Quantum homogeneous spaces and quasi-Hopf algebras

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Abstract. We propose a formulation of the quantization problem of Manin quadruples, and show that a solution to this problem yields a quantization of the corresponding Poisson homogeneous spaces. We then solve both quantization problems in an example related to quantum spheres.

Keywords: Manin triples, Manin pairs, Poisson-Lie groups, homogeneous spaces, Hopf algebras, quasi-Hopf algebras

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To the memory of Moshé Flato

Introduction

According to a theorem of Drinfeld, formal Poisson homogeneous spaces over a formal Poisson-Lie group G_+ with Lie algebra $\mathfrak{g}_+$ correspond bijectively to G_+-conjugation classes of Lagrangian (i.e., maximal isotropic) Lie subalgebras $\mathfrak{h}$ of the double Lie algebra $\mathfrak{g}$ of $\mathfrak{g}_+$. The formal Poisson homogenous space is then $G_+/(G_+ \cap H)$, where H is the formal Lie group with Lie algebra $\mathfrak{h}$. The corresponding quantization problem is to deform the algebra of functions over the homogeneous space to an algebra-module over the quantized enveloping algebra of $\mathfrak{g}_+$.

In this paper, we show that there is a Poisson homogeneous structure on the formal homogeneous space G/H, such that the embedding of $G_+/(G_+ \cap H)$ in G/H is Poisson, where G is the formal Lie group with Lie algebra $\mathfrak{g}$. It is therefore natural to seek a quantization of the function algebra of $G_+/(G_+ \cap H)$ as a quotient of a quantization of G/H.

The data of $(\mathfrak{g}, \mathfrak{h})$ and the r-matrix of $\mathfrak{g}$ constitute an example of a quasitriangular Manin pair (see Section 1.1). We introduce the notion of the quantization of a quasitriangular Manin pair, which consists of a quasitriangular Hopf algebra quantizing the Lie bialgebra $\mathfrak{g}$, quasi-Hopf algebras quantizing the Manin pair

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$(\mathfrak{g}, \mathfrak{h})$, and a twist element relating both structures. We then show (Theorem 2.1) that this data gives rise to a quantization of G/H.

The quadruple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, \mathfrak{h})$ formed by adjoining $\mathfrak{h}$ to the Manin triple of $\mathfrak{g}_+$ is called a Manin quadruple. The quantization of a Manin quadruple is the additional data of Hopf algebras quantizing the Manin triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$, subject to compatibility conditions with the quantization of the underlying quasitriangular Manin pair. We show that any quantization of a given quadruple gives rise to a quantization of the corresponding homogeneous space (Theorem 3.1).

Finally, in Section 4.1, we explicitly solve the problem of quantizing a Manin quadruple in a situation related to quantum spheres.

1. Manin quadruples and Poisson homogeneous spaces

In this section, we define Lie-algebraic structures, i.e., quasitriangular Manin pairs and Manin quadruples, and the Poisson homogeneous spaces naturally associated to them.

1.1. Quasitriangular Manin pairs

Recall that a *quasitriangular Lie bialgebra* is a triple $(\mathfrak{g}, r, \langle\ ,\ \rangle_{\mathfrak{g}})$, where $\mathfrak{g}$ is a complex Lie algebra, $\langle\ ,\ \rangle_{\mathfrak{g}}$ is a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$, and r is an element of $\mathfrak{g}\otimes\mathfrak{g}$ such that $r + r^{(21)}$ is the symmetric element of $\mathfrak{g}\otimes\mathfrak{g}$ defined by $\langle\ ,\ \rangle_{\mathfrak{g}}$, and r satisfies the classical Yang-Baxter equation, $[r^{(12)}, r^{(13)}] + [r^{(12)}, r^{(23)}] + [r^{(13)}, r^{(23)}] = 0$. (Such Lie bialgebras are also called factorizable.)

Assume that $\mathfrak{g}$ is finite dimensional, and let G be a Lie group with Lie algebra $\mathfrak{g}$. Then G is equipped with the Poisson-Lie bivector $P_G = r^L - r^R$, where, for any element a of $\mathfrak{g}\otimes\mathfrak{g}$, we denote the right- and left-invariant 2-tensors on G corresponding to a by a^L and a^R. If $\mathfrak{g}$ is an arbitrary Lie algebra, the same statement holds for its formal Lie group.

We will call the pair $(\mathfrak{g}, r, \langle\ ,\ \rangle_{\mathfrak{g}}, \mathfrak{h})$ of a quasitriangular Lie bialgebra $(\mathfrak{g}, r, \langle\ ,\ \rangle_{\mathfrak{g}})$ and a Lagrangian Lie subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ a *quasitriangular Manin pair*.

Assume that $(\mathfrak{g}, r, \langle\ ,\ \rangle_{\mathfrak{g}}, \mathfrak{h})$ is a quasitriangular Manin pair, and that L is a Lagrangian complement of $\mathfrak{h}$ in $\mathfrak{g}$. Let (ε^i) and (ε_i) be dual bases of $\mathfrak{h}$ and L, and set $r_{\mathfrak{h},L} = \sum_i \varepsilon^i \otimes \varepsilon_i$. The restriction to L of the Lie bracket of $\mathfrak{g}$ followed by the projection to the first factor in $\mathfrak{g} = \mathfrak{h} \oplus L$ yields an element $\varphi_{\mathfrak{h},L}$ of $\wedge^3\mathfrak{h}$. Let us set $f_{\mathfrak{h},L} = r_{\mathfrak{h},L} - r$; then $f_{\mathfrak{h},L}$ belongs to $\wedge^2\mathfrak{g}$.

Then the twist of the Lie bialgebra $(\mathfrak{g}, \partial r)$ by $f_{\mathfrak{h},L}$ is the quasitriangular Lie quasi-bialgebra $(\mathfrak{g}, \partial r_{\mathfrak{h},L}, \varphi_{\mathfrak{h},L})$. The cocycle $\partial r_{\mathfrak{h},L}$ maps $\mathfrak{h}$ to $\wedge^2\mathfrak{h}$, so $(\mathfrak{h}, (\partial r_{\mathfrak{h},L})_{|\mathfrak{h}}, \varphi_{\mathfrak{h},L})$ is a sub-Lie quasi-bialgebra of $(\mathfrak{g}, \partial r_{\mathfrak{h},L}, \varphi_{\mathfrak{h},L})$.

1.2. Manin quadruples

1.2.1. *Definition*

A *Manin quadruple* is a quadruple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, \mathfrak{h})$, where $\mathfrak{g}$ is a complex Lie algebra, equipped with a nondegenerate invariant symmetric bilinear form $\langle\ ,\ \rangle_{\mathfrak{g}}$, and $\mathfrak{g}_+, \mathfrak{g}_-$ and $\mathfrak{h}$ are three Lagrangian subalgebras of $\mathfrak{g}$, such that $\mathfrak{g}$ is equal to the direct sum $\mathfrak{g}_+ \oplus \mathfrak{g}_-$ (see [11]).

In particular, $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is a Manin triple. This implies that for any Lie group G with subgroups G_+, G_- integrating $\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-$, we obtain a Poisson-Lie group structure P_G on G, such that G_+ and G_- are Poisson-Lie subgroups of (G, P_G) (see [6]). We denote the corresponding Poisson structures on G_+ and G_- by P_{G_+} and P_{G_-}.

Any Manin quadruple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, \mathfrak{h})$ gives rise to a quasitriangular Manin pair $(\mathfrak{g}, r, \langle\ ,\ \rangle_{\mathfrak{g}}, \mathfrak{h})$, where $r = r_{\mathfrak{g}_+,\mathfrak{g}_-} = \sum_i e^i \otimes e_i$, and $(e^i), (e_i)$ are dual bases of $\mathfrak{g}_+$ and $\mathfrak{g}_-$.

1.2.2. *Examples*

In the case where $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is the Manin triple associated with the Sklyanin structure on a semisimple Lie group G_+, the Manin quadruples were classified in [12]. (See also [4].) It was shown in [14] that the Lagrangian subalgebras $\mathfrak{h}$ of $\mathfrak{g}$ such that the intersection $\mathfrak{g}_+ \cap \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}_+$ correspond bijectively to the classical dynamical r-matrices for $\mathfrak{g}_+$. We will treat the quantization of this example in Section 4, in the case where $\mathfrak{g}_+ = \mathfrak{sl}_2$.

