The deformation quantization of certain super-Poisson brackets and BRST cohomology

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Abstract. On every split supermanifold equipped with the Rothstein super-Poisson bracket we construct a deformation quantization by means of a Fedosov-type procedure. In other words, the supercommutative algebra of all smooth sections of the dual Grassmann algebra bundle of an arbitrarily given vector bundle E (equipped with a fiber metric) over a symplectic manifold M will be deformed by a series of bidifferential operators having first order supercommutator proportional to the Rothstein superbracket. Moreover, we discuss two constructions related to the above result, namely the quantized BRST-cohomology for a locally free Hamiltonian Lie group action and the classical BRST cohomology in the general coisotropic (or reducible) case without using a 'ghosts of ghosts' scheme.

Keywords: Deformation quantization, even symplectic supermanifolds, Fedosov construction, BRST-cohomology, coisotropic submanifolds

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To the memory of Moshé Flato

Introduction

In the usual programme of deformation quantization (cf. [4]) the quantum mechanical multiplication is considered as a formal associative deformation (a so-called star-product) of the pointwise multiplication of the classical observables, viz. the algebra of smooth complex-valued functions on a given symplectic manifold. The deformation is such that to first order in the deformation parameter λ (corresponding to $\hbar$) the commutator of the deformed product is proportional to the Poisson bracket. The difficult question of existence of these star-products for every symplectic manifold was settled independently by De Wilde and Lecomte [14] and Fedosov [17, 18], and even for general Poisson manifolds by Kontsevich [29].

Four years ago, adequate formulations for star-products in the theory of supermanifolds, however, did not seem to have appeared in the literature although the geometric quantization scheme had found its suitable generalization to the super case (see, e.g., [22] and references therein). In order to elaborate our understanding of supermanifolds, (at Freiburg) I proposed to the diploma student Ralf Eckel to give a formulation thereof in terms of associated bundles of certain jet group bundles which he did rather nicely in his diploma thesis [16] in April 1996. He also provided a star-product formula for the case where the 'fermionic directions'

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formed a trivial vector bundle (see below for a precise statement). When preparing my habilitation thesis in May 1996, I suddenly realized that a simple Fedosov procedure could be set up for general vector bundles: However, I did not know in advance a possible super-Poisson bracket, so I first constructed the deformed algebra à la Fedosov and *a posteriori* computed the super-Poisson bracket as its first order supercommutator in [6], a result which I included in my habilitation thesis. A month later I was made aware by Amine El Gradechi that the super-Poisson bracket I had computed out of this quantization exactly coincided with Rothstein's super-Poisson bracket, see [31], found in 1991.

In this report I should like to give a detailed description of this Fedosov construction (thereby including an improved version of the preprint [6] without some rather embarrassing misprints). I shall also include sketches of two more recent constructions related to the above and done in collaboration with H.-C. Herbig and S. Waldmann (see [8, 7]), namely the quantum BRST cohomology for covariant star-products, and the classical BRST cohomology for arbitrary coisotropic constraint surfaces (the 'reducible first-class case' in the physics literature) where a so-called 'ghosts-for-ghosts'-scheme is no longer necessary.

The supermanifolds I shall deal with in this report will only be 'split', more precisely, the set-up will be as follows: Let (M,ω) be a $2m$-dimensional symplectic manifold and E be an arbitrary n-dimensional vector bundle over M. Then the algebra $\mathscr{C}_0$ of 'classical superobservables' can be considered as the space of all smooth sections of the complexified dual Grassmann algebra bundle of E (see, e.g., [3]), i.e.,

$$\mathscr{C}_0 := \mathbb{C}\Gamma(\Lambda E^*), \tag{1}$$

where the multiplication is the pointwise wedge product. Clearly, $\mathscr{C}_0$ is a $\mathbb{Z}_2$-graded supercommutative algebra, i.e., $\phi\wedge\psi = (-1)^{d_1d_2}\psi\wedge\phi$ for $\phi,\psi\in\Gamma(\Lambda E^*)$, ϕ of degree d_1 and ψ of degree d_2. A *super-Poisson bracket* for $\mathscr{C}_0$ is by definition a $\mathbb{Z}_2$-graded bilinear map $M_1 : \mathscr{C}_0\times\mathscr{C}_0\to\mathscr{C}_0$ which is superanticommutative, i.e., $M_1(\psi,\phi) = -(-1)^{d_1d_2}M_1(\phi,\psi)$, satisfies the superderivation rule $M_1(\phi,\psi\wedge\chi) = M_1(\phi,\psi)\wedge\chi + (-1)^{d_1d_2}\psi\wedge M_1(\phi,\chi)$, and the super-Jacobi identity, i.e., $(-1)^{d_1d_3}M_1(M_1(\phi,\psi),\chi) + cycl. = 0$ where $\chi\in\mathscr{C}_0$ is of degree d_3.

It is in general not difficult to find super-Poisson brackets of purely algebraic type, i.e., which vanish when one of their arguments is restricted to a smooth complex-valued function, by means of a fiber metric q in E (see, e.g., [4], p. 123, eq. 5-1):

$$M_1'(\phi,\psi) = q^{AB}(j(e_A)\phi)\wedge(i(e_B)\psi), \tag{2}$$

where q^{AB} are the components of the induced fiber metric in the dual bundle E^* in the dual basis to a local basis (e_A), $1\le A\le \dim E$, of sections of E, and $i(e_B)$ and $j(e_A)$ denote the usual interior product left antiderivation and right antiderivation, respectively, which are also often denoted by $\overrightarrow{\partial}_A$ and $\overleftarrow{\partial}_A$ in the literature on supermanifolds. The above definition does not depend on the choice of that local basis. In the case where M is $\mathbb{R}^{2m}$ with the standard Poisson bracket one can combine the standard bracket with the above super-Poisson bracket to get

$$M_1(\phi,\psi) = \frac{\partial\phi}{\partial q^i}\wedge\frac{\partial\psi}{\partial p_i} - \frac{\partial\phi}{\partial p_i}\wedge\frac{\partial\psi}{\partial q^i} + q^{AB}(j(e_A)\phi)\wedge(i(e_B)\psi). \qquad (3)$$

However, for nontrivial bundles it does not seem to be so obvious to generalize this bracket in the sense that it is equal to — at least in degree zero — the Poisson bracket of the base space M when restricted to the sections of degree zero. Some time ago M. Rothstein has given a formula for this more general situation, [31]:

$$\{\phi,\psi\}_R = \Lambda^{ij}((1-2\hat{R}^E)^{-1})_i^k\wedge\nabla^E_{\partial_k}\phi\wedge\nabla^E_{\partial_j}\psi + q^{AB}(j(e_A)(\phi))\wedge(i(e_B)(\psi)), \quad (4)$$

where ∇^E is a covariant derivative in the bundle E preserving the fiber metric q and $\hat{R}^E$ is a suitable section in the bundle $\Gamma(Hom(TM,TM)\otimes\Lambda^{\mathrm{even}}E^*)$ constructed out of the curvature of ∇^E (see Section 1 for details).

