Drinfeld realization of quantum superalgebras and $U_q(\mathfrak{o}\hat{\mathfrak{s}}\mathfrak{p}(1,2))$

Jintai Ding
Department of Mathematical Sciences, University of Cincinnati, Cincinnati, Ohio 45220, U.S.A.
e-mail: `ding@math.uc.edu`

Boris Feigin
Landau Institute of Theoretical Physics, Russian Academy of Sciences, 117940 Moscow, Russia
e-mail: `feigin@feigin.mccme.ru`

Abstract. In this paper, we extend the generalization of Drinfeld realization of quantum affine algebras to quantum affine superalgebras with its Drinfeld comultiplication and its Hopf algebra structure, which depends on a function $g(z)$ satisfying the relation: $g(z) = g(z^{-1})^{-1}$. In particular, we present the Drinfeld realization of $U_q(\mathfrak{o}\hat{\mathfrak{s}}\mathfrak{p}(1,2))$ and its Serre relations.

Keywords: quantum superalgebra, Drinfeld realization, Serre relation

Mathematics Subject Classifications (2000): 81R10, 17B37

Dedicated to our friend Moshé Flato

1. Introduction

Quantum groups as a noncommutative and noncocommutative Hopf algebras were discovered by Drinfeld [8] and Jimbo [16]. The standard definition of a quantum group is given as a deformation of universal enveloping algebra of a simple (super-) Lie algebra by the basic generators and the relations based on the data coming from the corresponding Cartan matrix. However, for the case of quantum affine algebras, there is a different aspect of the theory, namely their loop realizations. The first approach was given by Faddeev, Reshetikhin, and Takhtajan [12] and Reshetikhin and Semenov-Tian-Shansky [17], who obtained a realization of the quantum loop algebra $U_q(\mathfrak{g} \otimes \mathbb{C}[t,t^{-1}])$ via a canonical solution of the Yang-Baxter equation depending on a parameter $z \in \mathbb{C}$. On the other hand, Drinfeld [9] gave another realization of the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ and its special degeneration called the Yangian, which is widely used in constructions of special representation of affine quantum algebras [14]. In [9], Drinfeld only gave the realization of the quantum affine algebras as an algebra, and as an algebra this realization is equivalent to the approach above [3] through certain Gauss decomposition for the case of $U_q(\hat{\mathfrak{gl}}(n))$. Certainly, the most important aspect of the structures of the quantum groups is its Hopf algebra structure, especially its comultiplication. Drinfeld also constructed a new Hopf algebra structure for this loop realization. The new comultiplication

G. Dito and D. Sternheimer (eds.), Conférence Moshé Flato 1999, Vol. 2, 93 – 104.

in this formulation, which we call the Drinfeld comultiplication, is simple and has very important applications [7, 5].

In [4], we observe that in the Drinfeld realization of quantum affine algebras $U_q(\hat{\mathfrak{sl}}(n))$, the structure constants are certain rational functions $g_{ij}(z)$, whose functional property of $g_{ij}(z)$ decides completely the Hopf algebra structure. In particular, for the case of $U_q(\hat{\mathfrak{sl}}_2)$, its Drinfeld realization is given completely in terms of a function $g(z)$, which has the following function property: $g(z) = g(z^{-1})^{-1}$. This leads us to generalize this type of Hopf algebras. Namely, we can substitute $g_{ij}(z)$ by other functions that satisfy the functional property of $g_{ij}(z)$, to derive new Hopf algebras.

In this paper, we will further extend the generalization of the Drinfeld realization of $U_q(\hat{\mathfrak{sl}}_2)$ to derive quantum affine superalgebras. As an example, we will also present the quantum affine superalgebra $U_q(\mathfrak{o}\hat{\mathfrak{s}}\mathfrak{p}(1,2))$ in terms of the new formulation, in particular, we present the Serre relations in terms of the current operators.

The paper is organized as the following: in Section 2, we recall the main results in [4] about the generalization of Drinfeld realization of $U_q(\hat{\mathfrak{sl}}(2))$; in Section 3, we present the definition of the generalized Drinfeld realization of quantum superalgebras; in Section 4, we present the formulation of $U_q(\mathfrak{o}\hat{\mathfrak{s}}\mathfrak{p}(1,2))$.

2. Drinfeld realization of quantum algebras

In [4], we derive a generalization of Drinfeld realization of $U_q(\hat{\mathfrak{sl}}_n)$. For the case of $U_q(\mathfrak{sl}_2)$, we first present the complete definition.

Let $g(z)$ be an analytic functions satisfying the following property that $g(z) = g(z^{-1})^{-1}$ and $\delta(z)$ be the distribution with support at 1.