In [9], the following class of Manin quadruples was studied. Let $\bar{\mathfrak{g}}$ be a semisimple Lie algebra with nondegenerate invariant symmetric bilinear form $\langle\ ,\ \rangle_{\bar{\mathfrak{g}}}$, and Cartan decomposition $\bar{\mathfrak{g}} = \bar{\mathfrak{n}}_+ \oplus \bar{\mathfrak{h}} \oplus \bar{\mathfrak{n}}_-$. Let $\mathscr{K}$ be a commutative ring equipped with a nondegenerate symmetric bilinear form $\langle\ ,\ \rangle_{\mathscr{K}}$. Assume that $R \subset \mathscr{K}$ is a Lagrangian subring of $\mathscr{K}$, with Lagrangian complement Λ. Let us set $\mathfrak{g} = \bar{\mathfrak{g}} \otimes \mathscr{K}$, let us equip $\mathfrak{g}$ with the bilinear form $\langle\ ,\ \rangle_{\bar{\mathfrak{g}}} \otimes \langle\ ,\ \rangle_{\mathscr{K}}$ and let us set

$$\mathfrak{g}_+ = (\bar{\mathfrak{h}} \otimes R) \oplus (\bar{\mathfrak{n}}_+ \otimes \mathscr{K}), \quad \mathfrak{g}_- = (\bar{\mathfrak{h}} \otimes \Lambda) \oplus (\bar{\mathfrak{n}}_- \otimes \mathscr{K}), \quad \mathfrak{h} = \bar{\mathfrak{g}} \otimes R. \qquad (1)$$

More generally, extensions of these Lie algebras, connected with the additional data of a derivation ∂ of $\mathscr{K}$ leaving $\langle\ ,\ \rangle_{\mathscr{K}}$ invariant and preserving R, were considered in [9]. Examples of quadruples $(\mathscr{K}, \partial, R, \langle\ ,\ \rangle_{\mathscr{K}})$, where $\mathscr{K}$ is an infinite-dimensional vector space, arise in the theory of complex curves.

1.3. Formal Poisson homogeneous spaces

In what follows, all homogeneous spaces will be formal, so if $\mathfrak{a}$ is a Lie algebra and A is the associated formal group, the function ring of A is $\mathscr{O}_A := (U\mathfrak{a})^*$, and if $\mathfrak{b}$ is a Lie subalgebra of $\mathfrak{a}$, and B is the associated formal group, the function ring of A/B is $\mathscr{O}_{A/B} := (U\mathfrak{a}/(U\mathfrak{a})\mathfrak{b})^*$.

We will need the following result on formal homogeneous spaces.

Lemma 1.1 *Let $\mathfrak{a}$ be a Lie algebra, let $\mathfrak{a}_+$ and $\mathfrak{b}$ be Lie subalgebras of $\mathfrak{a}$, and let A, A_+, and B be the associated formal groups. The restriction map*

$(U\mathfrak{a}/(U\mathfrak{a})\mathfrak{b})^* \to (U\mathfrak{a}_+/(U\mathfrak{a}_+(\mathfrak{a}_+ \cap \mathfrak{b})))^*$ *is a surjective morphism of algebras from* $\mathscr{O}_{A/B}$ *to* $\mathscr{O}_{A_+/(A_+\cap B)}$.

Proof. Let L_+ be a complement of $\mathfrak{a}_+ \cap \mathfrak{b}$ in $\mathfrak{a}_+$, and let L be a complement of $\mathfrak{b}$ in $\mathfrak{a}$, containing L_+. When V is a vector space, we denote by $S(V)$ its symmetric algebra. The following diagram is commutative

$$\begin{array}{ccc} S(L_+) & \to & U\mathfrak{a}_+/(U\mathfrak{a}_+(\mathfrak{a}_+ \cap \mathfrak{b})) \\ \downarrow & & \downarrow \\ S(L) & \to & U\mathfrak{a}/(U\mathfrak{a})\mathfrak{b} \end{array}$$

The horizontal maps are the linear isomorphisms obtained by symmetrization. Since the natural map from $S(L_+)$ to $S(L)$ is injective, so is the right-hand vertical map, and its dual is surjective. ■

A *Poisson homogeneous space* (X, P_X) for a Poisson-Lie group (Γ, P_Γ) is a Poisson formal manifold (X, P_X), equipped with a transitive action of Γ, and such that the map $\Gamma \times X \to X$ is Poisson. Then there is a Lie subgroup Γ' of Γ such that $X = \Gamma/\Gamma'$. Following [8], we observe that these two conditions are equivalent: a) the projection map $\Gamma \to X$ is Poisson, b) the Poisson bivector P_X vanishes at one point of X. If either of these equivalent conditions is satisfied, (X, P_X) is said to be *of group type*. (See [11].)

1.4. Poisson homogeneous space structure on G/H

Let $(\mathfrak{g}, r, \langle\ ,\ \rangle_{\mathfrak{g}}, \mathfrak{h})$ be a quasitriangular Manin pair. Let H be the formal subgroup of G with Lie algebra $\mathfrak{h}$, and let $P_{G/H}$ be the 2-tensor on G/H equal to the projection of r^L.

Proposition 1.1 *Let* $(\mathfrak{g}, r, \langle\ ,\ \rangle_{\mathfrak{g}}, \mathfrak{h})$ *be a quasitriangular Manin pair. Then* $P_{G/H}$ *is a Poisson bivector on* G/H*, and* G/H *is a Poisson homogeneous space for* (G, P_G).

Proof. The only nonobvious property is the antisymmetry of the bracket defined by r^L. Let t denote the symmetric element of $\mathfrak{g} \otimes \mathfrak{g}$ defined by $\langle\ ,\ \rangle_{\mathfrak{g}}$. If f_1 and f_2 are right $\mathfrak{h}$-invariant functions on G, $\{f_1, f_2\} + \{f_2, f_1\} = t^L(df_1 \otimes df_2) = t^R(df_1 \otimes df_2)$, by the invariance of t. Since $\mathfrak{h}$ is Lagrangian, t belongs to $\mathfrak{h} \otimes \mathfrak{g} + \mathfrak{g} \otimes \mathfrak{h}$. Since moreover f_1 and f_2 are $\mathfrak{h}$-invariant, $\{f_1, f_2\} + \{f_2, f_1\}$ vanishes. ■

Remark 1 In the case where $(\mathfrak{g}, r, \langle\ ,\ \rangle_{\mathfrak{g}}, \mathfrak{h})$ corresponds to a Manin quadruple, this statement has been proved by Etingof and Kazhdan, who also constructed a quantization of this Poisson homogeneous space ([11]).

Remark 2 For g in G and x in $\mathfrak{g}$, let us denote the adjoint action of g on x by ${}^g x$. In the case of a quadruple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, \mathfrak{h})$, the Poisson homogeneous space $(G/H, P_{G/H})$ is of group type if and only if there exists g in G such that ${}^g\mathfrak{h}$ is *graded* for the Manin triple decomposition, i.e., such that ${}^g\mathfrak{h} = ({}^g\mathfrak{h} \cap \mathfrak{g}_+) \oplus ({}^g\mathfrak{h} \cap \mathfrak{g}_-)$.

1.5. Poisson homogeneous space structure on $G_+/(G_+\cap H)$

Let $(\mathfrak{g},\mathfrak{g}_+,\mathfrak{g}_-,\mathfrak{h})$ be a Manin quadruple. The inclusion $G_+\subset G$ induces an inclusion map $i : G_+/(G_+\cap H)\to G/H$. On the other hand, by Proposition 1.1, $(\mathfrak{g},\mathfrak{g}_+,\mathfrak{g}_-,\mathfrak{h})$ defines a Poisson structure on G/H.

Proposition 1.2 *There exists a unique Poisson structure $P_{G_+/(G_+\cap H)}$ on $G_+/(G_+\cap H)$ such that the inclusion i is Poisson. Then $(G_+/(G_+\cap H),P_{G_+/(G_+\cap H)})$ is a Poisson homogeneous space for (G_+,P_{G_+}).*

Proof. Let $\langle\ ,\ \rangle_{\mathfrak{g}\otimes\mathfrak{g}}$ denote the bilinear form on $\mathfrak{g}\otimes\mathfrak{g}$ defined as the tensor square of $\langle\ ,\ \rangle_{\mathfrak{g}}$. We must prove that for any g in G_+, $r_{\mathfrak{g}_+,\mathfrak{g}_-}$ belongs to $\mathfrak{g}_+\otimes\mathfrak{g}_+ + \mathfrak{g}\otimes{}^g\mathfrak{h} + {}^g\mathfrak{h}\otimes\mathfrak{g}$. The annihilator of this space in $\mathfrak{g}\otimes\mathfrak{g}$ with respect to $\langle\ ,\ \rangle_{\mathfrak{g}\otimes\mathfrak{g}}$ is $(\mathfrak{g}_+\cap{}^g\mathfrak{h})\otimes{}^g\mathfrak{h}+{}^g\mathfrak{h}\otimes(\mathfrak{g}_+\cap{}^g\mathfrak{h})$. Let us show that for any x in this annihilator, $\langle r_{\mathfrak{g}_+,\mathfrak{g}_-},x\rangle_{\mathfrak{g}\otimes\mathfrak{g}}$ is zero. Assume that $x=v\otimes w$, where $v\in{}^g\mathfrak{h}, w\in\mathfrak{g}_+\cap{}^g\mathfrak{h}$; then

$$\langle r_{\mathfrak{g}_+,\mathfrak{g}_-},x\rangle_{\mathfrak{g}\otimes\mathfrak{g}} = \langle\sum_i e^i\otimes e_i, v\otimes w\rangle_{\mathfrak{g}\otimes\mathfrak{g}} = \langle v,w\rangle_{\mathfrak{g}} = 0,$$

where the second equality follows from the facts that $\mathfrak{g}_+$ is isotropic and that $(e^i),(e_i)$ are dual bases, and the last equality follows from the isotropy of ${}^g\mathfrak{h}$. On the other hand, the isotropy of $\mathfrak{g}_+$ implies that $\langle r_{\mathfrak{g}_+,\mathfrak{g}_-},(\mathfrak{g}_+\cap{}^g\mathfrak{h})\otimes{}^g\mathfrak{h}\rangle_{\mathfrak{g}\otimes\mathfrak{g}}=0$. Therefore $r_{\mathfrak{g}_+,\mathfrak{g}_-}$ belongs to $\mathfrak{g}_+\otimes\mathfrak{g}_+ + \mathfrak{g}\otimes{}^g\mathfrak{h}+{}^g\mathfrak{h}\otimes\mathfrak{g}$; this implies the first part of the Proposition.