The paper is organized as follows: In the first part, I transfer Fedosov's Weyl algebra bundle to the above situation by simply tensoring with the dual Grassmann bundle ΛE^*. The fiberwise multiplication has also a component in ΛE^* built by means of the fiber metric in E. Then Fedosov's procedure can completely be imitated without further difficulties: We show the existence of a Fedosov connection D of square zero whose kernel in the space of antisymmetric degree zero is in linear 1-1 correspondence to the space of formal power series in λ with coefficients in $\mathscr{C}_0$

$$\mathscr{C} := \mathscr{C}_0[[\lambda]], \qquad (5)$$

which immediately gives rise to the desired quantum deformation (Theorem 1.3). Then I explicitly compute the super-Poisson bracket M_1 as the term proportional to $(\mathbf{i}\lambda)/2$ by means of Fedosov's recursion formulas (Theorem 1.4) and show that it is equal to the Rothstein superbracket. We evaluate the formulas a little bit further in Section 2 in the case where the connection ∇^E is flat: Using a local basis of covariantly constant sections the quantum multiplication is a sort of tensor product of a star-product on (the smooth complex-valued functions on) M and a formal Clifford multiplication. Section 3 is concerned with a sketch of a quantized BRST formulation (see [8]). In Section 4 I shall sketch joint work with Hans-Christian Herbig where we have found a classical BRST complex for general coisotropic (reducible first class) constraint manifolds using the Rothstein superbracket, see also [7].

Notation: In all of this paper the Einstein summation convention is used: Two equal indices are automatically summed up over their range unless stated otherwise. Moreover, we widely make use of Fedosov's notation in [18] with the following exceptions: We use the symbol ∇ to denote the covariant derivative and not Fedosov's ∂ and describe the occurring symmetric tensor fields with $\vee$ products (see, e.g., [23], p. 209–226) and use the symmetric substitution operator i_s instead of Fedosov's functions of y and derivatives with respect to y.

1. A star-product for sections of Grassmann algebra bundles

This Section is — up to some corrected typos and an additional remark — identical to [6].

1.1. The Fedosov construction

Let (M,ω) be a $2m$-dimensional symplectic manifold and E an n-dimensional real vector bundle over M with a fixed nondegenerate fiber metric q. For the computations that will follow we shall use coordinates $(x^1,\cdots,x^{2m})$ in a chart U of M. The base fields $\frac{\partial}{\partial x^i}$ will be denoted by ∂_i $(1\leq i\leq 2m)$ for short. For computations in E we shall use a local basis (e_A), $1\leq A\leq n$, of sections of E over U. Denote the dual basis in the dual bundle E^* of E by (e^A), $(1\leq A\leq n)$. Let $\Lambda\in\Gamma(\Lambda^2TM)$ denote the Poisson structure of (M,ω), i.e., the Poisson bracket of two smooth real valued functions f,g is given by $\{f,g\}:=\Lambda(df,dg)$. Denoting the components of ω and Λ in that chart by $\omega_{ij}:=\omega(\partial_i,\partial_j)$ and $\Lambda^{ij}:=\Lambda(dx^i,dx^j)$ we use the sign conventions of [1] where $\Lambda^{ik}\omega_{jk}=\delta^i_j$. Fix a torsion-free symplectic connection ∇^M in the tangent bundle of M. This is well known to always exist which can be seen by Heß's formula $\omega(\nabla^M_XY,Z):=\omega(\tilde{\nabla}_XY,Z)+\frac{1}{3}(\tilde{\nabla}_X\omega)(Y,Z)+\frac{1}{3}(\tilde{\nabla}_Y\omega)(X,Z)$ where X,Y,Z are arbitrary vector fields on M and $\tilde{\nabla}$ is an arbitrary torsion-free connection on M (see [25]). Fix a connection ∇^E in E which is compatible with q, i.e., $X(q(e_1,e_2))=q(\nabla^E_Xe_1,e_2)+q(e_1,\nabla^E_Xe_2)$ for an arbitrary vector field X on M and sections e_1,e_2 of E. This is also well known to always exist which can be seen by the formula $q(\nabla^E_Xe_1,e_2):=q(\tilde{\nabla}^E_Xe_1,e_2)+\frac{1}{2}(\tilde{\nabla}^E_Xq)(e_1,e_2)$ for an arbitrary connection $\tilde{\nabla}^E$ in E.

We are now forming the *Fedosov algebra* $\mathscr{W}\otimes\Lambda$:

$$\mathscr{W}\otimes\Lambda:=\left(\times_{s=0}^{\infty}\Gamma(\mathbb{C}(\vee^sT^*M\otimes\Lambda E^*\otimes\Lambda T^*M))\right)[[\lambda]]. \tag{6}$$

This is to say that the elements of $\mathscr{W}\otimes\Lambda$ are formal sums $\sum_{s,t=0}^{\infty}w_{st}\lambda^t$ where the w_{st} are smooth sections in the complexification of the vector bundle $\vee^sT^*M\otimes\Lambda E^*\otimes\Lambda T^*M$. In what follows we shall sometimes use the following factorized sections $F:=f\otimes\phi\otimes\alpha\lambda^{t_1}$ and $G:=g\otimes\psi\otimes\beta\lambda^{t_2}$ where $f\in\Gamma(\vee^{s_1}T^*M)$, $g\in\Gamma(\vee^{s_2}T^*M)$, $\phi\in\Gamma(\Lambda^{d_1}E^*)$, $\psi\in\Gamma(\Lambda^{d_2}E^*)$, $\alpha\in\Gamma(\Lambda^{a_1}T^*M)$, and $\beta\in\Gamma(\Lambda^{a_2}T^*M)$. Let $deg_s,deg_E,deg_a,deg_\lambda$ be the obvious degree maps from $\mathscr{W}\otimes\Lambda$ to itself, i.e., those $\mathbb{C}$-linear maps for which the above factorized sections $f\otimes\phi\otimes\alpha\lambda^{t_1}$ are eigenvectors to the eigenvalues s_1,d_1,a_1,t_1 respectively and which we refer to as the symmetric degree, the E-degree, the antisymmetric degree, and the λ-degree, respectively. Moreover, let P_E and P_λ be the corresponding maps $(-1)^{deg_E}$ and $(-1)^{deg_\lambda}$ which we refer to as the E-parity and the λ-parity, respectively. We say that a $\mathbb{C}$-linear endomorphism Φ of $\mathscr{W}\otimes\Lambda$ is of ζ-degree k $(\zeta=s,a,E,\lambda)$ iff $[deg_\zeta,\Phi]=k\Phi$. Analogously, Φ is said to be of ζ-parity $(-1)^k$ $(\zeta=E,\lambda)$ iff $P_\zeta\Phi P_\zeta=(-1)^k\Phi$. Let C denote the complex conjugation of sections in $\mathscr{W}\otimes\Lambda$. We shall sometimes write $\mathscr{W}$ for the space of elements of $\mathscr{W}\otimes\Lambda$ having zero antisymmetric degree and $\mathscr{W}\otimes\Lambda^a$ for the space of those elements having antisymmetric degree a. The space $\mathscr{W}\otimes\Lambda$ is an associative algebra with respect to the usual pointwise product where we do *not* use the graded tensor product of the two Grassmann algebras involved. More precisely, for the above factorized sections the pointwise or undeformed multiplication is simply given by

$$(f\otimes\phi\otimes\alpha\lambda^{t_1})(g\otimes\psi\otimes\beta\lambda^{t_2}) := (f\vee g)\otimes(\phi\wedge\psi)\otimes(\alpha\wedge\beta)\lambda^{t_1+t_2}. \tag{7}$$

