On positive deformations of $*$-algebras[‡]

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Abstract. Motivated by deformation quantization we consider $*$-algebras over ordered rings and their deformations: we investigate formal associative deformations compatible with the $*$-involution and discuss a cohomological description in terms of a Hermitian Hochschild cohomology. As an ordered ring allows for a meaningful definition of positive functionals and as the formal power series with coefficients in an ordered ring are again an ordered ring we define a deformation to be positive if any positive linear functional of the undeformed algebra can be deformed into a positive linear functional of the deformed algebra. We discuss various examples and prove in particular that star-products on symplectic manifolds are positive deformations.

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Dedicated to the memory of Moshé Flato

1. Introduction

In this paper we discuss several aspects of deformations of algebras with a $*$-involution which are defined over an ordered ring. Our main example is deformation quantization as introduced by Bayen, Flato, Frønsdal, Lichnerowicz, and Sternheimer in [3], see e.g. [41, 39] for recent surveys. In deformation quantization the algebra of classical observables, the smooth complex-valued functions $C^\infty(M)$ on a Poisson manifold, is deformed into the quantum mechanical observable algebra by introducing a new, $\hbar$-dependent associative product, the star product, such that in zeroth order of $\hbar$ the star-product coincides with the pointwise product and the commutator yields in first order i times the Poisson bracket. In deformation quantization the star-product usually is regarded as formal, i.e., based on formal power series in $\hbar$, and thus a star-product is a formal associative deformation in the sense of Gerstenhaber [26]. The existence of star-products was shown by De Wilde and Lecomte [22], Fedosov [23, 24], and Omori, Maeda, and Yoshioka [35] for the case of symplectic manifolds, and by Kontsevich [30] for the general case of a

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Poisson manifold. The classification up to equivalence is due to Nest and Tsygan [32, 33], Bertelson, Cahen, and Gutt [4], Deligne [21], Weinstein and Xu [42], and Kontsevich [30]. On the other hand the formal character of these star products is certainly not sufficient from the physical point of view whence one also has to study convergence properties with respect to $\hbar$. This can be done either by starting with a formal star product and then finding suitable subalgebras where one can make sense out of convergence, see e.g. [16, 17, 18, 19, 5, 8, 9, 7] and references therein, or directly by formulating the deformation within the framework of e.g. C^*-algebras from the beginning, which leads e.g. to Rieffel's notion of strict deformation quantization, see [38] and Landsman's book [31] for related approaches and references. In many cases the formal approach can be seen as an 'asymptotic expansion' in $\hbar$ of a convergent situation, which can be made more precise in several examples as e.g. for cotangent bundles [8, 9, 7], and it is very useful to think of formal deformations in that way even if it is not clear if there exists a corresponding strict deformation (see also [21, 36] for a more sheaf theoretic approach). This leads to the following program, namely trying to find 'formal analogs' of various techniques known in C^*-algebra theory in such a way that they can be viewed as 'asymptotic expansions' and thus to understand the classical and semi-classical limit of these constructions which possibly opens also an asymptotic point of view to Connes' non-commutative geometry [20]. This approach has turned out to be interesting both from the physical and mathematical point of view: in [14, 13] Bordemann and Waldmann have introduced a notion of formal GNS representations for star-products on pre-Hilbert spaces over the ring of formal power series by noticing that the ring of real formal power series is in fact ordered and thus allows to define positive functionals etc. as in the C^*-algebra theory. This approach (re-)produces many physically interesting results for the star-product algebras in a purely formal framework, as e.g. for cotangent bundles the Schrödinger-like differential operator representations, Aharonov-Bohm-like effects and representations on sections of line bundles when magnetic monopoles are present [8, 9, 7]. Also thermodynamical KMS states and their representations have been investigated, see [1, 2] as well as [10, 11] and [40] for a related 'formal' Tomita-Takesaki theory. On the other hand the framework of $*$-algebras and pre-Hilbert spaces over ordered rings seems to be mathematically interesting and turns out to have a rich structure. In [15] we started to develop the notions of algebraic Rieffel induction and formal Morita equivalence for such $*$-algebras in order to understand the (semi-)classical limits of the corresponding constructions for C^*-algebras, as e.g. used by Landsman [31] to describe quantum analogous of phase space reduction. As this also has been done in the framework of deformation quantization, see e.g. [25, 6], it would be very interesting to compare these approaches. During our investigations we needed the notion of a *positive deformation* of a $*$-algebra in the sense that all positive linear functionals can be deformed into positive linear functionals of the deformed algebra. This is in some sense 'dual' to Landsman's notion of a positive quantization map, but less restrictive, and it seems to be more suitable for our purposes.