Let us prove that $(G_+/(G_+\cap H),P_{G_+/(G_+\cap H)})$ is a Poisson homogeneous space for (G_+,P_{G_+}). We have a commutative diagram

$$\begin{array}{ccc} G_+\times(G_+/(G_+\cap H)) & \stackrel{a_{G_+}}{\to} & G_+/(G_+\cap H) \\ i_G\times i\downarrow & & i\downarrow \\ G\times(G/H) & \stackrel{a_G}{\to} & G/H \end{array}$$

where i_G is the inclusion map of G_+ in G and a_G (resp., a_{G_+}) is the action map of G on G/H (resp., of G_+ on $G_+/(G_+\cap H)$). The maps $i_G\times i$, i and a_G are Poisson maps; since i induces an injection of tangent spaces, it follows that a_{G_+} is a Poisson map. ∎

In [8], Drinfeld defined a Poisson bivector $P'_{G_+/(G_+\cap H)}$ on $G_+/(G_+\cap H)$, which can be described as follows. When V is a Lagrangian subspace of $\mathfrak{g}$, identify $(\mathfrak{g}_+\cap V)^\perp$ with a subspace of $\mathfrak{g}_-$, and define $\bar{\xi}_V:(\mathfrak{g}_+\cap V)^\perp\to\mathfrak{g}_+/(\mathfrak{g}_+\cap V)$ to be the linear map which, to any element a_- of $(\mathfrak{g}_+\cap V)^\perp\subset\mathfrak{g}_-$, associates the class of an element $a_+\in\mathfrak{g}_+$ such that a_++a_- belongs to V. Then there is a unique element $\xi_V\in(\mathfrak{g}_+/(\mathfrak{g}_+\cap V))^{\otimes 2}$, such that $(a_-\otimes id)(\xi_V)=\bar{\xi}_V(a_-)$, for any $a_-\in(\mathfrak{g}_+\cap V)^\perp$.

For any g in G_+, identify the tangent space of $G_+/(G_+\cap H)$ at $g(G_+\cap H)$ with $\mathfrak{g}_+/(\mathfrak{g}_+\cap{}^g\mathfrak{h})$ via left-invariant vector fields. The element ξ_g of $(\mathfrak{g}_+/(\mathfrak{g}_+\cap{}^g\mathfrak{h}))^{\otimes 2}$ corresponding to the value of $P'_{G_+/(G_+\cap H)}$ at $g(G_+\cap H)$ is then $\xi_{{}^g\mathfrak{h}}$.

Proposition 1.3 *The bivectors $P_{G_+/(G_+\cap H)}$ and $P'_{G_+/(G_+\cap H)}$ are equal.*

Proof. We have to show that the injection $i : G_+/(G_+\cap H) \to G/H$ is compatible with the bivectors $P'_{G_+/(G_+\cap H)}$ and $P_{G/H}$. The differential of i at $g(G_+\cap H)$, where g belongs to G_+, induces the canonical injection ι from $\mathfrak{g}_+/(\mathfrak{g}_+\cap {}^g\mathfrak{h})$ to $\mathfrak{g}/{}^g\mathfrak{h}$. Let us show that for any g in G_+, the injection $\iota\otimes\iota$ maps $\xi_g \in (\mathfrak{g}_+/(\mathfrak{g}_+\cap {}^g\mathfrak{h}))^{\otimes 2}$ to the class of $r_{\mathfrak{g}_+,\mathfrak{g}_-}$ in $(\mathfrak{g}/{}^g\mathfrak{h})^{\otimes 2}$. We have to verify the commutativity of the diagram

$$\begin{array}{ccc} (\mathfrak{g}_+\cap {}^g\mathfrak{h})^\perp & \stackrel{\iota^*}{\leftarrow} & ({}^g\mathfrak{h})^\perp \\ \bar{\xi}_g\downarrow & & \bar{r}_g\downarrow \\ \mathfrak{g}_+/(\mathfrak{g}_+\cap {}^g\mathfrak{h}) & \stackrel{\iota}{\to} & \mathfrak{g}/{}^g\mathfrak{h} \end{array}$$

where the horizontal maps are the natural injection and restriction maps, and $\bar{r}_g$ is defined by $\bar{r}_g(a) =$ the class of $\langle r_{\mathfrak{g}_+,\mathfrak{g}_-}, a\otimes id\rangle_{\mathfrak{g}\otimes\mathfrak{g}}$ mod ${}^g\mathfrak{h}$, for $a \in ({}^g\mathfrak{h})^\perp$. Let a belong to $({}^g\mathfrak{h})^\perp$. By the maximal isotropy of ${}^g\mathfrak{h}$, the element a can be identified with an element of ${}^g\mathfrak{h}$. Let us write $a = a_+ + a_-$, with $a_\pm \in \mathfrak{g}_\pm$. Then $\iota^*(a) = a_-$, $\bar{\xi}_g(a_-) = a_+ + (\mathfrak{g}_+\cap {}^g\mathfrak{h})$ by the definition of $\bar{\xi}_g$ and because $a \in {}^g\mathfrak{h}$, and $\iota(a_+ + \mathfrak{g}_+\cap {}^g\mathfrak{h}) = a_+ + {}^g\mathfrak{h}$. On the other hand, $\bar{r}_g(a) = a_+ + {}^g\mathfrak{h}$, so the diagram commutes. ■

Moreover, it is a result of Drinfeld ([8], Remark 2) that the formal Poisson homogeneous spaces for (G_+, P_{G_+}) are all of the type described in Proposition 1.2.

Remark 3 The (G_+, P_{G_+})-Poisson homogeneous space

$$(G_+/(G_+\cap H), P_{G_+/(G_+\cap H)})$$

is of group type if and only if for some $g \in G_+$, ${}^g\mathfrak{h}$ is graded for the Manin triple decomposition, see Remark 2.

Remark 4 *Conjugates of Manin quadruples.* Let us denote the conjugate of an element x in G by an element h in G by ${}^gx = gxg^{-1}$. For $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, \mathfrak{h})$ a Manin quadruple, and g in G, $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, {}^g\mathfrak{h})$ is also a Manin quadruple. If g belongs to H, this is the same quadruple; and if g belongs to G_+, the Poisson structure induced by $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, {}^g\mathfrak{h})$ on $G_+/(G_+\cap {}^gH)$ is isomorphic to that induced by $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, \mathfrak{h})$ on $G_+/(G_+\cap H)$, via conjugation by g. It follows that Poisson homogeneous space structures induced by $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, \mathfrak{h})$ are indexed by elements of the double quotient $G_+\setminus G/H$.

Remark 5 The first examples of Poisson homogeneous spaces which are not of group type were studied in [3] and [13], under the name of affine Poisson structures. In these cases, the stabilizer of a point is trivial.

2. Quantization of G/H

In this section, we introduce axioms for the quantization of a quasitriangular Manin pair. We then show that any solution to this quantization problem leads to a quantization of the Poisson homogeneous space G/H constructed in Section 1.4.

2.1. Definition of quantization of Poisson homogeneous spaces

Let (Γ, P_Γ) be a Poisson-Lie group and let (X, P_X) be a formal Poisson homogeneous space over (Γ, P_Γ). Let $(\mathscr{A}, \Delta_{\mathscr{A}})$ be a quantization of the enveloping algebra of the Lie algebra of Γ, and let $\mathscr{A}^{opp}$ denote the opposite algebra to $\mathscr{A}$.

Definition 2.1 *A* quantization of the Poisson homogeneous space (X, P_X) *is a* $\mathbb{C}[[\hbar]]$*-algebra* $\mathscr{X}$*, such that*
1) $\mathscr{X}$ *is a quantization of the Poisson algebra* $(\mathscr{O}_X, P_X)$ *of formal functions on X, and*
2) $\mathscr{X}$ *is equipped with an algebra-module structure over* $(\mathscr{A}^{opp}, \Delta)$*, whose reduction mod* $\hbar$ *coincides with the algebra-module structure of* $\mathscr{O}_X$ *over* $((U\mathfrak{g})^{opp}, \Delta_0)$*, where* Δ_0 *is the coproduct of* $U\mathfrak{g}$.

Condition 1 means that there is an isomorphism of $\mathbb{C}[[\hbar]]$-modules from $\mathscr{X}$ to $\mathscr{O}_X[[\hbar]]$, inducing an algebra isomorphism between $\mathscr{X}/\hbar\mathscr{X}$ and $\mathscr{O}_X$, and inducing on $\mathscr{O}_X$ the Poisson structure defined by P_X.