Note that the above four degree maps are derivations and the above two parity maps are automorphisms of this multiplication. Moreover, $\mathscr{W}\otimes\Lambda$ is supercommutative in the sense that

$$GF = (-1)^{d_1d_2+a_1a_2}FG. \tag{8}$$

A linear endomorphism Φ of $\mathscr{W}\otimes\Lambda$ of E-parity $(-1)^{d'}$ and antisymmetric degree a' is said to be a superderivation of type $((-1)^{d'},a')$ of the undeformed algebra $\mathscr{W}\otimes\Lambda$ iff $\Phi(FG)=(\Phi F)G+(-1)^{d'd_1+a'a_1}F(\Phi G)$. Let σ denote the linear map

$$\sigma:\mathscr{W}\otimes\Lambda\to\Gamma(\Lambda E^*\otimes\Lambda T^*M)[[\lambda]] \tag{9}$$

which projects onto the component of symmetric degree zero and clearly is a homomorphism for the undeformed multiplication.

We now combine the two covariant derivatives ∇^M_X in TM and ∇^E_X in E into a covariant derivative ∇_X in $TM\otimes E$ in the usual fashion and extend it canonically to a connection ∇ in $\mathscr{W}\otimes\Lambda$. On the above factorized sections we get in a chart:

$$\nabla(f\otimes\phi\otimes\alpha) = \big((\nabla^M_{\partial_i}f)\otimes\phi + f\otimes(\nabla^E_{\partial_i}\phi)\big)\otimes(dx^i\wedge\alpha) + f\otimes\phi\otimes d\alpha. \tag{10}$$

In order to define a deformed fiberwise associative multiplication consider the following insertion maps for a vector field X on M and a section e of E: Let $i_a(X)$ and $i(e)$ denote the usual inner product antiderivations in $\Gamma(\Lambda T^*M)$ and $\Gamma(\Lambda E^*)$, respectively, and extend them in a canonical manner to superderivations of type $(1,-1)$ and $(-1,0)$ of the undeformed algebra $\mathscr{W}\otimes\Lambda$, respectively. Let $j(e)$ be defined by $P_E i(e)$. Moreover, let $i_s(X)$ denote the corresponding inner product derivation (or symmetric substitution, [23], p. 209–226) in $\times_{s=0}^{\infty}\Gamma(\vee^s T^*M)$, again extended to a derivation of the undeformed algebra $\mathscr{W}\otimes\Lambda$ in the canonical way. Let q^{AB} denote the components of the induced fiber metric q^{-1} in E^*, i.e., $q^{AB} := q^{-1}(e^A,e^B)$. Note that q^{AB} is the inverse matrix to $q(e_A,e_B)$. Then for two elements F,G of $\mathscr{W}\otimes\Lambda$ we can now define the fiberwise deformed multiplication $\circ$:

$$\begin{aligned} F\circ G := \sum_{k,l=0}^{\infty}\frac{(\mathbf{i}\lambda/2)^{k+l}}{k!l!}\Lambda^{i_1j_1}\cdots\Lambda^{i_kj_k}q^{A_1B_1}\cdots q^{A_lB_l}\times \\ \times\big(i_s(\partial_{i_1})\cdots i_s(\partial_{i_k})j(e_{A_1})\cdots j(e_{A_l})F\big)\big(i_s(\partial_{j_1})\cdots i_s(\partial_{j_k})i(e_{B_1})\cdots i(e_{B_l})G\big). \end{aligned} \tag{11}$$

Moreover, let δ and δ^* be the canonical superderivations of the undeformed algebra $\mathscr{W}\otimes\Lambda$ of type $(1,1)$ and $(1,-1)$, respectively, which are induced by the identity map of T^*M to T^*M, where in the case of δ the preimage of the identity is regarded as being part of $\vee T^*M$ and the image as being part of ΛT^*M, and vice versa in the case of δ^*. On the above factorized sections these maps read in coordinates

$$\begin{aligned}\delta(f\otimes\phi\otimes\alpha) &= (i_s(\partial_i)f)\otimes\phi\otimes(dx^i\wedge\alpha), &(12)\\ \delta^*(f\otimes\phi\otimes\alpha) &= (dx^i\vee f)\otimes\phi\otimes(i_a(\partial_i)\alpha). &(13)\end{aligned}$$

Define the total degree derivation Deg:

$$Deg := 2deg_\lambda + deg_s + deg_E. \tag{14}$$

A $\circ$-superderivation of type $((-1)^{d'}, a')$ is defined in an analogous manner as for the undeformed multiplication. Note that $(\mathscr{W}\otimes\Lambda,\circ)$ is not a graded associative algebra in the strict sense since it is equal to the Cartesian product of the eigenspaces of Deg and not to the direct sum of these eigenspaces. It is, however, filtered by those complex subspaces of $\mathscr{W}\otimes\Lambda$ (indexed by a nonnegative integer r) which are given by the images of the maps $Deg(Deg-1)\cdots(Deg-(r-1))$.

We collect some properties of the above structures in the following Proposition:

Proposition 1.1 *With the above definitions and notations we have the following:*

1. $\delta^2 = 0 = (\delta^*)^2$ *and* $\delta\delta^* + \delta^*\delta = deg_s + deg_a$.
2. $\delta\nabla + \nabla\delta = 0$.
3. $Ker(\delta)\cap Ker(deg_a) = \mathscr{C}$.
4. P_E *is a* $\circ$*-automorphism and* deg_a *is a* $\circ$*-derivation which equips the Fedosov algebra* $(\mathscr{W}\otimes\Lambda,\circ)$ *with the structure of a* $\mathbb{Z}_2\times\mathbb{Z}$*-graded associative algebra.*
5. δ, ∇*, and Deg are* $\circ$*-superderivations of type* $(1,1)$, $(1,1)$, *and* $(1,0)$, *respectively.*
6. *The parity map* P_λ *and the complex conjugation C are graded* $\circ$*-antiautomorphisms, i.e.,* $\Phi(F\circ G) = (-1)^{d_1d_2+a_1a_2}\Phi(G)\circ\Phi(F)$ *for* $\Phi = P_\lambda, C$.