Definition 2.1 *$U_q(g, f\mathfrak{sl}_2)$ is an associative algebra with unit 1 and the generators: $x^{\pm}(z)$, $\varphi(z)$, $\psi(z)$, a central element c and a nonzero complex parameter q, where $z \in \mathbb{C}^*$. $\varphi(z)$ and $\psi(z)$ are invertible. In terms of the generating functions: the defining relations are*

$$
\begin{aligned}
&\varphi(z)\varphi(w) = \varphi(w)\varphi(z),\\
&\psi(z)\psi(w) = \psi(w)\psi(z),\\
&\varphi(z)\psi(w)\varphi(z)^{-1}\psi(w)^{-1} = \frac{g(z/wq^{-c})}{g(z/wq^{c})},\\
&\varphi(z)x^{\pm}(w)\varphi(z)^{-1} = g(z/wq^{\mp\frac{1}{2}c})^{\pm 1}x^{\pm}(w),\\
&\psi(z)x^{\pm}(w)\psi(z)^{-1} = g(w/zq^{\mp\frac{1}{2}c})^{\mp 1}x^{\pm}(w),\\
&[x^{+}(z), x^{-}(w)] = \frac{1}{q-q^{-1}}\left(\delta(\frac{z}{w}q^{-c})\psi(wq^{\frac{1}{2}c}) - \delta(\frac{z}{w}q^{c})\varphi(zq^{\frac{1}{2}c})\right),\\
&x^{\pm}(z)x^{\pm}(w) = g(z/w)^{\pm 1}x^{\pm}(w)x^{\pm}(z).
\end{aligned}
$$

Theorem 2.1 *The algebra $U_q(g, f\mathfrak{sl}_2)$ has a Hopf algebra structure, which are given by the following formulas.*

Coproduct: Δ

$$\begin{aligned}
&(0)\quad \Delta(q^c) = q^c \otimes q^c,\\
&(1)\quad \Delta(x^+(z)) = x^+(z) \otimes 1 + \varphi(zq^{\frac{c_1}{2}}) \otimes x^+(zq^{c_1}),\\
&(2)\quad \Delta(x^-(z)) = 1 \otimes x^-(z) + x^-(zq^{c_2}) \otimes \psi(zq^{\frac{c_2}{2}}),\\
&(3)\quad \Delta(\varphi(z)) = \varphi(zq^{-\frac{c_2}{2}}) \otimes \varphi(zq^{\frac{c_1}{2}}),\\
&(4)\quad \Delta(\psi(z)) = \psi(zq^{\frac{c_2}{2}}) \otimes \psi(zq^{-\frac{c_1}{2}}),
\end{aligned}$$

where $c_1 = c \otimes 1$ and $c_2 = 1 \otimes c$.

Counit: ε

$$\begin{aligned}
\varepsilon(q^c) = 1 \quad & \varepsilon(\varphi(z)) = \varepsilon(\psi(z)) = 1,\\
& \varepsilon(x^{\pm}(z)) = 0.
\end{aligned}$$

Antipode: a

$$\begin{aligned}
&(0)\quad a(q^c) = q^{-c},\\
&(1)\quad a(x^+(z)) = -\varphi(zq^{-\frac{c}{2}})^{-1} x^+(zq^{-c}),\\
&(2)\quad a(x^-(z)) = -x^-(zq^{-c})\psi(zq^{-\frac{c}{2}})^{-1},\\
&(3)\quad a(\varphi(z)) = \varphi(z)^{-1},\\
&(4)\quad a(\psi(z)) = \psi(z)^{-1}.
\end{aligned}$$

Strictly speaking, $U_q(g, f\mathfrak{sl}_2)$ is not an algebra. This concept, which we call a functional algebra, has already been used before [18], etc.

The Drinfeld realization for the case of $U_q(\hat{\mathfrak{sl}}_2)$ [9] as a Hopf algebra is different, and it an algebra and Hopf algebra defined with current operators in terms of formal power series.

Let $g(z)$ be an analytic functions that satisfying the following property that $g(z) = g(z^{-1})^{-1} = G^+(z)/G^-(z)$, where $G^{\pm}(z)$ is an analytic function without poles except at 0 or ∞ and $G^{\pm}(z)$ have no common zero point. Let $\delta(z) = \sum_{n\in\mathbb{Z}} z^n$, where z is a formal variable.

Definition 2.2 *The algebra $U_q(g, \mathfrak{sl}_2)$ is an associative algebra with unit 1 and the generators: $\bar{a}(l)$, $\bar{b}(l)$, $x^{\pm}(l)$, for $l \in \mathbb{Z}$ and a central element c. Let z be a formal variable and*

$$x^{\pm}(z) = \sum_{l\in\mathbb{Z}} x^{\pm}(l) z^{-l},$$

$$\varphi(z) = \sum_{m\in\mathbb{Z}} \varphi(m) z^{-m} = \exp[\sum_{m\in\mathbb{Z}_{\le 0}} \bar{a}(m) z^{-m}] \exp[\sum_{m\in\mathbb{Z}_{>0}} \bar{a}(m) z^{-m}]$$

and

$$\psi(z)=\sum_{m\in\mathbb{Z}}\psi(m)z^{-m}=\exp[\sum_{m\in\mathbb{Z}_{\leq 0}}\bar{b}(m)z^{-m}]\exp[\sum_{m\in\mathbb{Z}_{>0}}\bar{b}(m)z^{-m}].$$