The first part of condition 2 means that $\mathscr{X}$ has a module structure over $\mathscr{A}^{opp}$, such that for $x, y \in \mathscr{X}, a \in \mathscr{A}$, and $\Delta(a) = \sum a^{(1)} \otimes a^{(2)}$, $a(xy) = \sum a^{(1)}(x) a^{(2)}(y)$.

Conventions
We will say that a $\mathbb{C}[[\hbar]]$-module V is *topologically free* if it is isomorphic to $W[[\hbar]]$, where W is a complex vector space. We denote the canonical projection of V onto $V/\hbar V$ by $v \mapsto v$ mod $\hbar$. In what follows, all tensor products of $\mathbb{C}[[\hbar]]$-modules are $\hbar$-adically completed. When E is a $\mathbb{C}[[\hbar]]$-module, we denote by E^* its dual $\mathrm{Hom}_{\mathbb{C}[[\hbar]]}(E, \mathbb{C}[[\hbar]])$. For a subset $\mathscr{S}$ of an algebra $\mathscr{A}$, we denote by $\mathscr{S}^\times$ the group of invertible elements of $\mathscr{S}$. When $\mathscr{A}, \mathscr{B}$ are two Hopf or quasi-Hopf algebras with unit elements $1_{\mathscr{A}}, 1_{\mathscr{B}}$ and counit maps $\varepsilon_{\mathscr{A}}, \varepsilon_{\mathscr{B}}$, and $\mathscr{S}$ is a subset of $\mathscr{A} \otimes \mathscr{B}$, we denote by $\mathscr{S}_0^\times$ the subgroup of $\mathscr{S}^\times$ with elements x such that $(\varepsilon_{\mathscr{A}} \otimes id)(x) = 1_{\mathscr{B}}$ and $(id \otimes \varepsilon_{\mathscr{B}})(x) = 1_{\mathscr{A}}$. We also denote $\mathrm{Ker}\,\varepsilon_{\mathscr{A}}$ by $\mathscr{A}_0$.

2.2. Quantization of quasitriangular Manin pairs

Definition 2.2 *A* quantization of a quasitriangular Manin pair $(\mathfrak{g}, r, \langle\ ,\ \rangle_{\mathfrak{g}}, \mathfrak{h})$ *is the data of*
1) a quasitriangular Hopf algebra $(A, \Delta, \mathscr{R})$ *quantizing* $(\mathfrak{g}, r, \langle\ ,\ \rangle_{\mathfrak{g}})$*,*
2) a subalgebra $B \subset A$ *and an element* F *in* $(A \otimes A)_0^\times$*, such that*
a) $B \subset A$ *is a flat deformation of the inclusion* $U\mathfrak{h} \subset U\mathfrak{g}$*,*
b) $F\Delta(B)F^{-1} \subset B \otimes B$*, and* $F^{(12)}(\Delta \otimes id)(F)\left(F^{(23)}(id \otimes \Delta)(F)\right)^{-1} \in B^{\otimes 3}$*,*
c) there exists a Lagrangian complement L *of* $\mathfrak{h}$ *in* $\mathfrak{g}$*, such that*

$$\left(\frac{1}{\hbar}(F - F^{(21)}) \text{ } mod \text{ } \hbar\right) = r - r_{\mathfrak{h},L},$$

where $r_{\mathfrak{h},L} = \sum_i \varepsilon^i \otimes \varepsilon_i$, *and* $(\varepsilon^i), (\varepsilon_i)$ *are dual bases of* $\mathfrak{h}$ *and* L.

Observe that condition 1 implies that $(\frac{1}{\hbar}(\mathscr{R}-1) \bmod \hbar) = r$. In condition 2, $(A \otimes A)_0^\times = 1 + \hbar(A_0 \otimes A_0)$. Condition 2a means that the $\mathbb{C}[[\hbar]]$-module isomorphism between A and $U\mathfrak{g}[[\hbar]]$ arising from condition 1 can be chosen in such a way that it induces an isomorphism between B and $U\mathfrak{h}[[\hbar]]$.

Let (A, Δ_B, Φ_B) be the quasi-Hopf algebra obtained by twisting the Hopf algebra (A, Δ) by F. Then, by definition, $\Delta_B(x) = F\Delta(x)F^{-1}$, for any $x \in A$, and $\Phi_B = F^{(12)}(\Delta \otimes id)(F)\left(F^{(23)}(id \otimes \Delta)(F)\right)^{-1}$. Condition 2b expresses the fact that $B \subset A$ is a sub-quasi-Hopf algebra of A.

Condition 2c expresses the fact that the classical limit of (B, Δ_B, Φ_B) is $\mathfrak{h}$ equipped with the Lie quasi-bialgebra structure associated with L.

2.3. Quantization of $(G/H, P_{G/H})$

Let us denote the counit map of A by ε.

Theorem 2.1 *Assume that* $((A, \Delta, \mathscr{R}), B, F)$ *is a quantization of the quasitriangular Manin pair* $(\mathfrak{g}, r, \langle\ ,\ \rangle_{\mathfrak{g}}, \mathfrak{h})$. *Let* $(A^*)^B$ *be the subspace of* A^* *consisting of the forms* ℓ *on A such that* $\ell(ab) = \ell(a)\varepsilon(b)$ *for any* $a \in A$ *and* $b \in B$.

a) For ℓ, ℓ' *in* $(A^*)^B$, *define* $\ell * \ell'$ *to be the element of* A^* *such that*

$$(\ell * \ell')(a) = (\ell \otimes \ell')(\Delta(a)F^{-1}),$$

for any a in A. Then $*$ *defines an associative algebra structure on* $(A^*)^B$.

b) For $a \in A$ *and* $\ell \in (A^*)^B$, *define* $a\ell$ *to be the form on A such that* $(a\ell)(a') = \ell(aa')$, *for any* $a' \in A$. *This map defines on* $((A^*)^B, *)$ *a structure of an algebra-module over the Hopf algebra* (A^{opp}, Δ).

c) The algebra $((A^*)^B, *)$ *is a quantization of the Poisson algebra* $(\mathscr{O}_{G/H}, P_{G/H})$.

With its algebra-module structure over (A^{opp}, Δ), $((A^*)^B, *)$ *is a quantization of the Poisson homogeneous space* $(G/H, P_{G/H})$.

Proof. Let ℓ and ℓ' belong to $(A^*)^B$. Then for any $a \in A, b \in B$,

$$\begin{aligned}(\ell * \ell')(ab) &= (\ell \otimes \ell')(\Delta(a)\Delta(b)F^{-1}) = (\ell \otimes \ell')(\Delta(a)F^{-1}\Delta_B(b)) \\ &= \varepsilon(b)(\ell \otimes \ell')(\Delta(a)F^{-1}) = \varepsilon(b)(\ell * \ell')(a),\end{aligned}$$

where the third equality follows from the fact that $\Delta_B(B) \subset B \otimes B$ and $(\varepsilon \otimes \varepsilon) \circ \Delta_B = \varepsilon$. It follows that $\ell * \ell'$ belongs to $(A^*)^B$.

Let ℓ, ℓ' and ℓ'' belong to $(A^*)^B$. Then for any a in A,

$$((\ell * \ell') * \ell'')(a) = (\ell \otimes \ell' \otimes \ell'')\left((\Delta \otimes id) \circ \Delta(a)(\Delta \otimes id)(F^{-1})(F^{(12)})^{-1}\right), \quad (2)$$

and

$$(\ell * (\ell' * \ell''))(a) = (\ell \otimes \ell' \otimes \ell'')\left((id \otimes \Delta) \circ \Delta(a)(id \otimes \Delta)(F^{-1})(F^{(23)})^{-1}\right).$$

By the coassociativity of Δ and the definition of Φ_B, this expression is equal to $(\ell \otimes \ell' \otimes \ell'')\left((\Delta \otimes id) \circ \Delta(a)(\Delta \otimes id)(F^{-1})(F^{(12)})^{-1}\Phi_B^{-1}\right)$, and since Φ_B belongs to $B^{\otimes 3}$ and $\varepsilon^{\otimes 3}(\Phi_B) = 1$, this is equal to the right-hand side of (2). It follows that $(\ell * \ell') * \ell'' = \ell * (\ell' * \ell'')$, so $*$ is associative. Moreover, ε belongs to $(A^*)^B$ and is the unit element of $((A^*)^B, *)$. This proves part a of the theorem.

Part b follows from the definitions.