Proof. 1. Straightforward.
2. This follows from the vanishing torsion of ∇^M.
3. Without the factor ΛE^* the kernel of δ in the space of antisymmetric degree zero consists of the constants, which proves this statement.
4. The associativity of $\circ$ is known, see, e.g., [4], p. 123, eq. 5-2, and can be done by a long straightforward computation. We shall sketch a shorter proof: $\circ$ is defined on each fiber (for $m\in M$) $\mathscr{W}_m := (\times_{i=0}^\infty(\vee^i T_mM^*\otimes\Lambda E_m^*\otimes\Lambda T_mM^*))[[\lambda]]$ on which we can rewrite the multiplication in the more compact form ($F,G\in\mathscr{W}_m$)

$$F\circ G = \mu(e^{\frac{i\lambda}{2}(R+S)}(F\otimes G)), \tag{15}$$

where the tensor product is over $\mathbb{C}[[\lambda]]$, μ denotes the undeformed fiberwise multiplication, and $R := \Lambda_m^{ij} i_s(\partial_i) \otimes i_s(\partial_j)$, $S := q_m^{AB} j(e_A) \otimes i(e_B)$. Due to the derivation properties of $i_s(\partial_i)$, $i(e_A)$, and $j(e_B)$ we get formulas like

$$
\begin{aligned}
R\,\mu \otimes 1 &= \mu \otimes 1\,(R_{13}+R_{23}),\\
R\,1 \otimes \mu &= 1 \otimes \mu\,(R_{12}+R_{13}),\\
S\,\mu \otimes 1 &= \mu \otimes 1\,(S_{13}(P_E)_2+S_{23}),\\
S\,1 \otimes \mu &= 1 \otimes \mu\,(S_{12}+S_{13}(P_E)_2).
\end{aligned}
$$

where the index notation is borrowed from Hopf algebras and indicates on which of the three tensor factors of $\mathscr{W}_m$ the maps R, S, and P_E should act, e.g., $R_{23} := 1 \otimes R$, $(P_E)_2 := 1 \otimes P_E \otimes 1$. These "pull through formulas" can be used to pull through the corresponding formal exponentials. Since all the maps $i_s(\partial_i)$ commute with $j(e_A)$ and $i(e_B)$ and since the $j(e_A)$ commute with all $i(e_B)$ whereas $j(e_A)$ and $j(e_B)$ anticommute as well as $i(e_A)$ and $i(e_B)$ we can conclude that all the six maps R_{12}, R_{13}, R_{23}, S_{12}, $S_{13}(P_E)_2$, and S_{23} pairwise commute. This is the essential step for associativity. The gradation properties are immediate.

5. The derivation properties of δ and Deg are clear, for the corresponding statement for ∇ the fact that ∇^M preserves the Poisson structure Λ and that ∇^E preserves the dual fiber metric q^{-1} is crucial.

6. Straightforward. ■

Due to the first part of this proposition we can construct a $\mathbb{C}[[\lambda]]$-linear endomorphism δ^{-1} of the Fedosov algebra in the following way: On the above factorized sections F we put

$$
\delta^{-1}F := \begin{cases} \frac{1}{s_1+a_1}\delta^* F & \text{if } s_1+a_1 \geq 1 \\ 0 & \text{if } s_1+a_1 = 0 \end{cases}. \tag{16}
$$

Since $\mathscr{W} \otimes \Lambda$ is an $\mathbb{Z}_2 \times \mathbb{Z}$-graded associative algebra we can form the $\mathbb{Z}_2 \times \mathbb{Z}$-graded super Lie bracket which reads on the above factorized sections:

$$
[F,G] := ad(F)G := F \circ G - (-1)^{d_1 d_2 + a_1 a_2} G \circ F. \tag{17}
$$

It follows from the associativity of $\circ$ that $ad(F)$ is $\circ$-superderivation of the Fedosov algebra $(\mathscr{W} \otimes \Lambda, \circ)$ of type $((-1)^{d_1}, a_1)$. Note that the map $\frac{\mathbf{i}}{\lambda} ad(F)$ which we shall often use in what follows is always well defined because of the supercommutativity of the undeformed multiplication (8).

Consider now the curvature tensors R^M of ∇^M and R^E of ∇^E, i.e., for three vector fields X,Y,Z on M and a section e of E we have $R^M(X,Y)Z = \nabla^M_X \nabla^M_Y Z - \nabla^M_Y \nabla^M_X Z - \nabla^M_{[X,Y]} Z$ and $R^E(X,Y)e = \nabla^E_X \nabla^E_Y e - \nabla^E_Y \nabla^E_X e - \nabla^E_{[X,Y]} e$. Define elements $R^{(M)}$ and $R^{(E)}$ of the Fedosov algebra which are contained in $\Gamma(\vee^2 T^*M \otimes \Lambda^2 T^*M)$

and $\Gamma(\Lambda^2 E^* \otimes \Lambda^2 T^*M)$, respectively, as follows where V, W are vector fields on M and e_1, e_2 are sections of E:

$$R^{(M)}(V,W,X,Y) := \omega(V, R^M(X,Y)W), \tag{18}$$
$$R^{(E)}(e_1,e_2,X,Y) := -q(e_1, R^E(X,Y)e_2). \tag{19}$$

Note that this is well defined: Since ∇^M preserves ω and ∇^E preserves q it follows that $R^{(M)}$ is symmetric in V,W and $R^{(E)}$ is antisymmetric in e_1, e_2. In coordinates these two elements of the Fedosov algebra can be written in the form $R^{(M)} = (1/4)R^{(M)}_{klij} dx^k \vee dx^l \otimes 1 \otimes dx^i \wedge dx^j$ and $R^{(E)} = (1/4)R^{(E)}_{ABij} 1 \otimes e^A \wedge e^B \otimes dx^i \wedge dx^j$. Set

$$R := R^{(M)} + R^{(E)}. \tag{20}$$

Then the following Proposition is immediate:

Proposition 1.2 *With the above definitions and notations we have:*

1. $\nabla^2 = \frac{\mathbf{i}}{\lambda} ad(R)$.

2. $P_E(R) = R$, $P_\lambda(R) = R$ *and* $C(R) = R$.

3. $\delta R = 0$.

4. $\nabla R = 0$.

Proof. 1. Straightforward computation.
2. Obvious.
3. This is a consequence of the vanishing torsion of ∇^M (first Bianchi identity).
4. This is a reformulation of the second Bianchi identity for linear connections in arbitrary vector bundles. ■

We shall now make the ansatz for a Fedosov connection D, i.e., we are looking for an element $r \in \mathscr{W} \otimes \Lambda^1$ *of even E-parity*, i.e., $P_E(r) = r$, such that the map

$$D := -\delta + \nabla + \frac{\mathbf{i}}{\lambda} ad(r) \tag{21}$$

has square zero, i.e., $D^2 = 0$. The following properties of D for any r are crucial:

Lemma 1.1 *Let r be an arbitrary element of* $\mathscr{W} \otimes \Lambda^1$ *of even E-parity. Then*

1. $D^2 = \frac{\mathbf{i}}{\lambda} ad(-\delta r + \nabla r + R + \frac{\mathbf{i}}{\lambda} r \circ r)$.

2. $D(-\delta r + \nabla r + R + \frac{\mathbf{i}}{\lambda} r \circ r) = 0$.