In this paper we shall discuss the deformations of $*$-algebras over ordered rings in general and apply our results to the main examples of deformation quantization obtaining the following results: after reviewing briefly the notions of ordered rings and $*$-algebras and discussing the notion of 'sufficiently many positive linear functionals' in Section 2, we set up the deformation program for $*$-algebras in Section 3 where the 'real part' of the Hochschild cohomology is relevant for Hermitian deformations, i.e., those deformations which do not deform the $*$-involution. In Section 4 we motivate the notion of a positive deformation and discuss some basic constructions proving in particular that positive deformations of $*$-algebras with sufficiently many positive linear functionals still have sufficiently many positive linear functionals. Finally, in Section 5 we prove that star-products on symplectic manifolds are positive deformations and that on arbitrary Poisson manifolds star-products have sufficiently many positive linear functionals.

2. $*$-algebras over ordered rings

In this section we shall recall some basic definitions and examples concerning ordered rings and $*$-algebras. For further references and details see e.g. [14, 15]. Let R be an *ordered ring*, i.e., an associative, commutative and unital ring with a subset $\mathsf{P} \subset \mathsf{R}$ such that R is the disjoint union $\mathsf{R} = -\mathsf{P} \cup \{0\} \cup \mathsf{P}$ and $\mathsf{P} \cdot \mathsf{P} \subseteq \mathsf{P}$, $\mathsf{P} + \mathsf{P} \subseteq \mathsf{P}$. The elements in P are called positive. Clearly R has characteristic zero and no zero divisors. Then consider the quadratic ring extension $\mathsf{C} = \mathsf{R}(\mathrm{i})$ where $\mathrm{i}^2 := -1$. Complex conjugation $z = a + \mathrm{i}b \mapsto \overline{z} := a - \mathrm{i}b$ with $a, b \in \mathsf{R}$ is defined as usual and clearly $z \in \mathsf{R} \subset \mathsf{C}$ iff $z = \overline{z}$ and $\overline{z}z \geq 0$.

Let $\mathscr{A}$ be an associative algebra over C endowed with a $*$-involution, i.e., an involutive C-antilinear anti-automorphism $^* : \mathscr{A} \to \mathscr{A}$, then $\mathscr{A}$ is called a $*$-*algebra* over C. As usual we define *Hermitian*, *unitary* and *normal elements* in C. Given such a $*$-algebra we define a linear functional $\omega : \mathscr{A} \to \mathsf{C}$ to be *positive* if $\omega(A^*A) \geq 0$ for all $A \in \mathscr{A}$. If $\mathscr{A}$ is unital then a positive linear functional is called a state if $\omega(\mathbf{1}) = 1$. For any positive linear functional one has the Cauchy-Schwarz inequality $\omega(A^*B)\overline{\omega(A^*B)} \leq \omega(A^*A)\omega(B^*B)$ and $\omega(A^*B) = \overline{\omega(B^*A)}$ for $A, B \in \mathscr{A}$. Next we consider positive algebra elements in $\mathscr{A}$. There are essentially two ways to define positivity [15, Def. 2.3]: firstly, we call $A \in \mathscr{A}$ *algebraically positive* if $A = a_1 A_1^* A_1 + \cdots + a_n A_n^* A_n$, where $n \in \mathbb{N}$, $a_i \geq 0$, and $A_i \in \mathscr{A}$ for $i = 1, \ldots, n$, and we denote the set of all algebraically positive elements by $\mathscr{A}^{++}$. Secondly, we define a Hermitian element A to be *positive* if for all positive linear functionals $\omega : \mathscr{A} \to \mathsf{C}$ one has $\omega(A) \geq 0$, and we denote the positive elements by $\mathscr{A}^+$. Then clearly $\mathscr{A}^{++} \subseteq \mathscr{A}^+$ and both sets are convex cones with the property that for any $C \in \mathscr{A}$ one has $C^*\mathscr{A}^{++}C \subseteq \mathscr{A}^{++}$ as well as $C^*\mathscr{A}^+C \subseteq \mathscr{A}^+$.

