = \nabla_\mu - \partial_\mu$. Then by (2.1)

$$A'_\mu = TA_\mu T^{-1} + T\partial_\mu T^{-1}.$$

The field strength is $F_{\mu\nu} = (\nabla_\mu, \nabla_\nu)$ and transforms as $F'_{\mu\nu} = TF_{\mu\nu}T^{-1}$ by (2.1). The nature of the gauge group is so far unspecified.

We now assume that $T \in SU_q(2)$. Then

$$T^t \varepsilon T = T\varepsilon T^t = \varepsilon \tag{2.2}$$

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where

$$\varepsilon = \begin{pmatrix} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{pmatrix}.$$

Then, if ψ is also a Dirac field, $\psi^t C\varepsilon\gamma^\mu\nabla_\mu\psi$ is invariant where ψ^t is the transpose of ψ while C and ε satisfy

$$L^t CL = C$$
$$T^t\varepsilon T = \varepsilon$$

where L is a Lorentz transformation and C is the charge conjugation matrix. The basic Lagrangian may be chosen to be

$$S = \int d^4x \left[Q_\ell F_{\mu\nu}F^{\mu\nu}Q_r + i\psi^t C\varepsilon\gamma^\mu\nabla_\mu\psi + \frac{1}{2}[(\nabla_\mu\varphi)^t\varepsilon\nabla^\mu\varphi + \varphi^t\varepsilon\varphi]\right], \quad (2.3)$$

where $Q'_\ell = Q_\ell T^{-1}$ and $Q'_r = TQ_r$, and where kinetic terms in Q_ℓ and Q_r are omitted. Here φ is a Lorentz scalar that is also a fundamental representation of $SU_q(2)$.

3. The $SU_q(2)$ algebra

Let

$$T(0) = \begin{pmatrix} \alpha_0 & \beta_0 \\ -q^{-1}\bar{\beta}_0 & \bar{\alpha}_0 \end{pmatrix}.$$

Then (2.2) implies

$$\begin{aligned} &\alpha_0\beta_0 = q\beta_0\alpha_0 && \alpha_0\bar{\alpha}_0 + \beta_0\bar{\beta}_0 = 1 \\ &\alpha_0\bar{\beta}_0 = q\bar{\beta}_0\alpha_0 && \bar{\alpha}_0\alpha_0 + q_1^2\bar{\beta}_0\beta_0 = 1 \\ &\beta_0\bar{\beta}_0 = \bar{\beta}_0\beta_0 && q_1 = q^{-1}. \end{aligned} \quad (3.1)$$

Let us next introduce

$$T(x) = \begin{pmatrix} e^{i\varphi(x)}\alpha_0 & e^{i\varphi(x)}\beta_0 \\ -q_1e^{-i\varphi(x)}\bar{\beta}_0 & e^{-i\varphi(x)}\bar{\alpha}_0 \end{pmatrix} = \begin{pmatrix} \alpha(x) & \beta(x) \\ -q_1\bar{\beta}(x) & \bar{\alpha}(x) \end{pmatrix}.$$

Then the elements of $T(0)$ and $T(x)$ satisfy the same relations (3.1). We shall write $(\alpha, \beta, \bar{\beta}, \bar{\alpha})$ for the space dependent matrix elements.

There is associated with the algebra (3.1) a state space. Define the ground state, $|0\rangle$, by $\alpha|0\rangle = 0$. Since β and $\bar{\beta}$ commute we may require $|0\rangle$ to be a common eigenstate of β and $\bar{\beta}$. Then

$$\beta|0\rangle = b|0\rangle$$
$$\bar{\beta}|0\rangle = \bar{b}|0\rangle$$

Note

$$(\bar{\alpha}\alpha + q_1^2\beta\bar{\beta})|0\rangle = |0\rangle \text{ or } |b|^2 = q^2. \tag{3.2}$$