Proof. This is straightforward using Proposition 1.2 and the fact that $r \circ r = \frac{1}{2}[r,r]$ for the above elements r of even E-parity and odd antisymmetric degree. ■

For an arbitrary element $w \in \mathscr{W} \otimes \Lambda$ we shall make the following decomposition according to the total degree Deg:

$$w = \sum_{k=0}^{\infty} w^{(k)} \quad \text{where } Deg(w^{(k)}) = kw^{(k)}. \tag{22}$$

Note that each $w^{(k)}$ is always a *finite* sum of sections in some $\Gamma(\vee^s T^*M \otimes \Lambda E^* \otimes \Lambda T^*M)$. The subspaces of all elements of $\mathscr{W} \otimes \Lambda$, $\mathscr{W}$, $\mathscr{W} \otimes \Lambda^a$, and $\mathscr{C}$ of total degree k will be denoted by $\mathscr{W}^{(k)} \otimes \Lambda$, $\mathscr{W}^{(k)}$, $\mathscr{W}^{(k)} \otimes \Lambda^a$, and $\mathscr{C}^{(k)}$, respectively.

As in Fedosov's paper [18] there is the following Theorem:

Theorem 1.1 *With the above definitions and notations: Let $r \in \mathscr{W} \otimes \Lambda^1$ be defined by the following recursion:*

$$\begin{aligned} r^{(3)} &:= \delta^{-1} R, \\ r^{(k+3)} &:= \delta^{-1} \left(\nabla r^{(k+2)} + \frac{\mathbf{i}}{\lambda} \sum_{l=1}^{k-1} r^{(l+2)} \circ r^{(k-l+2)} \right). \end{aligned}$$

Then r has the following properties: It is real ($C(r) = r$), depends only on λ^2 ($P_\lambda(r) = r$), has even E-parity, and is in the kernel of δ^{-1}. Moreover, the corresponding Fedosov derivation $D = -\delta + \nabla + (\mathbf{i}/\lambda) ad(r)$ has square zero.

Proof. The behavior of r under the parity transformations and complex conjugation immediately follows from the fact that they commute with δ^{-1} and from their (anti)homomorphism properties (Prop.1.1, 3., 5.; Prop.1.2, 2.).
Let $A := -\delta r + \nabla r + R + \frac{\mathbf{i}}{\lambda} r \circ r =: -\delta r + R + B$. Recall the equation $\delta\delta^{-1} + \delta^{-1}\delta = 1$ on the subspace of the Fedosov algebra where $deg_s + deg_a$ have nonzero eigenvalues. Clearly, $A^{(2)} = -\delta r^{(3)} + R = 0$ because $\delta R = 0$ (Prop.1.2,3.) hence $R = \delta\delta^{-1} R$. Suppose $A^{(l)} = 0$ for all $2 \le l \le k+1$. By Lemma 1.1, 2. we have $0 = (DA)^{(k+1)} = -\delta A^{(k+2)} = -\delta B^{(k+2)}$. Hence $B^{(k+2)} = \delta\delta^{-1} B^{(k+2)} = \delta r^{(k+3)}$ proving $A^{k+2} = 0$ which inductively implies $D^2 = 0$ since we had already shown that r is of even E-parity. ■

We shall now compute the kernel of the Fedosov derivation. More precisely, define

$$\mathscr{W}_D := Ker(D) \cap Ker(deg_a). \tag{23}$$

As in Fedosov's paper [18] we have the important characterization:

Theorem 1.2 *With the above definitions and notations: $\mathscr{W}_D$ is a subalgebra of the Fedosov algebra $(\mathscr{W} \otimes \Lambda, \circ)$. Moreover, the map σ (9) restricted to $\mathscr{W}_D$ is a $\mathbb{C}[[\lambda]]$-linear bijection onto $\mathscr{C}$.*

Proof. The kernel of a superderivation is always a subalgebra. Since D and σ are $\mathbb{C}[[\lambda]]$-linear the subalgebra $\mathscr{W}_D$ is a $\mathbb{C}[[\lambda]]$-submodule of $\mathscr{W}$.
Let $w \in \mathscr{W}$. Decompose $w = w_0 + w_+$ where $w_0 := \sigma(w)$ and $w_+ := (1-\sigma)(w)$. We shall prove by induction over the total degree k that $w \in \mathscr{W}$ is in $\mathscr{W}_D$ iff for all nonnegative integers k $w_0^{(k)}$ is arbitrary in $\mathscr{C}^{(k)}$ and $w_+^{(k)}$ is uniquely given by the equation

$$w_+^{(k)} = \delta^{-1}\left(\nabla w^{(k-1)} + \frac{\mathbf{i}}{\lambda}\sum_{l=1}^{k-2}\left[r^{(l+2)}, w^{(k-1-l)}\right]\right) =: \delta^{-1}((Aw)^{(k-1)}), \qquad (24)$$

where of course an empty sum is defined to be zero and $w_+^{(0)} = 0$. Note that $Dw = -\delta w + Aw$ and that the $\mathbb{C}$-linear map A does not lower the total degree of w.

Now the equation $(Dw)^{(k)} = 0$ is equivalent to the inhomogeneous equation $\delta(w^{(k+1)}) = (Aw)^{(k)}$. A necessary condition for this equation to be solvable for $w^{(k+1)}$ clearly is $\delta((Aw)^{(k)}) = 0$. But this is also sufficient since then $(Aw)^{(k)} = \delta\delta^{-1}(Aw)^{(k)}$ and we have the particular solution $w_+^{(k+1)} = \delta^{-1}(Aw)^{(k)}$ (since $\sigma\delta^{-1} = 0$) which satisfies (24). To this particular solution any solution to the homogeneous equation $\delta(w^{(k+1)}) = 0$ can be added which precisely is the space $\mathscr{C}^{(k+1)}$.

It remains to show that conversely every initial piece $w' := w_0^{(0)} + w_0^{(1)} + w_+^{(1)} + \cdots + w_0^{(k)} + w_+^{(k)}$ where $w_0^{(l)}$ was arbitrarily chosen in $\mathscr{C}^{(l)}$, $w_+^{(l)}$ is determined by (24) for all $0 \le l \le k$, and $(Dw')^{(l)} = 0$ for all $-1 \le l \le k-1$ can be continued to $w'' := w' + w_0^{(k+1)} + w_+^{(k+1)}$ with $w_0^{(k+1)}$ arbitrary in $\mathscr{C}^{(k+1)}$, $w_+^{(k+1)}$ determined by (24), and $(Dw'')^{(k)} = 0$. By induction, this will eventually lead to $w \in \mathscr{W}_D$ characterized by the above properties. Indeed, since $D^2 = 0$ we have $0 = (D^2w')^{(k-1)} = -\delta((Dw')^{(k)}) = -\delta((Aw')^{(k)})$. Define $w_+^{(k+1)}$ by $\delta^{-1}((Aw')^{(k)})$ and choose any $w_0^{(k+1)} \in \mathscr{C}^{(k+1)}$. It follows at once that $w_+^{(k+1)}$ satisfies (24) and that we get $(Dw'')^{(k)} = 0$ which proves the induction and the Theorem. ■