In terms of the formal variables z, w, the defining relations are

$$\begin{aligned}
&a(l)a(m)=a(m)a(l),\\
&b(l)b(m)=b(m)b(l),\\
&\varphi(z)\psi(w)\varphi(z)^{-1}\psi(w)^{-1}=\frac{g(z/wq^{-c})}{g(z/wq^{c})},\\
&\varphi(z)x^{\pm}(w)\varphi(z)^{-1}=g(z/wq^{\mp\frac{1}{2}c})^{\pm 1}x^{\pm}(w),\\
&\psi(z)x^{\pm}(w)\psi(z)^{-1}=g(w/zq^{\mp\frac{1}{2}c})^{\mp 1}x^{\pm}(w),\\
&[x^{+}(z),x^{-}(w)]=\frac{1}{q-q^{-1}}\left(\delta(\frac{z}{w}q^{-c})\psi(wq^{\frac{1}{2}c})-\delta(\frac{z}{w}q^{c})\varphi(zq^{\frac{1}{2}c})\right),\\
&G^{\mp}(z/w)x^{\pm}(z)x^{\pm}(w)=G^{\pm}(z/w)x^{\pm}(w)x^{\pm}(z),
\end{aligned}$$

where by $g(z)$ we mean the Laurent expansion of $g(z)$ in a region $r_1>|z|>r_2$.

Theorem 2.2 *The algebra $U_q(g,\mathfrak{sl}_2)$ has a Hopf algebra structure. The formulas for the coproduct Δ, the counit ε and the antipode a are the same as given in Theorem 2.1.*

Here, one has to be careful with the expansion of the structure functions $g(z)$ and $\delta(z)$, for the reason that the relations between $x^{\pm}(z)$ and $x^{\pm}(z)$ are different from the case of the functional algebra above.

Example 2.1 Let $\bar{g}(z)$ be a an analytic function such that $\bar{g}(z^{-1})=-z^{-1}\bar{g}(z)$. Let $g(z)=q^{-2}\frac{\bar{g}(q^2z)}{\bar{g}(q^{-2}z)}$. Then $g(z)=g(z^{-1})^{-1}$. With this $g(z)$, we can define an algebra $U_q(g,\mathfrak{sl}_2)$ by Definition 2.2.

Case I. Let $\bar{g}(z)$=$1-z$. $\varphi(m)=\psi(-m)=0$ for $m\in\mathbb{Z}_{>0}$ and $\varphi(0)\psi(0)=1$. Then this algebra is $U_q(\hat{\mathfrak{sl}}_2)$.

Case II. Let $\bar{g}(z)=\theta_p(z)=\prod_{j>0}(1-p^j)(1-p^{j-1}z)(1-p^jz^{-1})$ be the Jacobi's theta function. $\theta_p(z^{-1})=-z^{-1}\theta_p(z)=\theta_p(pz)$. We will call this algebra $U_q(\theta,\mathfrak{sl}_2)$. We take the expansion of $g(z)$ in the region $|q^2|>|z|>|q^2p|$. If we take the limit that p goes to zero, we would recover $U_q(\hat{\mathfrak{sl}}_2)$.

3. Drinfeld realization of quantum superalgebras

Similarly, we can use the same idea to define quantum affine superalgebras, which is an extension of the generalization in the section above. Again, we will start from the same function $g(z)$, namely, let $g(z)$ be an analytic functions satisfying the following property that $g(z) = g(z^{-1})^{-1}$ and $\delta(z)$ be the distribution with support at 1.

Definition 3.1 *$U_q(g, f\mathfrak{s})$ is an a $\mathbb{Z}_2$-graded associative algebra with unit 1 and the generators: $x^{\pm}(z)$, $\varphi(z)$, $\psi(z)$, a central element c and a nonzero complex parameter q, where $z \in \mathbb{C}^*$, $x^{\pm}(z)$ are graded 1 (mod 2), and $\varphi(z)$, $\psi(z)$ and c are graded 0 (mod 2). $\varphi(z)$ and $\psi(z)$ are invertible. In terms of the generating functions: the defining relations are*

$$\begin{aligned}
&\varphi(z)\varphi(w) = \varphi(w)\varphi(z),\\
&\psi(z)\psi(w) = \psi(w)\psi(z),\\
&\varphi(z)\psi(w)\varphi(z)^{-1}\psi(w)^{-1} = \frac{g(z/wq^{-c})}{g(z/wq^{c})},\\
&\varphi(z)x^{\pm}(w)\varphi(z)^{-1} = g(z/wq^{\mp\frac{1}{2}c})^{\pm 1}x^{\pm}(w),\\
&\psi(z)x^{\pm}(w)\psi(z)^{-1} = g(w/zq^{\mp\frac{1}{2}c})^{\mp 1}x^{\pm}(w),\\
&\{x^{+}(z), x^{-}(w)\} = \frac{1}{q-q^{-1}}\left(\delta(\frac{z}{w}q^{-c})\psi(wq^{\frac{1}{2}c}) - \delta(\frac{z}{w}q^{c})\varphi(zq^{\frac{1}{2}c})\right),\\
&x^{\pm}(z)x^{\pm}(w) = -g(z/w)^{\pm 1}x^{\pm}(w)x^{\pm}(z),
\end{aligned}$$

where $\{x, y\} = xy + yx$.