Define $|n\rangle = \lambda_n \bar{\alpha}^n|0\rangle$. Then

$$\begin{aligned}\beta|n\rangle &= \lambda_n \beta\bar{\alpha}^n|0\rangle \\ &= \lambda_n q^n \bar{\alpha}^n \beta|0\rangle\end{aligned}$$

by iterating (3.1). Then $\beta|n\rangle = q^n b|n\rangle$. Likewise $\bar{\beta}|n\rangle = q^n\bar{b}|n\rangle$. One also finds

$$\lambda_n = \prod_0^{n-1}(1-|b|^2q^{2s})^{-1/2}$$

Finally define the analogue of the Hamiltonian of the oscillator:

$$H = \frac{1}{2}(\alpha\bar{\alpha} + \bar{\alpha}\alpha).$$

By (3.1)

$$H = 1 - \frac{1}{2}(1+q_1^2)\bar{\beta}\beta.$$

Then

$$H|n\rangle = \left[1 - \frac{1}{2}(1+q^2)q^{2n}\right]|n\rangle$$

by (3.2). Here the $|n\rangle$ are eigenstates of $\beta, \bar{\beta}$ and H. The levels depend on q^{2n} rather than on n, i.e. they are arranged in geometric rather than arithmetic progression and in this respect resemble the $\langle n\rangle$ order of the q-oscillator.

4. An Internal Space [1]

The existence of the algebra (3.1) leads to novel features of q-systems, as one sees in the q-deformations of familiar elementary systems such as the harmonic oscillator and the hydrogen atom.

In these examples one finds that the wave functions are not numerically valued, but lie in the q-algebra. To interpret these results according to the usual rules, however, it is necessary to calculate numerically valued transition probabilities between states represented by these wave functions. There is a natural procedure for doing this by utilizing a Fock space associated with the algebra (3.1). If such a path is followed, however, these simple one-particle problems like the oscillator and the hydrogen atom are endowed with internal degrees of freedom or partially promoted to the complexity of quantum field problems. In a similar way a quantum field-theoretic problem will acquire a second Fock space or internal degrees of freedom that can be given a non-local interpretation. We shall now see how this might work out for any q-deformed field theory, including Yang-Mills.

A general field lying in the q-algebra will have the following expansion

$$\psi_\mu(x) = \sum_\rho \left[f_\mu(\rho,x)a(\rho) + g_\mu(\rho,x)\bar{b}(\rho)\right]$$

where

$$\begin{aligned}
\rho &= (\vec{p}, r, s) \\
\sum_\rho &= \left(\frac{1}{2\pi}\right)^{3/2} \int \frac{d\vec{p}}{(2p_o)^{1/2}} \sum_{r,s} \\
f_\mu(\rho,x) &= \sum_s U_\mu(\vec{p},r,s)e^{-ipx}\tau_s \\
g_\mu(\rho,x) &= \sum_s V_\mu(\vec{p},r,s)e^{ipx}\tau_s
\end{aligned}$$

Here μ is a generic tensor index, $\bar{a}$ and $\bar{b}$ are creation operators for particles and antiparticles, the r-sum is the sum over different polarizations and the s-sum is the sum over generators of the q-algebra.

We may set

$$\psi_\mu = \sum_{s=1}^{4} \psi_{\mu s}\tau_s$$

where the τ_s may be chosen as follows:

$$\tau_s \equiv \tau_{ss'} = T_{jk}\delta_{s,j}\delta_{s',k}.$$

This choice of the τ_s implies a privileged gauge. Any other linear combination is allowed. A general gauge transformation will carry ψ into a space where the basis elements are no longer linear in τ_s.

5. Energy and mass of a q-scalar particle

Let us consider the contribution of the scalar field in (2.3). Then the energy operator is

$$\mathscr{E} = \int T_{00}d\vec{x} = \frac{1}{2}\int [(\nabla\varphi)^t\varepsilon\nabla\varphi + m_0^2\varphi^t\varepsilon\psi]d\vec{x} \tag{5.1}$$

where

$$\varphi = \left(\frac{1}{2\pi}\right)^{3/2} \int \frac{d\vec{p}}{(2p_0)^{1/2}} \sum_s U(p,s)\left[e^{-ipx}a(p,s) + e^{ipx}\bar{a}(p,s)\right]\tau_s \tag{5.2}$$

with the usual commutators for the a and $\bar{a}$. Here the s-sum is over the generators of the q-gauge group. We shall compute just the contribution of the free field, i.e. without the vector interaction.

Denote the basis states in Fock space by $|Nn\rangle$. Then

$$\bar{a}(\vec{p},s)a(\vec{p},s')\beta|N(\vec{p},s)n\rangle = N(\vec{p},s)q^n b|N(\vec{p},s)n\rangle\delta(s,s') \tag{5.3}$$

where $N(\vec{p},s)$ is the population number of the state $(\vec{p},s)$.