Let

$$\tau : \mathscr{C} \to \mathscr{W}_D \subset \mathscr{W} \qquad (25)$$

be the inverse of the restriction of σ to $\mathscr{W}_D$. For $\phi \in \Gamma(\Lambda E^*)$ we shall speak of $\tau(\phi)$ as the *Fedosov-Taylor series of* ϕ and refer to the components $\tau(\phi)^{(k)}$ as the Fedosov-Taylor coefficients. We collect some of the properties of τ in the following Proposition:

Proposition 1.3 *With the above definitions and notations:*

1. *τ commutes with P_E, P_λ, and C.*

2. *Let* $\phi = \sum_{d=0}^{n} \phi^{(d)} \in \Gamma(\Lambda E^*)$ *where* $n := \dim E$. *Then* $Deg(\phi^{(d)}) = d\phi^{(d)} = deg_E(\phi^{(d)})$. *Moreover*

$$\tau(\phi)^{(0)} = \phi^{(0)}, \tag{26}$$

$$\tau(\phi)^{(1)} = \delta^{-1}(\nabla\phi^{(0)}) + \phi^{(1)}, \tag{27}$$

$$\vdots \qquad \vdots$$

$$\tau(\phi)^{(n)} = \delta^{-1}\left(\nabla(\tau(\phi)^{(n-1)}) + \frac{\mathbf{i}}{\lambda}\sum_{l=1}^{n-2}\left[r^{(l+2)}, \tau(\phi)^{(n-1-l)}\right]\right) + \phi^{(n)}, \tag{28}$$

$$\tau(\phi)^{(n+1)} = \delta^{-1}\left(\nabla(\tau(\phi)^{(n)}) + \frac{\mathbf{i}}{\lambda}\sum_{l=1}^{n-1}\left[r^{(l+2)}, \tau(\phi)^{(n-l)}\right]\right), \tag{29}$$

$$\vdots \qquad \vdots$$

$$\tau(\phi)^{(k+1)} = \delta^{-1}\left(\nabla(\tau(\phi)^{(k)}) + \frac{\mathbf{i}}{\lambda}\sum_{l=1}^{k-1}\left[r^{(l+2)}, \tau(\phi)^{(k-l)}\right]\right), \tag{30}$$

where $k \geq n$. *The Fedosov-Taylor series* $\tau(\phi)$ *depends only on* λ^2.

3. *For any nonnegative integer k the map* $\phi \mapsto \tau(\phi)^{(k)}$ *is a polynomial in* λ *whose coefficients are differential operators from* $\Gamma(\Lambda E^*)$ *into some* $\Gamma(\vee^s T^*M \otimes \Lambda E^*)$ *of order k.*

Proof. Since r is invariant under the parity maps and complex conjugation, it follows that D commutes with these three maps, hence $\mathscr{W}_D$ is stable under these maps. Since σ obviously commute with them, so does the inverse of its restriction to $\mathscr{W}_D$, τ. The rest is a consequence of the preceding Theorem and a straightforward induction. ■

Define the following $\mathbb{C}[[\lambda]]$-bilinear multiplication on $\mathscr{C}$: for $\phi, \psi \in \mathscr{C}$

$$\phi * \psi := \sigma(\tau(\phi) \circ \tau(\psi)). \tag{31}$$

We shall call $*$ the *Fedosov star-product associated to* $(M, \omega, \nabla^M, E, q, \nabla^E)$. For $\phi, \psi \in \Gamma(\Lambda E^*)$ the star-product $\phi * \psi$ will be a formal power series in λ which we shall write in the following form:

$$\phi * \psi =: \sum_{t=0}^{\infty}\left(\frac{\mathbf{i}\lambda}{2}\right)^t M_t(\phi, \psi). \tag{32}$$

We list some important properties of the Fedosov star-product in the following

Theorem 1.3 *With the above definitions and notations:*

1. *The Fedosov star-product is associative and $\mathbb{Z}_2$-graded, i.e., P_E is an automorphism of $(\mathscr{C},*)$. The map P_λ and the complex conjugation C are graded antiautomorphisms of $(\mathscr{C},*)$.*

2. *The $\mathbb{C}$-bilinear maps M_t are all bidifferential, real, vanish on the constant functions in each argument for $t \geq 1$, and have the following symmetry property:*
$$M_t(\psi,\phi) = (-1)^t(-1)^{d_1 d_2} M_t(\phi,\psi). \tag{33}$$

3. *The term of order 0 is equal to the pointwise Grassmann multiplication. Hence $(\mathscr{C},*)$ is a formal associative deformation of the supercommutative algebra $(\mathscr{C}_0,\wedge)$.*

Proof. Basically, every stated property is easily derived from the definitions (31) and (32) and the corresponding behavior of the fiberwise multiplication $\circ$ under P_E, P_λ, and C. The reality of the M_t follows easily from the graded antihomomorphism property of C once Equation (33) is proved by means of the graded antihomomorphism property of the λ-parity. Since $\tau(1)$ is easily seen to be equal to 1 we have $1 * \psi = \psi = \psi * 1$, and the M_t must vanish on 1 for $t \geq 1$. Finally, each M_t obviously depends on only a finite number of Fedosov-Taylor coefficients whence it must be bidifferential. ■

1.2. Computation of the super-Poisson bracket

In this section we are going to compute an explicit expression for the term M_1 of the Fedosov star-product defined in the last section (compare (32) and Theorem 1.3). Only by means of the graded associativity of the deformed algebra $(\mathscr{C},*)$ we can derive the following Lemma:

Lemma 1.2 *Let ϕ,ψ,χ be sections in $\mathscr{C}_0$ of E-degree d_1,d_2,d_3, respectively. Then*

$$\begin{aligned} M_1(\psi,\phi) &= -(-1)^{d_1 d_2} M_1(\phi,\psi), && (34)\\ M_1(\phi,\psi\wedge\chi) &= M_1(\phi,\psi)\wedge\chi + (-1)^{d_1 d_2}\psi\wedge M_1(\phi,\chi), && (35)\\ 0 &= (-1)^{d_1 d_3} M_1(M_1(\phi,\psi),\chi) + \text{cycl.} && (36) \end{aligned}$$

Hence M_1 is a super-Poisson bracket *on $\mathscr{C}_0$.*

Proof. The first property is a particular case of (33). Consider now the graded commutator $[\phi,\psi] := \phi * \psi - (-1)^{d_1 d_2}\psi * \phi$ on $\mathscr{C}$. Because of the graded associativity of $*$ we have the superderivation property $[\phi,\psi * \chi] = [\phi,\psi] * \chi + (-1)^{d_1 d_2}\psi * [\phi,\chi]$. Writing this out with the M_t and taking the term of order λ we get the second property. For the third, take the term of order λ^2 in the super-Jacobi identity for the graded commutator. ■