The basic examples of such $*$-algebras are given by C itself, in which case $\mathsf{C}^+ = \mathsf{C}^{++}$ coincides with the non-negative elements in R. More generally, one can consider $M_n(\mathsf{C})$, the $n \times n$ matrices over C with the usual $*$-structure. As in the case of $\mathsf{C} = \mathbb{C}$ one can show that the positive linear functionals can be written as $A \mapsto \mathsf{tr}(\rho A)$ where tr is the trace functional and ρ is a Hermitian matrix such that $\langle v, \rho v\rangle \geq 0$ for all $v \in \mathsf{C}^n$, where $\langle \cdot, \cdot \rangle$ denotes the usual Hermitian product on C^n. Moreover, $A = A^* \in M_n(\mathsf{C})^+$ if and only if $\langle v, Av\rangle \geq 0$ for all $v \in \mathsf{C}^n$. In the case of deformation quantization we consider the space of complex-valued smooth functions $C_0^\infty(M)$ with compact support (only for technical reasons) on a Poisson manifold as the space of classical observables with the obvious $*$-algebra structure. Then the positive linear functionals are positive Borel measures with finite volume for every compact set and $C_0^\infty(M)^+$ consists of functions with $f(x) \geq 0$ for all $x \in M$ as expected. Nevertheless we should also mention that the above definitions lead to a somehow 'pathological' characterization of positive elements in the case of polynomial algebras, see e.g. [37]. The final example we would like to mention is the Grassmann algebra $\bigwedge(\mathsf{C}^n)$, where we define a $*$-involution by requiring $\mathbf{1}^* = \mathbf{1}$ and $e_i^* = e_i$, where $e_1, \ldots, e_n$ is the canonical basis of C^n. Then one easily shows that $\omega : \bigwedge(\mathsf{C}^n) \to \mathsf{C}$ is positive if and only if $\omega(\mathbf{1}) \geq 0$ and $\omega(e_{i_1} \wedge \cdots \wedge e_{i_r}) = 0$ for all $r \geq 1$ and $i_1, \ldots, i_r$. Thus in this example one has (up to normalization) only one positive linear functional.

To avoid such examples we state the following definition [15, Def. 2.7]: A $*$-algebra $\mathscr{A}$ has sufficiently many positive linear functionals if for every Hermitian element $H \neq 0$ there exists a positive linear functional ω such that $\omega(H) \neq 0$. Clearly $M_n(\mathsf{C})$ and $C_0^\infty(M)$ have sufficiently many positive linear functionals but $\bigwedge(\mathsf{C}^n)$ does not. This condition turns out to be rather strong as it implies already some very C^*-algebra like properties: one can show that if $\mathscr{A}$ has sufficiently many positive linear functionals then, e.g. $A^*A = 0$ implies $A = 0$ as well as $H^n = 0$ for a normal element H implies $H = 0$. Moreover, there exists a faithful $*$-representation on a pre-Hilbert space over C which can be constructed by means of the GNS construction, see [15, Prop. 2.8].

3. Deformations of $*$-algebras

Before studying deformations of $*$-algebras over ordered rings one notices that if R is an ordered ring then the ring of formal power series $\mathsf{R}[[\lambda]]$ is again ordered in a canonical way, namely one defines $a = \sum_{r=r_0}^\infty \lambda^r a_r$ to be positive if $a_{r_0} > 0$. Thus the deformation program stays in the same category of $*$-algebras over ordered rings (and their quadratic ring extension by 'i'). In the case of deformation quantization the deformation parameter λ plays the role of $\hbar$ and may be substituted $\lambda \mapsto \hbar$ in *convergent* situations.