Now substitute (5.2) in (5.1) and use (5.3). Then the eigenvalue of $\mathscr{E}$ is given by

$$\mathscr{E}|Nn\rangle = \int d\vec{p} \sum_s U(\vec{p}.s)^t \varepsilon U(\vec{p},s) p_o N(\vec{p},s) \tau_s^2 |Nn\rangle, \tag{5.4}$$

where $p_o^2 = \vec{p}^{\,2} + m^2$. Choose $\tau_s = \delta_{s1}\tau_1$, where $\tau_1 = \beta$. By (5.4)

$$\mathscr{E}|N(\vec{p},1)n\rangle = \int d\vec{p} \left[U(\vec{p},1)^t \varepsilon U(\vec{p},1)\right] p_o N(\vec{p},1)\beta^2 |N(\vec{p},1)n\rangle.$$

The factor $[U(p,1)^t \varepsilon U(p,1)]$ is numerical and may be normalized to unity. Then the eigenvalues of the partial energy contributed by the free scalar field are

$$\mathscr{E}' = \int d\vec{p} N(\vec{p},1) p_o q^{2n+1}.$$

If q is near unity, say $1 + \frac{\varepsilon}{2}$, then the mass is

$$\begin{aligned} m_o q^{2n+1} &= m_o \left(1 + \frac{\varepsilon}{2}\right)^{2n+1} \\ &= m_o \left(1 + \frac{1}{2}\varepsilon + n\varepsilon\right) \end{aligned}$$

for low n. It follows that a point particle with mass m_o in a $q = 1$ theory becomes a particle with a mass spectrum and internal degrees of freedom in a $q \neq 1$ theory. For small values of n the mass spectrum resembles a string spectrum with the tension determined by q and m_o. If $q < 1$ spectrum will be inverted and bounded. In the present $SU_q(2)$ example, $q < 1$.

One may construct similar arguments for the mass terms of other q-fields. In every case, however, the actual dependence of the field on the q-algebra is gauge dependent. To make sense of this procedure one requires the introduction of a privileged gauge, but that is also the way in which the Higgs mechanism and the symmetry breaking vacuum completes the original Yang-Mills theory.

Alternative to introducing an internal state space is averaging over the q-group space with the aid of the Woronowicz integral [2]. In this alternative formulation one replaces the action (2.3) which lies in the q-algebra, and is therefore not numerically valued, by the following average over the algebra:

$$S = h \int d^4x\, L$$

where h stands for the Haar measure, or the Woronowicz integral, which is a linear functional that may be evaluated term by term according to

$$h[\alpha^s \beta^n \bar{\beta}^m] = \delta^{s0}\delta^{mn} \frac{q^n}{[m+1]_q}.$$

Thus h projects out a power of $\beta\bar{\beta}$ that is evaluated as the same power of q. This result can be easily compared with the very similar result obtained by the alternative procedure that employs the state space.

The apparent similarity of a local gauge theory based on $SU_q(2)$ to one based on $SU(2)$ is misleading, however, since only the latter has closure, i.e. if T_1 and $T_2 \in SU(2)$ then $T_1T_2 \in SU(2)$; but if T_1 and $T_2 \in SU_q(2)$ then $T_1T_2 \notin SU_q(2)$ unless the elements of T_1 commute with the elements of T_2. In other words T_1 and T_2 must belong to different copies of the algebra. For example, it is possible that $T_1 = T(x_1)$ and $T_2 = T(x_2)$ where x_1 and x_2 both lie in spacetime but are not causally related; another possibility is offered by a Kaluza-Klein theory if x_1 and x_2 do not both lie in 4-dimensional spacetime. Such a collection of transformations may be regarded as a non-local group.

Alternatively one can assume that there are only two "gauges" and that T is a duality transformation. One example of this kind of duality is realized by the q-harmonic oscillator which has equivalent descriptions in terms of either (x,p) or $(a,\bar{a})$. These two representations are related by the q-canonical transformation:

$$\begin{pmatrix} a \\ \bar{a} \end{pmatrix} = T \begin{pmatrix} \frac{i}{\hbar}p \\ x \end{pmatrix} \qquad T \in SU_q(2).$$