Here we remark that the above relations are basically the same as in that of Definition 2.1 except the relation between $x^{\pm}(z)$ and $x^{\pm}(w)$ respectively, which differs by a negative sign.

Accordingly we have that, for the tensor algebra, the multiplication is defined for homogeneous elements a, b, c, d by $(a \otimes b)(c \otimes d) = (-1)^{[b][c]}\,(ac \otimes bd)$, where $[a] \in \mathbb{Z}_2$ denotes the grading of the element a.

Similarly we have:

Theorem 3.1 *The algebra $U_q(g, f\mathfrak{s})$ has a graded Hopf algebra structure, whose coproduct, counit and antipode are given by the same formulas of $U_q(q, f\mathfrak{sl}_2)$ in Theorem 2.1.*

As for the case of $U_q(g, f\mathfrak{sl}_2)$ is not a graded algebra but rather a graded functional algebra.

Let $g(z) = g(z^{-1})^{-1} = G^{+}(z)/G^{-}(z)$, where $G^{\pm}(z)$ is an analytic function without poles except at 0 or ∞ and $G^{\pm}(z)$ have no common zero point.

Definition 3.2 *The algebra $U_q(g, \mathfrak{s})$ is $\mathbb{Z}_2$ graded associative algebra with unit 1 and the generators: $\bar{a}(l)$, $\bar{b}(l)$, $x^{\pm}(l)$, for $l \in \mathbb{Z}$ and a central element c, where $x^{\pm}(l)$ are graded 1 (mod 2) and the rest are graded 0 (mod 2). Let z be a formal variable and*

$$x^{\pm}(z) = \sum_{l \in \mathbb{Z}} x^{\pm}(l) z^{-l},$$

$$\varphi(z) = \sum_{m\in\mathbb{Z}} \varphi(m)z^{-m} = \exp[\sum_{m\in\mathbb{Z}_{\leq 0}} \bar{a}(m)z^{-m}]\exp[\sum_{m\in\mathbb{Z}_{>0}} \bar{a}(m)z^{-m}]$$

and

$$\psi(z) = \sum_{m\in\mathbb{Z}} \psi(m)z^{-m} = \exp[\sum_{m\in\mathbb{Z}_{\leq 0}} \bar{b}(m)z^{-m}]\exp[\sum_{m\in\mathbb{Z}_{>0}} \bar{b}(m)z^{-m}].$$

In terms of the formal variables z, w, the defining relations are

$$\begin{aligned}
&\varphi(z)\varphi(w) = \varphi(w)\varphi(z),\\
&\psi(z)\psi(w) = \psi(w)\psi(z),\\
&\varphi(z)\psi(w)\varphi(z)^{-1}\psi(w)^{-1} = \frac{g(z/wq^{-c})}{g(z/wq^{c})},\\
&\varphi(z)x^{\pm}(w)\varphi(z)^{-1} = g(z/wq^{\mp\frac{1}{2}c})^{\pm 1}x^{\pm}(w),\\
&\psi(z)x^{\pm}(w)\psi(z)^{-1} = g(w/zq^{\mp\frac{1}{2}c})^{\mp 1}x^{\pm}(w),\\
&\{x^{+}(z),x^{-}(w)\} = \frac{1}{q-q^{-1}}\left(\delta(\frac{z}{w}q^{-c})\psi(wq^{\frac{1}{2}c}) - \delta(\frac{z}{w}q^{c})\varphi(zq^{\frac{1}{2}c})\right),\\
&(G^{\mp}(z/w))x^{\pm}(z)x^{\pm}(w) = -(G^{\pm}(z/w))x^{\pm}(w)x^{\pm}(z),
\end{aligned}$$

where by $g(z)$ we mean the Laurent expansion of $g(z)$ in a region $r_1 > |z| > r_2$.

The above relations are basically the same as in that of Definition 2.2 except the relation between $x^{\pm}(z)$ and $x^{\pm}(w)$ respectively, which differs by a negative sign. The expansion direction of the structure functions $g(z)$ and $\delta(z)$ is very important, for the reason that the relations between $x^{\pm}(z)$ and $x^{\pm}(z)$ are different from the case of the functional algebra above.