Denote by $\mu_0 : \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$ the C-bilinear associative product of $\mathscr{A}$ and by $I_0 : \mathscr{A} \to \mathscr{A}$ the $*$-involution, which we shall frequently refer to as the 'classical' structures. Then a *formal $*$-algebra deformation* of $\mathscr{A}$ is a formal deformation $\mu = \mu_0 + \sum_{r=1}^\infty \lambda^r \mu_r$ of μ_0 to an associative $\mathsf{C}[[\lambda]]$-bilinear product μ for $\mathscr{A}[[\lambda]]$ together with a formal deformation $I = I_0 + \sum_{r=1}^\infty \lambda^r I_r$ of the $*$-involution I_0 to a

$\mathsf{C}[[\lambda]]$-antilinear involutive anti-automorphism of μ. Two such deformations (μ, I) and $(\tilde{\mu}, \tilde{I})$ are called *equivalent* if there exists a formal series $T = \mathsf{id} + \sum_{r=1}^{\infty} \lambda^r T_r$ of C-linear maps $T_r : \mathscr{A} \to \mathscr{A}$ such that $T(\mu(A \otimes B)) = \tilde{\mu}(TA \otimes TB)$ and $TI = \tilde{I}T$. Due to the physical interpretation in deformation quantization of the Hermitian elements with respect to the classical $*$-involution as observables (which are the real-valued functions), we are mainly interested in those $*$-algebra deformations where the classical $*$-involution is *not* deformed, since the classical observables should be observable in the 'quantized' theory, too. This particular kind of $*$-algebra deformations shall be called *Hermitian deformations* of $\mathscr{A}$ (or symmetric, see [3] for this definition in the context of star products). Equivalence of Hermitian deformations now simply means that there is a real equivalence transformation.

We shall now briefly sketch the deformation theory for Hermitian deformations in terms of a Hermitian Hochschild cohomology for $\mathscr{A}$, see [26] for the general cohomological approach to deformations of algebras. On the Hochschild complex $C^\bullet(\mathscr{A}) = \bigoplus_{n=0}^{\infty} \mathsf{Hom}(\mathscr{A}^{\otimes n}, \mathscr{A})$ one defines an action of the $*$-involution (we sometimes use again the symbol $*$ instead of I_0) by

$$\varphi^*(A_1 \otimes \cdots \otimes A_n) := (\varphi(A_n^* \otimes \cdots \otimes A_1^*))^*, \tag{3.1}$$

where $A_1, \ldots, A_n \in \mathscr{A}$ and φ is a n-cochain. Clearly φ^* is again an n-cochain and the map $\varphi \mapsto \varphi^*$ is C-antilinear and involutive. Note that we have also reverted the order of the arguments. Then one finds by a straightforward computation the following relation for the Gerstenhaber product $\varphi \diamondsuit \psi$ of two cochains $\varphi \in C^n(\mathscr{A})$, $\psi \in C^m(\mathscr{A})$:

$$(\varphi \diamondsuit \psi)^* = (-1)^{(n-1)(m-1)}\, \varphi^* \diamondsuit \psi^*. \tag{3.2}$$

The fact that $*$ is an anti-automorphism of μ_0 is expressed by the relation

$$\mu_0^* = \mu_0, \tag{3.3}$$

whence we obtain for the Hochschild differential with respect to μ_0 for an n-cochain φ

$$(\delta\varphi)^* = (-1)^{n-1} \delta\varphi^*, \tag{3.4}$$

where δ is the Gerstenhaber bracket with μ_0, i.e., the graded $\diamondsuit$-commutator $\delta\varphi = (-1)^{n-1}[\mu_0, \varphi]$. We call a cochain φ *Hermitian* if $\varphi^* = \varphi$ and anti-Hermitian if $\varphi^* = -\varphi$, and denote the *Hermitian cochains* by $C_H^\bullet(\mathscr{A})$. They are a $\mathbb{Z}$-graded R-submodule of $C^\bullet(\mathscr{A})$. For a 0-cochain this coincides with the notion of a Hermitian algebra element while a 1-cochain is Hermitian if $\varphi(A^*) = \varphi(A)^*$ for all $A \in \mathscr{A}$. Moreover, for $\varphi \in C_H^n(\mathscr{A})$ the cochain $\delta\varphi$ is Hermitian if n is odd and anti-Hermitian if n is even. This observation leads to the following definition of the Hermitian Hochschild cohomology of $\mathscr{A}$. Define $Z_H^n(\mathscr{A})$ to be the set of Hermitian cocycles and define $B_H^n(\mathscr{A})$ to be the set of Hermitian coboundaries $\varphi = \delta\psi$ where ψ is Hermitian if n even and anti-Hermitian if n is odd. Then