Both sets satisfy q-commutation rules, namely:

$$a\bar{a} - q\bar{a}a = 1,$$
$$qxp - px = i\hbar.$$

One may promote the oscillator to the role of a scalar field by introducing the conjugate fields $\pi(x)$ and $\psi(x)$ satisfying

$$q\psi(x')\pi(x) - \pi(x)\psi(x') = i\hbar\delta(x - x').$$

If the Fourier components of $\pi(x)$ and $\psi(x)$ are p_k and q_k and if we set

$$\begin{pmatrix} a_k \\ \bar{a}_k \end{pmatrix} = T \begin{pmatrix} \frac{i}{\hbar}p_k \\ q_k \end{pmatrix}$$

then the field oscillators will satisfy q-commutators:

$$a_k\bar{a}_{k'} - q\bar{a}_{k'}a_k = \delta_{kk'}. \tag{5.5}$$

If one now calculates the energy of the free field one finds the eigenvalues

$$\langle n_k \rangle \hbar\omega_k$$

instead of

$$n_k\hbar\omega_k.$$

If q is close to unity and we set $q = 1 + \varepsilon$, then

$$\langle n\rangle = \frac{q^n - 1}{q - 1}$$
$$\frac{\langle n\rangle}{n} = 1 + \frac{n-1}{2}\ \varepsilon$$

Hence

$$\langle n\rangle\hbar\omega = n\left[\hbar\omega + \frac{n-1}{2}\varepsilon\hbar\omega\right].$$

This result may be interpreted to describe either a field with populations of normal modes increasing as $\langle n\rangle$ with fixed mass or as populations increasing linearly with n but with increasing mass. This example differs from the preceding case in that here there is no "internal" quantum number and in addition the field commutators were normal in the other case. Both examples illustrate how the introduction of the q-algebra may alter the particle spectrum.

To complete the discussion of this case it is again necessary to introduce the internal space in order to compute transition probabilities. Then the field acquires an internal mass spectrum as in the preceding example. In both cases the internal $(\alpha, \bar{\alpha})$ space is associated with the algebra (3.1). In the present case the $(\alpha, \bar{\alpha})$ space results from (5.5). In the previous (q-Yang-Mills) case the dynamical fields lie in the algebra (3.1) because of the postulated $SU_q(2)$ invariance.

6. Transition amplitudes

We may consider the emission of a q-vector by a q-spinor induced by the following interaction appearing in (2.3):

$$\int i\psi^t(x)C\varepsilon\gamma^\mu A_\mu(x)\psi(x)d\vec{x}$$

where

$$A_\mu(x) = \frac{1}{(2\pi)^{3/2}}\int\frac{d\vec{p}}{(2p_o)^{1/2}}\sum_p e_\mu(\rho)\left[e^{-ipx}A(\rho) + e^{ipx}\bar{a}(\rho)\right]$$

and

$$e_\mu(\rho) = \sum_s e_\mu(\rho, s)\tau_s \qquad \rho = (\vec{p}, r).$$

Here $(\vec{p}, r)$ labels momentum and polarization of the vector particle while τ_s is to be summed over a selected portion of the internal algebra generated by $(\alpha, \bar{\alpha}, \beta, \bar{\beta})$. Similarly we set

$$\psi(x) = \sum_s \psi_s(x)\hat{\tau}_s.$$

Then the general matrix element of interest is

$$\int d\vec{x}\langle N'_\psi N'_A n'_\psi n'_A|i\psi^t(x)C\varepsilon\gamma^\mu A_\mu(x)\psi(x)|N_\psi N_A n_\psi n_A\rangle$$

where N and n refer to the external and internal Fock spaces.

The emission of a specific vector particle (ρ) by the fermionic source will increase the corresponding population of the field from N to $N+1$. If a single fermion makes a transition from n_ψ to n'_ψ in the same event then the amplitude for the joint transition is

$$\int d\vec{x}\langle n'_\psi n'_A|i\psi^t(x)C\varepsilon\gamma^\mu\psi(x)A_\mu(x)|n_\psi n_A\rangle(N_A+1)^{1/2}$$

or

$$\int d\vec{x}\langle n'_\psi|i\psi^t(x)C\varepsilon\gamma^\mu\psi(x)|n_\psi\rangle\langle n'_A|A_\mu(x)|n_A\rangle(N_A+1)^{1/2}.$$

In the $q=1$ theories there is no need of internal quantum numbers like n_ψ and n_A.

Note that if τ_s or $\hat{\tau}_s$ is a function of β and $\bar{\beta}$ and does not contain free factors of α and $\bar{\alpha}$, then there is no change in internal quantum number in the transition. On the other hand one has

$$\langle n'|\bar{\alpha}|n\rangle = \delta(n',n+1)(1-q^{2n+2})^{1/2}.$$

The ratio of the emission to the absorption probabilities of a vector particle is

$$R = \frac{|\langle n+1|\bar{\alpha}|n\rangle|^2}{|\langle n-1|\alpha|n\rangle|^2} = \frac{1-q^{2(n+1)}}{1-q^{2n}}$$

multiplied by the corresponding factor for the spinor transition.

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