Let us prove that $(A^*)^B$ is a flat deformation of $(U\mathfrak{g}/(U\mathfrak{g})\mathfrak{h})^*$. Let us consider a Lagrangian complement L of $\mathfrak{h}$ in $\mathfrak{g}$. Let Sym denote the symmetrisation map from the symmetric algebra $S(\mathfrak{g})$ of $\mathfrak{g}$ to $U\mathfrak{g}$, i.e., the unique linear map such that $\mathrm{Sym}(x^l) = x^l$ for any $x \in \mathfrak{g}$ and $l \geq 0$, and let us define $\widetilde{L}$ to be $\mathrm{Sym}(S(L))$. Thus $\widetilde{L}$ is a linear subspace of $U\mathfrak{g}$, and inclusion of $\widetilde{L} \otimes U\mathfrak{h}$ in $U\mathfrak{g} \otimes U\mathfrak{g}$ followed by multiplication induces a linear isomorphism from $\widetilde{L} \otimes U\mathfrak{h}$ to $U\mathfrak{g}$. It follows that the restriction to $\widetilde{L}$ of the projection $U\mathfrak{g} \to U\mathfrak{g}/(U\mathfrak{g})\mathfrak{h}$ is an isomorphism, which defines a linear isomorphism between $\mathscr{O}_{G/H} = (U\mathfrak{g}/(U\mathfrak{g})\mathfrak{h})^*$ and $\widetilde{L}^*$.

Let us fix an isomorphism of $\mathbb{C}[[\hbar]]$-modules from A to $U\mathfrak{g}[[\hbar]]$, inducing an isomorphism between B and $U\mathfrak{h}[[\hbar]]$ and let us define C to be the preimage of $\widetilde{L}[[\hbar]]$. Thus C is isomorphic to $\widetilde{L}[[\hbar]]$, and inclusion followed by multiplication induces a linear isomorphism between $C \otimes B$ and A, as does any morphism between two topologically free $\mathbb{C}[[\hbar]]$-modules E and F, which induces an isomorphism between $E/\hbar E$ and $F/\hbar F$. Therefore, restriction of linear forms to C induces an isomorphism between $(A^*)^B$ and C^*. It follows that $(A^*)^B$ is isomorphic to $(\widetilde{L}[[\hbar]])^*$, which is in turn isomorphic to $\widetilde{L}^*[[\hbar]]$. Since $\widetilde{L}^*$ is isomorphic to $\mathscr{O}_{G/H}$, $(A^*)^B$ is isomorphic to $\mathscr{O}_{G/H}[[\hbar]]$.

Let us fix ℓ, ℓ' in $(A^*)^B$, and let us compute $(\frac{1}{\hbar}(\ell * \ell' - \ell' * \ell) \bmod \hbar)$. Let us set $f = (\frac{1}{\hbar}(F-1) \bmod \hbar)$. For a in A,

$$\begin{aligned}\frac{1}{\hbar}(\ell * \ell' - \ell' * \ell)(a) &= (\ell \otimes \ell')\left(\frac{1}{\hbar}(\Delta(a)F^{-1} - \Delta'(a)(F^{(21)})^{-1})\right)\\ &= (\ell \otimes \ell')\left(\frac{1}{\hbar}(\Delta(a)F^{-1} - \mathscr{R}\Delta(a)\mathscr{R}^{-1}(F^{(21)})^{-1})\right)\\ &= (\ell \otimes \ell')\left(-r\Delta_0(a_0) + \Delta_0(a_0)(r + f^{(21)} - f)\right) + o(\hbar),\end{aligned}$$

where Δ_0 is the coproduct of $U\mathfrak{g}$ and a_0 is the image of a in $A/\hbar A = U\mathfrak{g}$. Since, by condition 2c of Definition 2.2, $r - f + f^{(21)}$ is equal to $r_{\mathfrak{h},L}$, it belongs to $\mathfrak{h} \otimes \mathfrak{g}$ and since ℓ and ℓ' are right B-invariant, $(\frac{1}{\hbar}(\ell * \ell' - \ell' * \ell)(a) \bmod \hbar) = (\ell \otimes \ell')(-r\Delta_0(a_0))$, which is the Poisson bracket defined by r^L on G/H. This ends the proof of part c of the theorem.

Since $(F \bmod \hbar) = 1$, the reduction modulo $\hbar$ of the algebra-module structure of $((A^*)^B, *)$ over (A^{opp}, Δ) is that of $\mathscr{O}_X$ over $((U\mathfrak{g})^{opp}, \Delta_0)$. This ends the proof of the theorem. ∎

Remark 6 If $((A,\Delta,\mathscr{R}),B,F)$ is a quantization of a quasitriangular Manin pair $(\mathfrak{g},r,\langle\ ,\ \rangle_{\mathfrak{g}},\mathfrak{h})$, and if F_0 is an element of $(B\otimes B)_0^{\times}$, then $((A,\Delta,\mathscr{R}),F_0F)$ is a quantization of the same Manin pair. We observe that the product $*$ on $(A^*)^B$ is independent of such a modification of F.

Remark 7 In the case of a Manin quadruple $(\mathfrak{g},\mathfrak{g}_+,\mathfrak{g}_-,\mathfrak{h})$ where $\mathfrak{h}$ is graded for the Manin triple decomposition (see Remark 2), $r_{\mathfrak{g}_+,\mathfrak{g}_-}-r_{\mathfrak{h},L}=f-f^{(21)}$ belongs to $\mathfrak{h}\otimes\mathfrak{g}+\mathfrak{g}\otimes\mathfrak{h}$. The corresponding quantum condition is that

$$F\in 1+\hbar(B_0\otimes A_0+A_0\otimes B_0). \tag{3}$$

This is the case is [9], where F belongs to $(B\otimes A)_0^{\times}=1+\hbar(B_0\otimes A_0)$.

When condition (3) is fulfilled, the product $*$ is the restriction to $(A^*)^B$ of the usual product on A^*, defined as the dual map to Δ.

2.4. Relations in $(A^*)^B$

It is well known that the matrix coefficients of the representations of a quasitriangular Hopf algebra can be organized in L-operators, satisfying the so-called *RLL* relations. We recall this construction and introduce analogues of these matrix coefficients and of the *RLL* relations for the algebra $(A^*)^B$.

Recall that there is an algebra structure on A^*, where the product is $(\ell,\ell')\mapsto\ell\ell'$, such that for any $a\in A$, $(\ell\ell')(a)=(\ell\otimes\ell')(\Delta(a))$.

Let $\mathrm{Rep}(A)$ be the category of modules over A, which are free and finite dimensional over $\mathbb{C}[[\hbar]]$. There is a unique map

$$\oplus_{V\in\mathrm{Rep}(A)}(V^*\otimes V)\to A^*,\quad \kappa\mapsto\ell_\kappa,$$

such that for any object (V,π_V) in $\mathrm{Rep}(A)$, and any $\xi\in V^*$ and $v\in V$, $\ell_{\xi\otimes v}(a)=\xi(\pi_V(a)v)$, for any $a\in A$. Define $\mathrm{Coeff}(A)$ to be the image of this map. Then $\mathrm{Coeff}(A)$ is a subalgebra of A^*. Moreover, there is a Hopf algebra structure on $\mathrm{Coeff}(A)$, with coproduct $\Delta_{\mathrm{Coeff}(A)}$ and counit $\varepsilon_{\mathrm{Coeff}(A)}$, uniquely determined by the rules

$$\Delta_{\mathrm{Coeff}(A)}(\ell_{\xi\otimes v})=\sum_i \ell_{\xi\otimes v_i}\otimes\ell_{\xi^i\otimes v},\quad \varepsilon_{\mathrm{Coeff}(A)}(\ell_{\xi\otimes v})=\xi(v),$$

where (v_i) and (ξ^i) are dual bases of V and V^*. The duality pairing between A and A^* then induces a Hopf algebra pairing between (A,Δ) and $(\mathrm{Coeff}(A),\Delta_{\mathrm{Coeff}(A)})$ (see [1]).

For V an object of $\mathrm{Rep}(A)$, define L_V to be the element of $\mathrm{End}(V)\otimes\mathrm{Coeff}(A)$ equal to $\sum_i\kappa^i\otimes\ell_{\kappa_i}$, where (κ^i) and (κ_i) are dual bases of $\mathrm{End}(V)$ and $V^*\otimes V$. It follows from $\mathscr{R}\Delta=\Delta'\mathscr{R}$ that the relation

$$R_{V,W}^{(12)}L_V^{(1a)}L_W^{(2a)}=L_W^{(2a)}L_V^{(1a)}R_{V,W}^{(12)}$$

is satisfied in $\mathrm{End}(V) \otimes \mathrm{End}(W) \otimes \mathrm{Coeff}(A)$, where the superscripts 1, 2, and a refer to the successive factors of the tensor product. Moreover,

$$(id_V \otimes \Delta_{\mathrm{Coeff}(A)})(L_V) = L_V^{(1a)} L_V^{(1a')}$$

holds in $\mathrm{End}(V) \otimes \mathrm{Coeff}(A)^{\otimes 2}$, where the superscripts $1, a$ and a' refer to the successive factors of this tensor product.

For (V, π_V) an A-module, we set $V^B = \{v \in V | \forall b \in B, \pi_V(b)(v) = \varepsilon(b)v\}$. There is a unique map

$$\oplus_{V \in \mathrm{Rep}(A)} (V^* \otimes V^B) \to (A^*)^B, \quad \kappa \mapsto \widetilde{\ell}_\kappa$$

such that for $\xi \in V^*$ and $v \in V^B$, $\widetilde{\ell}_{\xi \otimes v}(a) = \xi(\pi_V(a)v)$, for any $a \in A$. Define $\mathrm{Coeff}(A,B)$ to be the image of this map.