$H^n_H(\mathscr{A}) := Z^n_H(\mathscr{A})\big/B^n_H(\mathscr{A})$ denotes the n-th *Hermitian Hochschild cohomology group*. As an ordered ring R has characteristic zero and no zero-divisors it is reasonable to assume in addition $\frac{1}{2} \in \mathsf{R}$ since one may pass to an appropriate quotient anyway. Now if $\frac{1}{2} \in \mathsf{R}$ then we can decompose any cochain R-linearly into a Hermitian and anti-Hermitian part, i.e.,

$$\varphi = \frac{1}{2}(\varphi + \varphi^*) + \mathrm{i}\frac{1}{2\mathrm{i}}(\varphi - \varphi^*), \tag{3.5}$$

and an easy check using (3.4) shows that the same holds on the level of cohomology whence we have the following R-linear isomorphism

$$\begin{aligned} H^\bullet(\mathscr{A}) &\cong H^\bullet_H(\mathscr{A}) \oplus \mathrm{i}H^\bullet_H(\mathscr{A}) \\ [\varphi] &\mapsto \left[\tfrac{1}{2}(\varphi + \varphi^*)\right] + \mathrm{i}\left[\tfrac{1}{2\mathrm{i}}(\varphi - \varphi^*)\right]. \end{aligned} \tag{3.6}$$

Now let us consider a deformation $\mu = \sum_{r=0}^\infty \lambda^r \mu_r$ of μ_0. Then the associativity condition for μ is well known to be equivalent to $[\mu, \mu] = 0$ which yields $[\mu_0, \mu_0] = 0$ in zeroth order and in higher orders of λ one has

$$\delta\mu_r = -\frac{1}{2}\sum_{s=1}^{r-1}[\mu_s, \mu_{r-s}]. \tag{3.7}$$

To obtain in addition a Hermitian deformation we need

$$\mu_r^* = \mu_r \tag{3.8}$$

for all r since λ is 'real' in $\mathsf{C}[[\lambda]]$. Thus we end up with the following observation that if one can solve both associativity and Hermiticity up to order $r-1$, then the obstructions to solve (3.7) and (3.8) are given by $H^3_H(\mathscr{A})$. If on the other hand one is able to solve the cohomological equation (3.7) for μ_r then there is even a Hermitian solution since an easy computation shows that the right hand side in (3.7) is anti-Hermitian and thus by (3.6) one may simply take the Hermitian part $\frac{1}{2}(\mu_r + \mu_r^*)$ instead of μ_r. Nevertheless note that this is in general not cohomologous to μ_r and thus there may be associative deformations of μ_0 which are *not* equivalent to a Hermitian deformation. This happens e.g. for star-products on symplectic manifolds with non-trivial second deRham cohomology, see [34]. We summarize these results in the following proposition:

Proposition 3.1 *Let $\mathscr{A}$ be a $*$-algebra and let $\mu_0 + \lambda\mu_1 + \ldots + \lambda^{r-1}\mu_{r-1}$ be a Hermitian deformation up to order $r-1$. Then the obstruction space for a Hermitian deformation up to order r is given by $H^3_H(\mathscr{A})$. If there exists an associative deformation up to order r then there also exists an Hermitian deformation up to order r.*

4. Positive deformations

Let us now discuss the main object of this work, the definition of positive deformations of $*$-algebras. In order to motivate this let us first recall the following example of [14, Sect. 2, p. 555]: Consider the phase space $\mathbb{R}^{2n}$ with the canonical coordinates $q^1,\ldots,q^n,p_1,\ldots,p_n$ and the symplectic form $\omega = \sum_k dq^k \wedge dp_k$. Then the Weyl-Moyal star-product, see [3], is given by

$$f *_{\mathrm{Weyl}} g = \mu_0 \circ \mathrm{e}^{\frac{\mathrm{i}\lambda}{2}\sum_k (\partial_{q^k}\otimes\partial_{p_k} - \partial_{p_k}\otimes\partial_{q^k})} f \otimes g, \tag{4.1}$$