Theorem 3.2 *The algebra $U_q(g,\mathfrak{s})$ has a Hopf algebra structure. The formulas for the coproduct Δ, the counit ε and the antipode a are the same as given in Theorem 2.1.*

Example 3.1 *Let $\bar{g}(z) = 1$. From [1, 19], we can see that $U_q(1,\mathfrak{s})$ is basically the same as $U_q(\hat{\mathfrak{gl}}(1,1))$.*

4. $U_q(o\hat{s}p(1,2))$ and its Serre relations

For a rational function $g(z)$ that satisfies $g(z) = g(z^{-1})^{-1}$, it is clear that $g(z)$ is determined by its poles and its zeros, which are paired to satisfy the relations above. For the simplest case (except $g(z) = 1$) that $g(z)$ has only one pole and one zero, we have

$$g(z) = \frac{zp-1}{z-p},$$

where p is the location of the pole of $g(z)$. Unfortunately, in this case, we do not know anyway to identify the algebra $U_q(g,\mathfrak{s})$ with other know structures.

Then, comes the second simplest case that $g(z)$ has two poles and two zeros. We know also that $g(z)$ must be in the form that

$$g(z)=\frac{zp_1-1}{z-p_1}\frac{zp_2-1}{z-p_2}.$$

In this section, we shall establish that for this case, $U_q(g,\mathfrak{s})$ is related to the affine quantum superalgebra $U_q(\mathfrak{o}\hat{\mathfrak{s}}\mathfrak{p}(1,2))$.

From now on, let us fix $g(z)$ to be $\frac{zp_1-1}{z-p_1}\frac{zp_2-1}{z-p_2}$.

As in [7, 6], for the case of quantum affine algebras, it is very important to understand the poles and zero of the product of current operators. We will start with the relations between $X^+(z)$ with itself. From the definition, we know that

$$(z-p_1w)(z-p_2w)X^+(z)X^+(w)=-(zp_1-w)(zp_2-w)X^+(w)X^+(z).$$

From this, we know that $X^+(z)X^+(w)$ has two poles, which are located at $(z-p_1w)=0$ and $(z-p_2w)=0$.

This also implies that

Proposition 4.1 $X^+(z)X^+(w)=0$, *when* $z=w$.

If we assume that $U_q(g,\mathfrak{s})$ is related to some quantized affine superalgebra, then we can see that the best chance we have is $U_q(\mathfrak{o}\hat{\mathfrak{s}}\mathfrak{p}(1,2))$ by looking at the number of zeros and poles of $X^+(z)X^+(w)$.

However, for the case of $U_q(\mathfrak{o}\hat{\mathfrak{s}}\mathfrak{p}(1,2))$, we know we need an extra Serre relation. For this, we will follow the idea of [13].

Let

$$\begin{aligned}
f(z_1,z_2)=&(z_1-p_1z_2)(z_1-p_2z_2),\\
Y^+(z,w)=&\frac{(z-p_1w)(z-p_2w)}{z-w}X^+(z)X^+(w),\\
Y^+(z_1,z_2,z_3)=&\frac{(z_1-p_1z_2)(z_1-p_2z_2)}{z_1-z_2}\frac{(z_1-p_1z_3)(z_1-p_2z_3)}{z_1-z_3}\times\\
&\frac{(z_2-p_1z_3)(z_2-p_2z_3)}{z_2-z_3}X^+(z_1)X^+(z_2)X^+(z_3),\\
F(z_1,z_2,z_3)=&(z_1-p_1z_2)(z_1-p_2z_2)(z_3-p_1z_1)(z_3-p_2z_1)(z_2-p_1z_3)(z_2-p_2z_3)\\
=&f(z_1,z_2)f(z_2,z_3)f(z_3,z_1),\\
\bar{F}(z_1,z_2,z_3)=&f(z_2,z_1)f(z_2,z_3)f(z_3,z_1).
\end{aligned}$$

Let $V_(z_1,z_2,z_3)$ be the algebraic variety of the zeros of $F(z_1,z_2,z_3)$. Let $V(a(z_1),a(z_2),a(z_3))$, be the image of the action of a on this variety, where $a\in$

S_3, the permutation group on z_1, z_2, z_3. Let $\bar{V}_(z_1, z_2, z_3)$ be the algebraic variety of the zeros of $\bar{F}(z_1, z_2, z_3)$. Let $\bar{V}(a(z_1), a(z_2), a(z_3))$, be the image of the action of a on this variety, where $a \in S_3$.

Proposition 4.2 *$Y^+(z,w)$ has no poles and is symmetry with respect to z and w. $Y^+(z_1, z_2, z_3)$ has no poles and is symmetry with respect to z_1, z_2, z_3.*

Following the idea in [13], we would like to define the following conditions that may be imposed on our algebra.

Zero Condition I: $Y^+(z_1, z_2, z_3)$ is zero on at least one line that crosses (0,0,0), and this line must lie in a $V(a(z_1), a(z_2), a(z_3))$ for some element $a \in S_3$.

Zero Condition II: $Y^+(z_1, z_2, z_3)$ is zero on at least one line that crosses (0,0,0), and this line must lie in a $\bar{V}(a(z_1), a(z_2), a(z_3))$ for some element $a \in S_3$.