For V an object of $\mathrm{Rep}(A)$, V^B is a free, finite-dimensional $\mathbb{C}[[\hbar]]$-module. It follows that the dual of $V^* \otimes V^B$ is $(V^B)^* \otimes V$, which may be identified with $\mathrm{Hom}_{\mathbb{C}[[\hbar]]}(V^B, V)$. Define $\widetilde{L}_V$ to be the element of $\mathrm{Hom}_{\mathbb{C}[[\hbar]]}(V^B, V) \otimes \mathrm{Coeff}(A,B)$ equal to $\sum_i \kappa^i \otimes \widetilde{\ell}_{\kappa_i}$, where (κ^i) and (κ_i) are dual bases of $\mathrm{Hom}_{\mathbb{C}[[\hbar]]}(V^B, V)$ and $V^* \otimes V^B$.

When (V, π_V) and (W, π_W) are objects of $\mathrm{Rep}(A)$, let $R_{V,W}$ be the element of $\mathrm{End}_{\mathbb{C}[[\hbar]]}(V \otimes W)$ equal to $(\pi_V \otimes \pi_W)(\mathscr{R})$, where $\mathscr{R}$ is the R-matrix of A. Recall that the twist of $\mathscr{R}$ by F is $\mathscr{R}_B = F^{(21)} \mathscr{R} F^{-1}$, and set $R_{B;V,W} = (\pi_V \otimes \pi_W)(\mathscr{R}_B)$.

Proposition 2.1 $\mathrm{Coeff}(A,B)$ *is a subalgebra of* $(A^*)^B$*. For any objects V and W in* $\mathrm{Rep}(A)$*, the relation*

$$R_{V,W}^{(12)} \widetilde{L}_V^{(1a)} \widetilde{L}_W^{(2a)} = \widetilde{L}_W^{(2a)} \widetilde{L}_V^{(1a)} (R_{B;V,W}^{(12)})_{|Z}$$

is satisfied in $\mathrm{Hom}_{\mathbb{C}[[\hbar]]}(Z, V \otimes W) \otimes \mathrm{Coeff}(A,B)$*, where Z is the intersection* $(V^B \otimes W^B) \cap R_{B;V,W}^{-1}(V^B \otimes W^B)$*. In this equality, the left-hand side is an element of* $\mathrm{Hom}_{\mathbb{C}[[\hbar]]}(V^B \otimes W^B, V \otimes W) \otimes \mathrm{Coeff}(A,B)$*, viewed as an element of* $\mathrm{Hom}_{\mathbb{C}[[\hbar]]}(Z, V \otimes W) \otimes \mathrm{Coeff}(A,B)$ *by restriction.*

Recall that an algebra-comodule $\mathscr{X}$ over a Hopf algebra $(\mathscr{A}, \Delta_{\mathscr{A}})$ is the data of an algebra structure over $\mathscr{X}$ and a left comodule structure of $\mathscr{X}$ over $(\mathscr{A}, \Delta_{\mathscr{A}})$, $\Delta_{\mathscr{X},\mathscr{A}} : \mathscr{X} \to \mathscr{A} \otimes \mathscr{X}$, which is also a morphism of algebras.

Proposition 2.2 *There is a unique algebra-comodule structure on* $\mathrm{Coeff}(A,B)$ *over* $(\mathrm{Coeff}(A), \Delta_{\mathrm{Coeff}(A)})$*, compatible with the algebra-module structure of* $(A^*)^B$ *over* (A^{opp}, Δ)*. The relation*

$$(id_V \otimes \Delta_{\mathrm{Coeff}(A,B),\mathrm{Coeff}(A)})(\widetilde{L}_V) = L_V^{(1a)} \widetilde{L}_V^{(1a')}$$

is satisfied in $\mathrm{End}(V) \otimes \mathrm{Coeff}(A) \otimes \mathrm{Coeff}(A,B)$*, where the superscripts* 1*, a, and a' refer to the successive factors of this tensor product.*

3. Quantization of $G_+/(G_+\cap H)$

In this section, we state axioms for the quantization of a Manin quadruple, and show that any such quantization gives rise to a quantization of the Poisson homogeneous space $G_+/(G_+\cap H)$ constructed in [8] (see Propositions 1.2 and 1.3).

3.1. Quantization of Manin quadruples

Let us fix a Manin quadruple $(\mathfrak{g},\mathfrak{g}_+,\mathfrak{g}_-,\mathfrak{h})$. Recall that $(\mathfrak{g},\mathfrak{g}_+,\mathfrak{g}_-,\mathfrak{h})$ gives rise to a quasitriangular Manin pair $(\mathfrak{g},r,\langle\ ,\ \rangle_{\mathfrak{g}},\mathfrak{h})$, if we set $r=r_{\mathfrak{g}_+,\mathfrak{g}_-}=\sum_i e^i\otimes e_i$, where (e^i) and (e_i) are dual bases of $\mathfrak{g}_+$ and $\mathfrak{g}_-$.

Definition 3.1 *A quantization of a Manin quadruple $(\mathfrak{g},\mathfrak{g}_+,\mathfrak{g}_-,\mathfrak{h})$ is the data of*
1) a quantization $((A,\Delta,\mathscr{R}),B,F)$ of the quasitriangular Manin pair

$$(\mathfrak{g},r_{\mathfrak{g}_+,\mathfrak{g}_-},\langle\ ,\ \rangle_{\mathfrak{g}},\mathfrak{h}),$$

2) a Hopf subalgebra A_+ of (A,Δ) such that
a) $A_+\subset A$ is a flat deformation of $U\mathfrak{g}_+\subset U\mathfrak{g}$,
b) $B\cap A_+\subset A_+$ is a flat deformation of the inclusion $U(\mathfrak{h}\cap\mathfrak{g}_+)\subset U\mathfrak{g}_+$,
c) F satisfies

$$F^{-1}\in\big((AB_0+A_+)\otimes A+A\otimes AB_0\big)\cap\big(AB_0\otimes A+A\otimes(AB_0+A_+)\big). \qquad (4)$$

It follows from condition 2c of Definition 2.2 and the beginning of the proof of Proposition 1.2 that $(\frac{1}{\hbar}(F-F^{(21)}) \bmod \hbar)$ belongs to $\mathfrak{g}_+\otimes\mathfrak{g}_+ + \mathfrak{h}\otimes\mathfrak{g}+\mathfrak{g}\otimes\mathfrak{h} = ((\mathfrak{h}+\mathfrak{g}_+)\otimes\mathfrak{g}+\mathfrak{g}\otimes\mathfrak{h})\cap(\mathfrak{h}\otimes\mathfrak{g}+\mathfrak{g}\otimes(\mathfrak{h}+\mathfrak{g}_+))$. Therefore condition (4) is natural. It is equivalent to the condition that F belong to the product of subgroups of $(A\otimes A)_0^\times$

$$\big(1+\hbar(AB_0\otimes A_0+A_0\otimes AB_0)\big)\big(1+\hbar(A_+)_0\otimes(A_+)_0\big).$$

Example
Recall that (1) is a graded Manin quadruple. In [9], a quantization of this quadruple was constructed for the case where $\bar{\mathfrak{g}}=\mathfrak{sl}_2$.

Remark 8 In the case where $\mathfrak{g}_+\cap\mathfrak{h}=0$, which corresponds to a homogeneous space over G_+ with trivial stabilizer, F automatically satisfies condition (4). Indeed, in that case, multiplication induces an isomorphism $A_+\otimes B\to A$, therefore $A=AB_0+A_+$.

3.2. Quantization of $(G_+/(G_+\cap H),P_{G_+/(G_+\cap H)})$

Let I_0 be the subspace of $\mathscr{O}_{G/H}$ equal to $\mathscr{O}_{G/H}\cap(U\mathfrak{g}_+)^\perp$. It follows from Lemma 1.1 that I_0 is an ideal of $\mathscr{O}_{G/H}$, and that the algebra $\mathscr{O}_{G_+/(G_+\cap H)}$ can be identified as a Poisson algebra with the quotient $\mathscr{O}_{G/H}/I_0$.

Let I be the subspace of $(A^*)^B$ defined as the set of all linear forms ℓ on A such that $\ell(a_+)=0$ for any $a_+\in A_+$. Therefore

$$I=(A^*)^B\cap A_+^\perp.$$

Theorem 3.1 *Assume that* $((A,\Delta,\mathscr{R}),A_+,B,F)$ *is a quantization of the Manin quadruple* $(\mathfrak{g},\mathfrak{g}_+,\mathfrak{g}_-,\mathfrak{h})$*. Then* I *is a two-sided ideal in* $(A^*)^B$*, the algebra* $(A^*)^B/I$ *is a flat deformation of* $\mathscr{O}_{G_+/(G_+\cap H)}$*, and is a quantization of the formal Poisson space* $(G_+/(G_+\cap H),P_{G_+/(G_+\cap H)})$*.*

Moreover, I *is preserved by the action of* A_+^{opp}*, and* $(A^*)^B/I$ *is an algebra-module over* (A_+^{opp},Δ)*. With this algebra-module structure,* $(A^*)^B/I$ *is a quantization of the* (G_+,P_{G_+})*-Poisson homogeneous space* $(G_+/(G_+\cap H),P_{G_+/(G_+\cap H)})$*.*

Proof. Let us fix ℓ in I and ℓ' in $(A^*)^B$. For any a_+ in A_+, we have

$$(\ell*\ell')(a_+) = (\ell\otimes\ell')(\Delta(a_+)F^{-1}) = 0$$

because $\Delta(A_+)\subset A_+\otimes A_+$ and by assumption (4) on F. In the same way, $(\ell'*\ell)(a_+)=0$. Therefore I is a two-sided ideal in $(A^*)^B$.