where $f,g \in C^\infty(\mathbb{R}^{2n})[[\lambda]]$, and μ_0 is the pointwise product. Clearly $\overline{(f *_{\mathrm{Weyl}} g)} = \overline{g} *_{\mathrm{Weyl}} \overline{f}$ and thus $*_{\mathrm{Weyl}}$ is a Hermitian deformation of the pointwise product with the complex conjugation as $*$-involution. Consider the Hamiltonian $H = \sum_k (q^k q^k + p_k p_k)$ of the harmonic oscillator and let $\delta : C^\infty(\mathbb{R}^{2n})[[\lambda]] \to \mathbb{C}[[\lambda]]$ be the δ-functional at 0 which is classically a positive linear functional. Then

$$\delta(H *_{\mathrm{Weyl}} H) = -\frac{\lambda^2}{2} < 0 \tag{4.2}$$

shows that δ is no longer a positive linear functional for the Weyl-Moyal star-product. On the other hand it is clear that the classical limit of a positive $\mathsf{C}[[\lambda]]$-linear functional of a deformed $*$-algebra $\mathscr{A}[[\lambda]]$ has to be positive with respect to the undeformed $*$-algebra structure. This motivates the following definition:

Definition 4.1 *Let $\mathscr{A}$ be a $*$-algebra over C and let $(\mathscr{A}[[\lambda]],\mu,I)$ be a $*$-algebra deformation of $\mathscr{A}$.*

i) The deformation is called positive if for any classically positive C-linear functional ω_0 there exist C-linear maps $\omega_r : \mathscr{A} \to \mathsf{C}$, $r \geq 1$, such that the $\mathsf{C}[[\lambda]]$-linear functional $\omega = \sum_{r=0}^\infty \lambda^r \omega_r : \mathscr{A}[[\lambda]] \to \mathsf{C}[[\lambda]]$ is positive with respect to the deformed $*$-algebra structure.

ii) The deformation is called strongly positive if any classically positive linear functional is positive for the deformed $*$-algebra, too.

Clearly positivity of a $*$-algebra deformation is a property of the whole corresponding equivalence class of $*$-algebra deformations as one can pull-back positive functionals with the equivalence transformation which yields again a positive functional without changing the classical limit. Nevertheless, this is no longer true for strongly positive deformations in general and it remains to find an appropriate notion of equivalence here.

As a first application we observe that having sufficiently many positive linear functionals is a property which is preserved under positive deformations:

Proposition 4.2 *Let $\mathscr{A}$ be a $*$-algebra over C having sufficiently many positive linear functionals. Then every positive deformation of $\mathscr{A}$ has sufficiently many positive linear functionals.*

Proof. Let $(\mathscr{A}[[\lambda]],\mu,I)$ be a positive deformation of $(\mathscr{A},\mu_0,I_0)$ and let $A = \sum_{r=r_0}^\infty \lambda^r A_r$ be a non-zero Hermitian element with respect to I with first non-vanishing order r_0. Then $A_{r_0} = I_0(A_{r_0})$ is Hermitian with respect to the classical $*$-involution and thus we find a positive C-linear functional ω_0 of $\mathscr{A}$ with $\omega_0(A_{r_0}) \neq 0$. Moreover, we find $\omega_r : \mathscr{A} \to \mathsf{C}$ such that $\omega = \sum_{r=0}^{\infty} \lambda^r \omega_r$ is a positive $\mathsf{C}[[\lambda]]$-linear functional of $\mathscr{A}[[\lambda]]$ with respect to the deformed $*$-algebra structure. But then clearly $\omega(A) \neq 0$, proving the proposition. ∎

As $*$-algebras with sufficiently many positive linear functionals behave in many aspects almost like C^*-algebras we are mainly interested in such positive deformations.

The main difficulty in discussing the existence of (strongly) positive deformations is that one has to deal with inequalities instead of equalities which seems to exclude a suitable cohomological approach. It may even be difficult to decide whether a given $*$-algebra deformation, as e.g. the Weyl-Moyal star-product, is positive or not. Nevertheless in the case of a Hermitian deformation we have the following simple but rather useful criterion:

Lemma 4.3 *Let $(\mathscr{A}[[\lambda]], \mu, I_0)$ be a Hermitian deformation of a $*$-algebra $\mathscr{A}$ over C. If for all $A \in \mathscr{A}$ and $r \geq 1$ one has $\mu_r(A^* \otimes A) \in \mathscr{A}^+$ then the deformation μ is strongly positive.*