Before we are going to compute M_1 directly it is useful to introduce the following notions:
For ϕ in $\mathscr{C}_0$ let ϕ_1 and ρ denote the component of symmetric degree one and λ-degree zero of the Fedosov-Taylor coefficient $\tau(\phi)$ and the section r (Theorem 1.1), respectively. Note that ϕ_1 is a smooth section in the bundle $T^*M\otimes\Lambda E^*$. Denote by $\Lambda_0 E^*$ the subbundle of the dual Grassmann bundle consisting of elements of even degree. Then ρ is a smooth section in $T^*M\otimes\Lambda_0 E^*\otimes T^*M$. Consider now the bundle $TM\otimes\Lambda_0 E^*\otimes T^*M$. There is an obvious fiberwise associative multiplication $\bullet$ in that bundle which comes from the identification of $TM\otimes T^*M$ with the bundle of linear endomorphism of TM: Let X,Y be vector fields on M, $\phi,\psi\in\Lambda_0 E^*$, and α,β one-forms on M. Then

$$(X\otimes\phi\otimes\alpha)\bullet(Y\otimes\psi\otimes\beta) := (\alpha(Y))X\otimes(\phi\wedge\psi)\otimes\beta. \tag{37}$$

Let $\hat{R}^E$ be the section in $\Gamma(TM\otimes\Lambda^2 E^*\otimes T^*M)$ whose components in a bundle chart read

$$\hat{R}^E := \frac{1}{4}\Lambda^{ik}R^{(E)}_{ABkj}\partial_i\otimes e^A\wedge e^B\otimes dx^j =: \partial_i\otimes(\hat{R}^E)^i_j\otimes dx^j, \tag{38}$$

and let $\hat{\rho}\in\Gamma(TM\otimes\Lambda_0 E^*\otimes T^*M)$ be defined by

$$\hat{\rho} := \partial_i\otimes\Lambda^{ik}i_s(\partial_k)\rho =: \partial_i\otimes\hat{\rho}^i_j\otimes dx^j. \tag{39}$$

Note that we can form arbitrary power series in $\hat{R}^E$ by using the multiplication $\bullet$ since $\hat{R}^E$ is nilpotent.

We have the following Lemma:

Lemma 1.3 *With the above notations and definitions:*

$$M_1(\phi,\psi) = \Lambda^{ij}(i_s(\partial_i)\phi_1)\wedge(i_s(\partial_j)\psi_1) + q^{AB}(j(e_A)(\phi))\wedge(i(e_B)(\psi)), \tag{40}$$

$$\phi_1 = dx^j((1-\hat{\rho})^{-1})^i_j\nabla^E_{\partial_i}\phi, \tag{41}$$

$$\hat{\rho} = 1-(1-2\hat{R}^E)^{1/2}. \tag{42}$$

where $(1-\hat{\rho})^{-1}$ and $(1-2\hat{R}^E)^{1/2}$ denote the corresponding power series with respect to the $\bullet$ multiplication.

Proof. The first equation is a straightforward computation.
For the second, use the Fedosov recursion for $\tau(\phi)$, (Proposition 1.3), note that $\phi_1^{(k)}$ is zero for $k\geq n+2$ and that only the component ρ of r matters since both $\tau(\phi)$ and r depend only on λ^2, sum over the total degree which yields the equation

$$\phi_1 = \delta^{-1}\nabla^E\phi + dx^j(\hat{\rho})^i_j(i_s(\partial_i)\phi_1)$$

which proves the second equation.
For the third, use the Fedosov recursion for r, (Theorem 1.1), take the component

of symmetric degree 1 and λ-degree zero, sum over the total degree, and arrive at the quadratic equation

$$\hat{\rho} - \hat{R}^E = \frac{1}{2}\hat{\rho} \bullet \hat{\rho}.$$

Since r and hence ρ does not contain components of symmetric degree zero, there is only one solution to this equation, namely the above third equation. ■

This Lemma immediately implies the desired formula for the super-Poisson bracket:

Theorem 1.4 *The super-Poisson bracket M_1 obtained by the Fedosov star-product takes the following form:*

$$\begin{aligned} M_1(\phi,\psi) &= \Lambda^{ij}((1-2\hat{R}^E)^{-1/2})_i^k \wedge ((1-2\hat{R}^E)^{-1/2})_j^l \wedge \nabla^E_{\partial_k}\phi \wedge \nabla^E_{\partial_l}\psi \\ &\quad + q^{AB}(j(e_A)(\phi)) \wedge (i(e_B)(\psi)) \end{aligned}$$

Proof. Clear from the Lemma! ■

Corollary 1.1 *The above super-Poisson bracket coincides with the Rothstein super-Poisson bracket $\{\ ,\ \}_R$, see (4) and [31].*

Proof. Since by definition $\Lambda^{ij}(\hat{R}^E)_i^k = \Lambda^{ki}(\hat{R}^E)_i^j$ the same relation holds for any power series (with respect to $\bullet$) $(f(\hat{R}^E))_i^k)$ whence the result. ■

Remarks:

1. In case (M,ω) is Kähler there exist star-products of Wick type on M (see [26, 12]): They are characterized by the property that for any two complex-valued smooth functions f,g on M the star-product $f *' g$ is made out of bidifferential operators which differentiate f in holomorphic directions only and g in antiholomorphic directions only. It seems to me very likely that super analogues of these star-products can readily be formulated for any complex holomorphic Hermitean vector bundle over M as it has been done in geometric quantization, see [22].

2. If the dual Grassmann bundle ΛE^* is replaced by the symmetric power $\vee E^*$ and the fiber metric q by some antisymmetric bilinear form on the fibers covariantly constant by some connection in E the whole construction can presumably carried through as well (see also Neumaier's related construction for differential operators in [10], Section 3). As we shall explain further down this can be interpreted as a particular case of a symplectic fibration.

3. It may also be interesting to compute this construction in the particular case where M is the cotangent bundle of an arbitrary semi-Riemannian manifold Q and E is the tangent bundle of Q pulled back to T^*Q by the bundle projection. Star-products on T^*Q are strongly related to (pseudo) differential operator calculus on Q, see [30, 9, 10, 11]. In that situation one could study asymptotic representation theory incorporating Dirac operators. T. Voronov has studied the algebra $\mathscr{C}$ using symbol calculus and its representations on the space of differential forms on Q (which is an intermediate step towards spinors), see [32].

Note added: The above Fedosov construction is not the full Fedosov construction one would expect in supermanifold theory as I have been made aware by the referee: There the super-Fedosov algebra should rather consist of a sort of completed tensor product of supersymmetric tensor fields (generalizing $\Gamma(\vee T^*M)$) and superdifferential forms (generalizing $\Gamma(\Lambda T^*M)$) which would include our $\mathscr{W} \otimes \Lambda$, but also -roughly speaking- additional symmetric tensors and differential forms 'in the purely fermionic directions'. Moreover the fiberwise multiplication would involve the full Rothstein superbracket. It is very probable that such a super Fedosov construction will go through without any big conceptual problem and, since to my best knowledge this has not yet been done in the literature, will be an interesting problem to attack.

I believe that the rôle of the above Fedosov construction can perhaps best be compared with the constructions which have been done in the meantime by B. Fedosov and O. Kravchenko for ordinary (i.e., non super) symplectic fibrations (see [20, 27]): They are using an *intermediate Fedosov construction* which starts with a 'purely vertical' star-product on the symplectic fibers satisfying some compatibility conditions which is supposed to already exist; in a second step the Fedosov construction proper is then only done for the base, but 'tensored' with the 'vertical algebras': The result is a star-product on the total space. The curvature of the fiber bundle underlying the symplectic fibration enters in the symplectic form of the total space when it is expressed in terms of the symplectic form on the base and on the fibers.