Let us prove that $(A^*)^B/I$ is a flat deformation of $\mathscr{O}_{G_+/(G_+\cap H)}$. To this end, we will identify the $\mathbb{C}[[\hbar]]$-modules $(A^*)^B/I$ with $(A_+^*)^{A_+\cap B}$, to which we apply the result of Theorem 2.1.

Recall that B_0 and $(A_+\cap B)_0$ denote the augmentation ideals of B and $A_+\cap B$. Thus $(A^*)^B$ is equal to $(A/AB_0)^*$, where AB_0 is the image of the product map $A\otimes B_0\to A$. In the same way, $(A_+^*)^{A_+\cap B}$ is equal to $(A_+/A_+(A_+\cap B)_0)^*$. Let us show that $(A^*)^B/I$ is equal to $(A_+/A_+(A_+\cap B)_0)^*$. Restriction of a linear form to A_+ induces a linear map $\rho:(A/AB_0)^*\to(A_+/A_+(A_+\cap B)_0)^*$. Moreover, the kernel of ρ is I, therefore ρ induces an injective map

$$\widetilde{\rho}:(A/AB_0)^*/I\to(A_+/A_+(A_+\cap B)_0)^*.$$

Let us now show that $\widetilde{\rho}$ is surjective. For this, it is enough to show that the restriction map $\rho:(A/AB_0)^*\to(A_+/A_+(A_+\cap B)_0)^*$ is surjective. $(A/AB_0)^*$ and $(A_+/A_+(A_+\cap B)_0)^*$ are topologically free $\mathbb{C}[[\hbar]]$-modules, and the map from $(A/AB_0)^*/\hbar(A/AB_0)^*$ to $(A_+/A_+(A_+\cap B)_0)^*/\hbar(A_+/A_+(A_+\cap B)_0)^*$ coincides with the canonical map from $\mathscr{O}_{G/H}$ to $\mathscr{O}_{G_+/(G_+\cap H)}$ which, by Lemma 1.1, is surjective. Therefore ρ is surjective, and so is $\widetilde{\rho}$. It follows that $(A_+^*)^{A_+\cap B}$ is a flat deformation of $\mathscr{O}_{G_+/(G_+\cap H)}$.

There is a commutative diagram of algebras

$$\begin{array}{ccc}(A^*)^B & \to & (A^*)^B/I\\ \downarrow & & \downarrow\\ \mathscr{O}_{G/H} & \to & \mathscr{O}_{G_+/(G_+\cap H)}\end{array}$$

where the vertical maps are projections $X\to X/\hbar X$. Since the projection $\mathscr{O}_{G/H}\to\mathscr{O}_{G_+/(G_+\cap H)}$ is a morphism of Poisson algebras, where $\mathscr{O}_{G/H}$ and $\mathscr{O}_{G_+/(G_+\cap H)}$ are equipped with $P_{G/H}$ and $P_{G_+/(G_+\cap H)}$, the classical limit of $(A^*)^B/I$ is $(\mathscr{O}_{G_+/(G_+\cap H)},P_{G_+/(G_+\cap H)})$.

Finally, the algebra-module structure of $(A^*)^B$ over (A^{opp},Δ) induces by restriction an algebra-module structure on $(A^*)^B$ over (A_+^{opp},Δ), and since I is preserved by the action of A_+^{opp}, $(A^*)^B/I$ is also an algebra-module over (A_+^{opp},Δ). ■

Remark 9 For $u \in A^\times$, set ${}^uB = uBu^{-1}$ and ${}^uF = (u\otimes u)\Delta_B(u)^{-1}F$. Let $((A^*)^{{}^uB}, *_u)$ be the algebra-module over (A^{opp},Δ) corresponding to $(A,{}^uB,{}^uF)$. There is an algebra-module isomorphism $i_u : ((A^*)^B,*) \to ((A^*)^{{}^uB},*_u)$, given by $(i_u\ell)(a) = \ell(au)$, for any $a\in A$.

If u does not lie in A_+, there is no reason for $A_+\cap {}^uB$ to be a flat deformation of its classical limit, nor for ${}^uF$ to satisfy (4). But, if the conditions of Theorem 3.1 are still valid for $(A,{}^uB,{}^uF)$, the resulting algebra-module over (A_+^{opp},Δ_+) can be different from the one arising from (A,B,F). We will see an example of this situation in Section 4.1.

Remark 10 In their study of preferred deformations, Bonneau *et al.* studied the case of quotients of compact, connected Lie groups ([2]). See also the article [15] and references that it cites.

In [5], Donin, Gurevich, and Shnider used quasi-Hopf algebra techniques to construct quantizations of some homogeneous spaces. More precisely, they classified the Poisson homogeneous structures on the semisimple orbits of a simple Lie group with Lie algebra $\mathfrak{g}_0$, and constructed their quantizations using Drinfeld's series $F_\hbar$ relating the Hopf algebra $U_\hbar\mathfrak{g}_0$ to a quasi-Hopf algebra structure on $U\mathfrak{g}_0[[\hbar]]$ involving the Knizhnik-Zamolodchikov associator.

In [16], Parmentier also used twists to propose a quantization scheme of Poisson structures on Lie groups, generalizing the affine Poisson structures.

4. Examples

In view of Remark 7, we can only find nontrivial applications of the above results in the case of a nongraded $\mathfrak{h}$. In this section, we shall construct quantizations of some nongraded Manin quadruples.

4.1. Finite-dimensional examples

Let us set $\bar{\mathfrak{g}} = \mathfrak{sl}_2(\mathbb{C})$; let $\bar{\mathfrak{g}} = \bar{\mathfrak{n}}_+\oplus\bar{\mathfrak{h}}\oplus\bar{\mathfrak{n}}_-$ be the Cartan decomposition of $\bar{\mathfrak{g}}$, and let $(\bar{e}_+,\bar{h},\bar{e}_-)$ be the Chevalley basis of $\bar{\mathfrak{g}}$, so $\bar{\mathfrak{n}}_\pm = \mathbb{C}e_\pm$ and $\bar{\mathfrak{h}} = \mathbb{C}\bar{h}$. Let $\langle\ ,\ \rangle_{\bar{\mathfrak{g}}}$ be the invariant symmetric bilinear form on $\bar{\mathfrak{g}}$ such that $\langle\bar{h},\bar{h}\rangle_{\bar{\mathfrak{g}}} = 1$. Set $\mathfrak{g} = \bar{\mathfrak{g}}\times\bar{\mathfrak{g}}$, $\langle(x,y),(x',y')\rangle_{\mathfrak{g}} = \langle x,x'\rangle_{\bar{\mathfrak{g}}} - \langle y,y'\rangle_{\bar{\mathfrak{g}}}$. Set $\mathfrak{g}_+ = \{(x,x), x\in\bar{\mathfrak{g}}\}$ and $\mathfrak{g}_- = \{(\eta+\xi_+,-\eta+\xi_-),\xi_\pm\in\bar{\mathfrak{n}}_\pm,\eta\in\bar{\mathfrak{h}}\}$. Then $(\mathfrak{g},\mathfrak{g}_+,\mathfrak{g}_-)$ is a Manin triple. Let $(e_+,h,e_-,e_+^*,h^*,e_-^*)$ be the basis of $\mathfrak{g}$, such that $x = (\bar{x},\bar{x})$ for $x\in\{e_+,h,e_-\}$, $e_+^* = (\bar{e}_+,0)$, $h^* = (\bar{h},-\bar{h})$ and $e_-^* = (0,\bar{e}_-)$.