For the line that crosses $(0,0,0)$ where $Y^+(z_1, z_2, z_3)$ is zero, we call it the zero line of $Y^+(z_1, z_2, z_3)$. Because $Y^+(z_1, z_2, z_3)$ is symmetric with respect to the action of S_3 on z_1, z_2, z_3, if a line is the zero line of $Y^+(z_1, z_2, z_3)$, then clearly the orbit of the line under the action of S_3 is also an zero line.

Remark 1 There is a simple symmetry that we would prefer to choose the function $F(z_1, z_2, z_3)$ to determine the variety $V(z_1, z_2, z_3)$. We have that

$$F(z_1, z_2, z_3) = f(z_1, z_2) f(z_2, z_3) f(z_3, z_1).$$

Let S_2^1 be the permutation group acting on z_1, z_2. Let S_2^2 be the permutation group acting on z_2, z_3. Let S_2^1 be the permutation group acting on z_3, z_1. Clearly, [13] we can choose from a family of varieties determined by the functions

$$f(a_1(z_1), a_1(z_2)) f(a_2(z_2), a_2(z_3)) f(a_3(z_3), a_3(z_1))$$

for $a_1 \in S_2^1$, $a_2 \in S_2^2$, $a_3 \in S_2^3$. For each such a function

$$f(a_1(z_1), a_1(z_2)) f(a_2(z_2), a_2(z_3)) f(a_3(z_3), a_3(z_1)),$$

we can attach a oriented diagram, whose nods are z_1, z_2, z_3, and the arrows are given by $(a_1(z_1) \to a_1(z_2))$, $(a_2(z_2) \to a_2(z_3))$ $(a_3(z_3) \to a_3(z_1))$. For example the diagram of $F(z_1, z_2, z_3)$ is given by

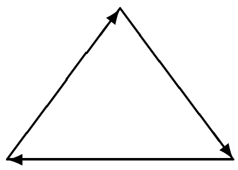

Diagram I

While $f(z_2, z_1) f(z_2, z_3) f(z_3, z_1)$ is denoted by

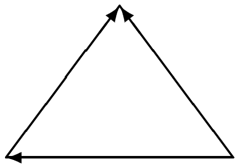

Diagram II

It is not difficult to see that the diagram for $F(z_1,z_2,z_3)$ is symmetric in the sense that all the points are equivalent, but for the second situation, the top point z_1 is different from the other two, in the sense that there are two arrows coming to z_1, one to z_3 and none to z_2. There is only one other such a diagram given by $F(z_1,z_3,z_2)$, which however comes from the S_3 action on $F(z_1,z_2,z_3)$.

From the above, we have the following:

Proposition 4.3 *Under the action of S_3 on the the family of varieties determined by the functions $f(a_1(z_1),a_1(z_2))f(a_2(z_2),a_2(z_3))f(a_3(z_3),a_3(z_1))$ for $a_1 \in S_2^1$, $a_2 \in S_2^2$, $a_3 \in S_2^3$, there are two orbits. One of the orbit consists of the two varieties determined by $F(z_1,z_2,z_3)$ and $F(z_1,z_2,z_3)$; and the rest forms another orbit.*

This shows that indeed we have two choices with respect the zero conditions: the Zero condition I and the Zero condition II.

We know that a zero line is always in the form

$$z_1 = q_1 z_2 = q_2 z_3.$$

Then we have

Proposition 4.4 *If we impose the Zero condition I on the algebra $U_q(g,\mathfrak{s})$, we have*

$$p_1 = p_2^{-2} \quad or \quad p_2 = p_1^{-2}.$$

Proof. The proof is very simple. Because of the action of S_3, we know that one of the line must lie in $V(z_1,z_2,z_3)$. Let us assume this line to be $z_1 = z_2 q_1 = q_2 z_3$.

We know immediately that q_1 must be p_1 or p_2.

Let us first deal with the case where $q_1 = p_1$. We also know that q_2 must be either p_1^{-1} or p_2^{-1} by looking at the relations between z_1 and z_3.

Case 1. Let $q_2 = p_1^{-1}$, which implies that p_1^{-2} must be either p_1 or p_2. Clearly, it cannot be p_1, which implies that the impossible condition $p_1 = 1$

Therefore, we have that $p_1^{-2} = p_2$, which is what we want.

Case 2. Let $q_2 = p_2^{-1}$, which implies that $p_1 p_2$ must be either p_1 or p_2. Clearly, it cannot be p_1 because it implies $p_2 = 1$, it cannot be neither be p_2, which implies that $p_1 = 1$. This completes the proof for $p_2 = p_1^{-2}$.

Similarly, if we have that $q_1 = p_2$, we can then show that $p_1 = p_2^{-2}$.

However from the algebraic point of view, the two condition are equivalent in the sense that p_1 and p_2 are symmetric.

Also we have that

Proposition 4.5 *If we impose the Zero condition II on the algebra $U_q(g,\mathfrak{s})$, we have*

$$p_1 = p_2^2 \quad or \quad p_2 = p_1^2.$$

However we also have that: **If we impose the Zero condition II on the algebra $U_q(g,\mathfrak{s})$, then $U_q(g,\mathfrak{s})$ is not a Hopf algebra anymore.**

The reason is that the Zero condition II cannot be satisfied by comultiplication, which can be checked by direct calculation.