Proof. Thanks to [9, Lem. A.5] it is sufficient to check the positivity of a $\mathsf{C}[[\lambda]]$-linear functional $\omega : \mathscr{A}[[\lambda]] \to \mathsf{C}[[\lambda]]$ on elements in $\mathscr{A}$ alone. But if $\omega_0 : \mathscr{A} \to \mathsf{C}$ is a positive linear functional then clearly $\omega_0(\mu(A^* \otimes A)) = \sum_{r=0}^{\infty} \lambda^r \omega_0(\mu_r(A^* \otimes A)) \geq 0$. ∎

Based on this lemma we have the following class of strongly positive deformations of $*$-algebras constructed by using commuting derivations:

Lemma 4.4 *Let $\mathscr{A}$ be a $*$-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$, where we assume $\mathbb{Q} \subseteq \mathsf{R}$, and let $D_1, \ldots, D_n$ be derivations of $\mathscr{A}$ such that $[D_i, D_j] = 0 = [D_i, D_j^*]$ for all $i, j = 1, \ldots, n$. Then*

$$\mu = \mu_0 \circ \mathrm{e}^{\lambda \sum_k D_k \otimes D_k^*} \tag{4.3}$$

defines a strongly positive Hermitian deformation of $\mathscr{A}$.

Proof. First we note that the exp-series is well defined thanks $\mathbb{Q} \subseteq \mathsf{R}$. As already observed , see (3.4), the C-linear maps D_i^* are again derivations and thus all occuring derivations commute which implies that (4.3) is associative. Clearly μ is Hermitian and one has $\mu_r(A^* \otimes A) \in \mathscr{A}^{++} \subseteq \mathscr{A}^+$ for all $r \geq 1$ whence we can apply the above lemma. ∎

5. Positive deformations in deformation quantization

Let us now discuss the situation in deformation quantization. The main example for a deformation of the type as in Lemma 4.4 is the Wick product $*_{\mathrm{Wick}}$ for $\mathbb{C}^n$: denote by $z^1, \ldots, z^n, \overline{z}^1, \ldots, \overline{z}^n$ the global (anti-) holomorphic coordinates then $*_{\mathrm{Wick}}$ is defined for $f, g \in C^\infty(\mathbb{C}^n)[[\lambda]]$ by

$$f *_{\mathrm{Wick}} g = \mu_0 \circ \mathrm{e}^{2\lambda \sum_k \partial_{z^k} \otimes \partial_{\overline{z}^k}} f \otimes g, \tag{5.1}$$

which is clearly of the above form and thus strongly positive and Hermitian. See [27, 28, 29, 12] for a more general treatment of such star-products with separation of variables on Kähler manifolds. From the above remark on equivalent Hermitian deformations and the well-known fact that on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ all star-products are equivalent we conclude that the Weyl-Moyal star-product $*_{\mathrm{Weyl}}$ is an example of a positive but *not* strongly positive deformation. More explicitly, we have the following: Using the explicit (and real) equivalence transformation $T = \mathrm{e}^{\lambda\Delta}$, where $\Delta = \sum_k \partial_{z^k} \partial_{\overline{z}^k}$ is the Laplacian and $f *_{\mathrm{Weyl}} g = T^{-1}(Tf *_{\mathrm{Wick}} Tg)$, we conclude that for any classically positive linear functional ω_0 the $\mathbb{C}[[\lambda]]$-linear functional $\omega = \omega_0 \circ T = \omega_0 \circ \mathrm{e}^{\lambda\Delta}$ is positive with respect to $*_{\mathrm{Weyl}}$. In particular, one obtains a deformation $\delta \circ \mathrm{e}^{\lambda\Delta}$ of the delta-functional which is positive. But note also that for some particular ω_0 there may be no need to deform it in order to obtain positivity, see e.g. [8, 9].