A quantization of the Manin triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is the algebra A with generators again denoted $(e_+, h, e_-, e_+^*, h^*, e_-^*)$, and relations

$$[h, e_\pm] = \pm 2e_\pm,\ [h^*, e_\pm^*] = 2e_\pm^*,\ [h, e_\pm^*] = \pm 2e_\pm^*,\ [h^*, e_\pm] = -2(e_\pm - 2e_\pm^*),$$
$$[e_+, e_-] = \frac{q^h - q^{-h}}{q - q^{-1}},\ [e_+, e_-^*] = \frac{q^h - q^{h^*}}{q - q^{-1}},\ [e_+^*, e_-] = \frac{q^{h^*} - q^{-h}}{q - q^{-1}},\ [e_+^*, e_-^*] = 0,$$
$$[h, h^*] = 0, \quad e_\pm^* e_\pm - q^{-2} e_\pm e_\pm^* = (1 - q^{-2})(e_\pm^*)^2,$$

where we set $q = \exp(\hbar)$. We define A_+ (resp., A_-) to be the subalgebra of A generated by e_+, h, e_- (resp., e_+^*, h^*, e_-^*). There is a unique algebra map $\Delta : A \to A \otimes A$, such that

$$\Delta(e_+) = e_+ \otimes q^h + 1 \otimes e_+, \Delta(e_-) = e_- \otimes 1 + q^{-h} \otimes e_-, \Delta(h) = h \otimes 1 + 1 \otimes h,$$
$$\Delta(e_+^*) = (e_+^* \otimes q^{h^*}) \left(1 - q^{-1}(q - q^{-1})^2 e_+^* \otimes e_-^*\right)^{-1} + 1 \otimes e_+^*,$$
$$\Delta(e_-^*) = e_-^* \otimes 1 + (q^{h^*} \otimes e_-^*) \left(1 - q^{-1}(q - q^{-1})^2 e_+^* \otimes e_-^*\right)^{-1},$$
$$\Delta(q^{h^*}) = (q^{h^*} \otimes q^{h^*}) \left(1 - q^{-1}(q - q^{-1})^2 e_+^* \otimes e_-^*\right)^{-1} \times$$
$$\times \left(1 - q^{-3}(q - q^{-1})^2 e_+^* \otimes e_-^*\right)^{-1}.$$

Then A_+ and A_- are Hopf subalgebras of A, and (A, A_+, A_-, Δ) is a quantization of the Manin triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$. The algebra A is the Drinfeld double of (A_+, Δ) and its R-matrix is

$$\mathscr{R} = \exp_{q^2}\left(-(q - q^{-1}) e_-^* \otimes e_+\right) q^{\frac{1}{2} h^* \otimes h} \exp_{q^2}\left(-(q - q^{-1}) e_+^* \otimes e_-\right)^{-1},$$

where $\exp_{q^2}(z) = \sum_{n \geq 0} \frac{z^n}{[n]!}$ and $[n]! = \prod_{k=1}^n (1 + q^2 + \cdots + q^{2k-2})$.

Let us fix $\alpha \in \mathbb{C}$ and define $\mathfrak{h}_\alpha$ to be the subalgebra $\mathrm{Ad}(e^{\alpha h^*})(\mathfrak{g}_+)$ of $\mathfrak{g}$. The linear space $\mathfrak{h}_\alpha$ is spanned by h and $e_\pm + \beta e_\pm^*$, where $\beta = e^{4\alpha} - 1$. When $\beta \neq 0$, $(\mathfrak{g}, \mathfrak{g}_\pm, \mathfrak{g}_\mp, \mathfrak{h}_\alpha)$ are nongraded Manin quadruples. We now assume $\beta \neq 0$.

Here are quantizations of these Manin quadruples. Let us define B_α to be the subalgebra of A generated by h and $e_\pm + \beta e_\pm^*$. Let us set

$$F_\alpha = \Psi_\alpha(e_+^* \otimes e_-^*), \text{ with } \Psi_\alpha(z) = \frac{\exp_{q^2}\left(-(q - q^{-1}) e^{-4\alpha} z\right)}{\exp_{q^2}\left(-(q - q^{-1}) z\right)}.$$

Proposition 4.1 *$((A, \Delta, \mathscr{R}), A_\pm, B_\alpha, F_\alpha)$ are quantizations of the Manin quadruples $(\mathfrak{g}, \mathfrak{g}_\pm, \mathfrak{g}_\mp, \mathfrak{h}_\alpha)$.*

Proof. We have $B_\alpha \cap A_+ = \mathbb{C}[h][[\hbar]]$, and $B_\alpha \cap A_- = \mathbb{C}[[\hbar]]$. Let us set $u_\alpha = e^{\alpha h^*}$. Then $\Delta(u_\alpha) = F^*(u_\alpha \otimes u_\alpha)(F^*)^{-1}$, where $F^* = \exp_{q^2}(-(q - q^{-1}) e_+^* \otimes e_-^*)$. It follows that $F_\alpha = (u_\alpha \otimes u_\alpha)\Delta(u_\alpha)^{-1}$, which implies that F_α satisfies the cocycle identity.

Moreover, $B_\alpha = u_\alpha A_+ u_\alpha^{-1}$, therefore $F_\alpha \Delta(B_\alpha) F_\alpha^{-1} \subset B_\alpha^{\otimes 2}$. In fact, A equipped with the twisted coproduct $F_\alpha \Delta F_\alpha^{-1}$ is a Hopf algebra, and thus B_α is a Hopf subalgebra of $(A, F_\alpha \Delta F_\alpha^{-1})$.

Let us now show that F_α satisfies both condition (4), and the similar condition where A_+ is replaced by A_-. This follows from the conjunction of

$$\Delta(u_\alpha) \in \left(1 + \hbar (A_-)_0 \otimes (A_+)_0\right)(u_\alpha \otimes u_\alpha)\left(1 + \hbar A_0 \otimes (A_+)_0\right) \tag{5}$$

and

$$\Delta(u_\alpha) \in \left(1 + \hbar (A_+)_0 \otimes (A_-)_0\right)(u_\alpha \otimes u_\alpha)\left(1 + \hbar (A_+)_0 \otimes A_0\right). \tag{6}$$

Then

$$\Delta(u_\alpha) = F'(u_\alpha \otimes u_\alpha)(F')^{-1} = F''(u_\alpha \otimes u_\alpha)(F'')^{-1}, \tag{7}$$

where

$$F' = \exp_{q^2}(-(q - q^{-1}) e_+^* \otimes e_-), \quad F'' = \exp_{q^2}(-(q - q^{-1}) e_+ \otimes e_-^*).$$

The first equality of (7) proves (5), and the second one proves (6). ■

One may expect that the algebra-module over (A_+^{opp}, Δ) constructed by means of $((A, \Delta, \mathscr{R}), A_+, B_\alpha, F_\alpha)$ in Theorem 3.1 is a formal completion of the function algebra over a Podleš sphere ([17]).

Remark 11 One shows that for $x \in A_-$, $\Delta(x) = F^* \widetilde{\Delta}(x)(F^*)^{-1}$, where $\widetilde{\Delta}$ is the coproduct on A_- defined by $\widetilde{\Delta}(e_+^*) = e_+^* \otimes q^{h^*} + 1 \otimes e_+^*$, $\widetilde{\Delta}(q^{h^*}) = q^{h^*} \otimes q^{h^*}$, and $\widetilde{\Delta}(e_-^*) = e_-^* \otimes 1 + q^{h^*} \otimes e_-^*$. The completion of the Hopf algebra (A_-, Δ) with respect to the topology defined by its augmentation ideal should be isomorphic to the formal completion at the identity of the quantum coordinate ring of SL_2. The Hopf algebra A was obtained by a method similar to that of Drinfeld's "new realizations".

4.2. The case of loop algebras

Let us return to the situation of the Manin quadruple (1), in the case where $\mathscr{K}$ is a field of Laurent series and R is a ring of functions on an affine curve. In this situation, one can consider the following problems.

1) If the double quotient $G_+ \setminus G/H$ is equipped with the zero Poisson structure, the projection $G/H \to G_+ \setminus G/H$ is Poisson. The ring of formal functions on this double quotient is $\mathscr{O}_{G_+ \setminus G/H} = (U\mathfrak{g}/((U\mathfrak{g})\mathfrak{h} + \mathfrak{g}_+ U\mathfrak{g}))^*$. On the other hand, $(A^*)^{A_+,B} = \{\ell \in A^* | \forall a_+ \in A_+, b \in B, \ell(a_+ab) = \ell(a)\varepsilon(a_+)\varepsilon(b)\}$ is a subalgebra of $(A^*)^B$. It is commutative because the R-matrix $\mathscr{R}$ of (A, Δ) belongs to $(A_+ \otimes A_-)_0^\times$ and the twisted R-matrix $F^{(21)} \mathscr{R} F^{-1}$ belongs to $(B \otimes A)_0^\times$ (see [9]).

It would be interesting to describe the algebra inclusion $(A^*)^{A_+,B} \subset (A^*)^B$, to see whether it is a flat deformation of $\mathscr{O}_{G_+ \setminus G/H}$, and when the level is critical, to describe the action of the quantum Sugawara field by commuting operators on $(A^*)^{A_+,B}$ and $(A^*)^B$. These operators could be related to the operators constructed in [10].

2) For g fixed in G, let ${}^g x$ denote the conjugate gxg^{-1} of an element x in G by g. One would like to describe the Poisson homogeneous spaces $G_\pm/(G_\pm \cap {}^g H)$, and to obtain quantizations of the Manin quadruples $(\mathfrak{g}, \mathfrak{g}_\pm, \mathfrak{g}_\mp, {}^g\mathfrak{h})$. A natural idea would be to start from the quantization of the Manin quadruple $(\mathfrak{g}, \mathfrak{g}_\pm, \mathfrak{g}_\mp, \mathfrak{h})$ obtained in [9], and to apply to B a suitable automorphism of A which lifts the automorphism $\mathrm{Ad}(g)$ of $U\mathfrak{g}$.

We hope to return to these questions elsewhere.

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