This is the most important reason that we will choose the Zero condition I to be imposed on the algebra $U_q(g,\mathfrak{s})$, which comes actually from the consideration of Hopf algebra structure. Namely, if we choose $V(z_1,z_2,z_3)$ or the equivalent ones which has the same diagram presentation as Diagram I to define the zero line of $Y^+(z_1,z_2,z_3)$, then, the quotient algebra derived from the **Zero Condition I** is still a Hopf algebra with the same Hopf algebra structure (comultiplication, counit and antipode).

From now on, we impose the Zero condition I on the algebra $U_q(g,\mathfrak{s})$, and let us fix the notation such that

$$p_1 = q^2, \qquad p_2 = q^{-1}.$$

Similarly, we define

$$Y^-(z,w) = \frac{(z-p_1^{-1}w)(z-p_2^{-1}w)}{z-w} X^+(z)X^+(w),$$

$$Y^-(z_1,z_2,z_3) = \frac{(z_1-p_1^{-1}z_2)(z_1-p_2^{-1}z_2)}{z_1-z_2}\frac{(z_1-p_1^{-1}z_3)(z_1-p_2^{-1}z_3)}{z_1-z_3}\times$$

$$\frac{(z_2-p_1^{-1}z_3)(z_2-p_2^{-1}z_3)}{z_2-z_3} X^+(z_1)X^+(z_2)X^+(z_3),$$

We now define the q-Serre relation.

***q*-Serre relations**

$Y^+(z_1,z_2,z_3)$ is zero on the line $z_1 = z_2q^{-1} = z_3q^{-2}$.

$Y^-(z_1,z_2,z_3)$ is zero on the line $z_1 = z_2q = z_3q^2$.

The q-Serre relations can also be formulated in more algebraic way.

Proposition 4.6 *The q-Serre relations are equivalent to the following two relations:*

$$\frac{(z_3-z_1q^{-1})(z_3-z_1q^3)(z_1-z_2q^2)}{z_3-z_1q}X^+(z_3)X^+(z_1)X^+(z_2)+$$
$$\frac{(z_2-z_1q^2)(z_2-z_1q^{-1})(z_3-z_1q^{-1})(z_3-z_1q^3)}{(z_1-z_2q^{-1})(z_3-z_1q)}X^+(z_3)X^+(z_2)X^+(z_1))-$$
$$\frac{((z_1-z_3q^2)(z_1-z_3q)(z_1q-z_3q^{-1})(z_1-z_2q^2)}{(z_1-z_3)(z_3-z_1q^2)}X^+(z_1)X^+(z_2)X^+(z_3)-$$
$$\frac{((z_1-z_3q^2)(z_1-z_3q)(z_1q-z_3q^{-1})(z_2-z_1q^2)(z_2-z_1q^{-1})}{(z_1-z_3)(z_3-z_1q^2)(z_1-z_2q^{-1})}\times$$
$$\times X^+(z_2)X^+(z_1)X^+(z_3))=0,$$

$$\frac{(z_3-z_1q)(z_3-z_1q^{-3})(z_1-z_2q^{-2})}{z_3-z_1q^{-1}}X^-(z_3)X^-(z_1)X^-(z_2)+$$
$$\frac{(z_2-z_1q^{-2})(z_2-z_1q)(z_3-z_1q)(z_3-z_1q^{-3})}{(z_1-z_2q)(z_3-z_1q^{-1})}X^-(z_3)X^-(z_2)X^-(z_1))-$$
$$\frac{((z_1-z_3q^{-2})(z_1-z_3q^{-1})(z_1q-z_3q)(z_1-z_2q^{-2})}{(z_1-z_3)(z_3-z_1q^{-2})}X^-(z_1)X^-(z_2)X^-(z_3)-$$
$$\frac{((z_1-z_3q^{-2})(z_1-z_3q^{-1})(z_1q-z_3q)(z_2-z_1q^{-2})(z_2-z_1q)}{(z_1-z_3)(z_3-z_1q^{-2})(z_1-z_2q)}\times$$
$$\times X^-(z_2)X^-(z_1)X^-(z_3))=0,$$

where the coefficient functions of the relations above are expanded in the region the expansion region of the corresponding monomial of the product of $X^{\pm}(z_j)$.

The proof is a simple calculation. (See also [11].) It is not very difficult to show that this relation will give us the classical Serre relations, but unfortunately, we still do not know how to write down a simple Serre relation like that of $U_q(\hat{\mathfrak{sl}}(3))$.