A local version of the above construction can even be used to show that all Hermitian star-products on a symplectic manifold are positive deformations:

Proposition 5.1 *Let $(M, *)$ be a symplectic manifold with a Hermitian star product. Then $*$ is a positive deformation.*

Proof. Let $\omega_0 : C_0^\infty(M) \to \mathbb{C}$ be a classically positive linear functional. Choose a locally finite open cover $\{O_\alpha\}_{\alpha \in I}$ of M by contractable charts and let $\{\chi_\alpha\}_{\alpha \in I}$ be a subordinate 'quadratic' partition of unity, i.e., $\sum_\alpha \overline{\chi}_\alpha \chi_\alpha = 1$. Endow each O_α with a local Wick star-product $*_\alpha$ and let T_α be a local and real equivalence transformation between $*|_{O_\alpha}$ and $*_\alpha$, i.e., $T_\alpha(f * g) = T_\alpha f *_\alpha T_\alpha g$ on O_α. Such an equivalence transformation exists since O_α is contractable, see e.g. [4]. Then it is easy to check that the $\mathbb{C}[[\lambda]]$-linear functional $\omega : C_0^\infty(M)[[\lambda]] \to \mathbb{C}[[\lambda]]$ defined by

$$\omega(f) := \sum_\alpha \omega_0 \left(T_\alpha \left(\overline{\chi}_\alpha * f * \chi_\alpha\right)\right) \tag{5.2}$$

is well defined, positive with respect to $*$, and a deformation of ω_0. ∎

Since $C_0^\infty(M)$ (or $C^\infty(M)$, respectively) has sufficiently many positive linear functionals we obtain from Propositions 4.2 and 5.1 the following useful corollary:

Corollary 5.2 *Let $(M, *)$ be a symplectic manifold with a Hermitian star product. Then the $\mathbb{C}[[\lambda]]$-algebra $(C_0^\infty(M)[[\lambda]], *)$ has sufficiently many positive $\mathbb{C}[[\lambda]]$-linear functionals.*

The case of a Hermitian star-product on an arbitrary Poisson manifold seems to be more involved since the proof of Proposition 5.1 uses at several points that the Poisson structure is symplectic. Thus it remains an open question whether or which star-products on arbitrary Poisson manifolds are positive deformations. Nevertheless we can show directly the weaker property that a Hermitian star-product on a Poisson manifold has sufficiently many positive linear functionals.

Proposition 5.3 *Let $(M,*)$ be a Poisson manifold with a Hermitian star-product. Then the $\mathbb{C}[[\lambda]]$-algebra $(C_0^\infty(M)[[\lambda]],*)$ has sufficiently many positive $\mathbb{C}[[\lambda]]$-linear functionals.*

Proof. Let $0 \neq f = \sum_{r=0}^\infty \lambda^r f_r \in C_0^\infty(M)[[\lambda]]$ be Hermitian then we may assume that $f_0 \neq 0$. Thus we find a point $x \in M$ and an open neighborhood $U \subseteq M$ of x such that $f|_U$ is either strictly positive or strictly negative. Choose now a smooth density $\mu \in \Gamma_0^\infty(|\bigwedge^n|T^*M)$ with $\mu(x) > 0$, $\mu \geq 0$, and $\operatorname{supp}\mu \subset U$. Then it follows by the same argument as in [14, Lem. 2] that the functional $\omega : g \mapsto \int_M g\mu$ is positive with respect to any star product on M and clearly we have $\omega(f) \neq 0$. ■

Let us conclude with a few remarks: It is clear that if a deformation ω of a classically positive linear functional ω_0 into a positive $\mathsf{C}[[\lambda]]$-linear functional of a deformed algebra $\mathscr{A}[[\lambda]]$ exists, the higher orders ω_r of the deformation are not necessarily unique. If for example the functional ω_0 is faithful, i.e., if $\omega_0(A^*A) > 0$ for all $A \neq 0$, then one can add *any* real linear functionals ω_r in higher orders and not lose positivity of ω. This raises the question of whether in a particular case there are in some sense 'natural' or 'minimal' corrections ω_r which deform ω_0 into a positive linear functional. In the case of star-products one can impose further conditions on ω_r by requiring e.g. continuity with respect to the canonical locally convex topologies of smooth functions. Moreover, one can demand that the corrections should not increase the support of the classical functional ω_0, see [40] for an extensive discussion on the involved locality structures. But even then a tremendous variety remains, since one can pull-back deformed positive linear functionals by $*$-automorphisms of the deformed algebra with the identity as classical limit. Then one ends up again with a deformation of the same classically positive linear functional. Hence a classification of positive deformations of a given classically positive linear functional modulo the above action of $*$-automorphisms would be highly desirable.

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