Definition 4.1 $U_q(g,\bar{\mathfrak{s}})$ *is the quotient algebra of* $U_q(g,\mathfrak{s})$ *with the kernel defined by the q-Serre relations,* $\bar{a}(m)=0$, $m>0$, $\bar{b}(m)$, $m<0$ *and* $a(0)=-b(0)$, *and* $g(z)$ *is expanded around 0.*

Theorem 4.7 $U_q(g,\bar{\mathfrak{s}})$ *is also a Hopf algebra, whose Hopf algebra structure is the same as that of* $U_q(g,\mathfrak{s})$.

This can be proven by calculation as in [4]

Another immediately result we can derive is that $U_q(g,\mathfrak{s})$ [2, 15] can be identified with $U_q(\mathfrak{o}\hat{\mathfrak{s}}\mathfrak{p}(1,2))$, where $U_q(\mathfrak{o}\hat{\mathfrak{s}}\mathfrak{p}(1,2))$ is derived from the FRTS realization using R-matrix and L-operators.

Theorem 4.8 *$U_q(g,\mathfrak{s})$ with the q-Serre relation is isomorphic to $U_q(\mathfrak{o}\hat{\mathfrak{s}}\mathfrak{p}(1,2)$.*

It is still an open and interesting question to study $U_q(g,\mathfrak{s})$ given by other function $g(z)$.

Acknowledgements. We would like to thank S. Khoroshkin and T. Hodges for their advice and encouragement.

References

1. Cai, J. F., Ju, G. X., Wang, S. K., and Wu, K.: Super Yangian double $DY(gl(1|1))$ with center extension, preprint q-alg/9703031.
2. Ding, J.: A remark on the FRTS realization and Drinfeld Realization of Quantum Affine Superalgebra $U_q(o\hat{s}p(1,2))$, *Lett. Math. Phys.* **49** (1999), 161–171.
3. Ding, J. and Frenkel, I. B.: Isomorphism of two realizations of quantum affine algebra $U_q(\hat{\mathfrak{gl}}(n))$, *Comm. Math. Phys.* **156** (1993), 277–300. Physics
4. Ding, J. and Iohara, K.: Generalization and deformation of the quantum affine algebras, *Lett. Math. Phys.* **41** (1997), 181-193.
5. Ding, J. and Iohara, K.: Drinfeld comultiplication and vertex operators, *J. Geom. Phys.* **23** (1997), 1–13.
6. Ding, J. and Khoroshkin, S.: Weyl group extension of quantized current algebras, *Transform. Groups* **5**, (2000), 35–59.
7. Ding, J. and Miwa, T.: Zeros and poles of quantum current operators and the condition of quantum integrability, *Publ. Res. Inst. Math. Sci.* **33** (1997), 277–284.
8. Drinfeld, V. G.: *Hopf algebra and the quantum Yang-Baxter equation*, *Dokl. Akad. Nauk. SSSR* **283** (1985), 1060–1064.
9. Drinfeld, V. G.: Quantum Groups, in: A. M. Gleason, (ed.), *Proceedings of the International Congress of Mathematicians (Berkeley 1986)*, Amer. Math. Soc., Providence, RI, 1987, pp. 798–820.
10. Drinfeld, V. G.: A new realization of Yangian and quantum affine algebra, *Soviet Math. Dokl.* **36** (1988), 212–216.
11. Enriquez, B.: On correlation functions of Drinfeld currents and shuffle algebras, preprint math/9809036.
12. Faddeev, L. D., Reshetikhin, N. Yu, and Takhtajan, L. A.: Quantization of Lie groups and Lie algebras, in: M. Jimbo, (ed.), *Yang-Baxter equation in integrable systems*, (Advanced Series in Mathematical Physics **10**) World Scientific, Teaneck, NJ, 1989, pp. 299–309.
13. Feigin, B. and Odesski, A.: Vector bundles on an elliptic curve and Sklyanin algebras, in: *Topics in quantum groups and finite-type invariants*, Amer. Math. Soc. Transl. Ser. 2, 185, Amer. Math. Soc., Providence, RI, 1998, pp. 65–84.
14. Frenkel, I.B. and Jing, N.: Vertex representations of quantum affine algebras, *Proc. Nat. Acad. Sci. U.S.A.* **85** (1988), 9373–9377.
15. Gould, M. and Zhang, Y.: On super-RS algebra and Drinfeld realization of quantum affine superalgebras, *Lett. Math. Phys.* **44** (1998), 291–308.
16. Jimbo, M.: A q-difference analogue of $U(\mathfrak{g})$ and Yang-Baxter equation, *Lett. Math. Phys.* **10** (1985), 63–69.
17. Reshetikhin, N. Yu. and Semenov-Tian-Shansky, M. A.: Central extensions of quantum current groups, *Lett. Math. Phys.* **19** (1990), 133–142.
18. Sklyanin, E. K.: Some algebraic structures connected with the Yang-Baxter equation, *Funktsional. Anal. i Prilozhen.* **16** (1982), 27–34.
19. Zhang, Y.: Comments on Drinfeld realization of quantum affine superalgebra $U_q[gl(m|n)^{(1)}]$ and its Hopf algebra structure, *J. Phys. A* **30** (1997), 8325–8335.