It seems to me that an even symplectic split supermanifold can be regarded as a 'supersymplectic fibration' with symplectic base and 'purely fermionic' fibers, and the simple nature of the Rothstein super symplectic form exactly corresponds to that picture. Moreover, in the Fedosov construction presented in this contribution the 'fermionic vertical direction', viz: The algebra $\Gamma(\Lambda E^*)$ already carries a simple explicit vertical star-product, namely a sort of formal Clifford multiplication (see also the next Section), and the construction is intermediate insofar that symmetric and antisymmetric tensor fields only come from the base. It is an interesting question under which circumstances the 'full' Fedosov construction for even symplectic supermanifolds (which will no doubt be much more complicated) reduces to the above 'intermediate construction'.

2. Flat vector bundles

An important particular case is given by a vector bundle E with fiber metric q on which there exists a *flat* covariant derivative ∇^E, for instance in the case of a trivial bundle $M \times \mathbb{R}^n$ with q being a nondegenerate bilinear form on $\mathbb{R}^n$ not depending on M.

Note first the standard fact for flat vector bundles that there is an open cover $(U_\alpha)_{\alpha \in I}$ of M together with a basis of local sections e_A, $1 \leq A \leq n$ defined on each U_α which are covariantly constant and which are related by constant transition matrices on the overlaps of any two of the U_α.

We have the following Lemma:

Lemma 2.1 *With the above additional assumptions the following holds:*

1. *The map r as defined in Theorem 1.1 does not depend on ΛE^*, i.e., is contained in $\times_{s=0}^{\infty}\Gamma(\mathbb{C}(\vee^s T^*M \otimes \Lambda^1 T^*M))[[\lambda]]$.*
2. *The Fedosov-Taylor series of a function $f \in C^\infty(M)$ does not depend on ΛE^*, i.e., is contained in $\times_{s=0}^{\infty}\Gamma(\mathbb{C}(\vee^s T^*M))[[\lambda]]$.*
3. *The Fedosov-Taylor series of a local covariantly constant section ϕ in $\Gamma(\Lambda E^*)$ is equal to ϕ.*

Proof. 1. Since $r^{(3)} = R^{(M)}$, and since δ^{-1} and ∇ preserve $\times_{s=0}^{\infty}\Gamma(\mathbb{C}(\vee^s T^*M \otimes \Lambda T^*M))[[\lambda]]$ which is a fiberwise subalgebra of $\mathscr{W} \otimes \Lambda$ the statement follows by induction using Theorem 1.1.
2. The proof is completely analogous to part 1. upon using the formulas in Proposition 1.3.
3. Again by induction using Proposition 1.3 where the fact is used that r supercommutes with $\Gamma(\Lambda E^*)$ according to 1. ∎

This immediately implies the following formula for the star-product:

Theorem 2.1 *We make the above assumptions. Let ϕ, ψ two sections in $\mathscr{C}$ and express them locally as $\phi = \sum_{d=0}^{n} \frac{1}{d!}\phi_{A_1\cdots A_d} e^{A_1} \wedge \cdots \wedge e^{A_d}$ and likewise for ψ where e^A, $1 \leq A \leq n$, is a local basis of covariantly constant sections of E^* and $\phi_{A_1\cdots A_d}$ are local C^∞-functions. Then*

$$\phi * \psi = \sum_{d,d'=0}^{n} \frac{1}{d!}\frac{1}{d'!}(\phi_{A_1\cdots A_d} *_{\mathrm{F}} \psi_{B_1\cdots B_{d'}})(e^{A_1} \wedge \cdots \wedge e^{A_d} *_{\mathrm{Cl}} e^{B_1} \wedge \cdots \wedge e^{B_{d'}}) \quad (43)$$

where $*_{\mathrm{F}}$ denotes the usual Fedosov star-product on M defined by the map r (restricted to $\times_{s=0}^{\infty}\Gamma(\mathbb{C}(\vee^s T^*M \otimes \Lambda^1 T^*M))[[\lambda]]$) and $*_{\mathrm{Cl}}$ denotes the formal tensorial Clifford multiplication in $\mathscr{C}$ defined by

$$\phi *_{\mathrm{Cl}} \psi := \sum_{l=0}^{n} \frac{(\mathbf{i}\lambda/2)^l}{l!} q^{A_1B_1}\cdots q^{A_lB_l}\big(j(e_{A_1})\cdots j(e_{A_l})\phi\big) \wedge \big(i(e_{B_1})\cdots i(e_{B_l})\psi\big).$$

Formula (43) does not depend on the chosen covariantly constant local trivializaton.

Proof. The subalgebra $\times_{s=0}^{\infty}\Gamma(\mathbb{C}(\vee^s T^*M \otimes \Lambda T^*M))[[\lambda]]$ of $\mathscr{W}$ is clearly preserved by the Fedosov derivative D whence it follows at once that $f * g = f *_{\mathrm{F}} g$ for all $f, g \in C^\infty(M)[[\lambda]]$. Moreover, for a covariantly constant section χ of $\mathscr{C}$ we clearly have $f\chi = \sigma(\tau(f)\tau(\chi)) = \sigma(\tau(f) \circ \tau(\chi))$ using the above Lemma and the properties of $\circ$ whence $f\chi = f * \chi = \chi * f$. Finally, note that $\chi * \chi' = \chi \circ \chi' = \chi *_{\mathrm{Cl}} \chi'$ for two covariantly constant sections, where the result is again covariantly constant, and therefore

$$(f\chi) * (g\chi') = f * \chi * g * \chi' = f * g * \chi * \chi' = (f *_{\mathrm{F}} g)(\chi *_{\mathrm{Cl}} \chi'),$$

which proves the above formula. Since the transition functions are constant it follows that (43) does not depend on the chosen local basis of covariantly constant sections. ∎

Conversely, it is easy to see that the above formula (43) always defines an associative $\mathbb{Z}_2$-graded deformation of $\mathscr{C}_0$ where $*_{\mathrm{F}}$ can be replaced by any given star-product on M: It is locally given by the tensor product over $\mathbb{C}[[\lambda]]$ of the associative algebra $(C^\infty(M)[[\lambda]]$ with the formal Clifford algebra $(\Lambda\mathbb{R}^n[[\lambda]], *_{\mathrm{Cl}})$.

For a trivial flat bundle without holonomy the above formula had been given by R. Eckel in his thesis [16], p. 66.

3. A quantum BRST complex for quantum covariant star-products

The results of this Section have been obtained in collaboration with Hans-Christian Herbig and Stefan Waldmann in [8].

Let (M, ω) a symplectic manifold. Suppose that a Lie group G (with Lie algebra $\mathfrak{g}$) symplectically and properly acts on M (e.g., when G is compact) allowing for a classical momentum map $J : M \to \mathfrak{g}^*$: For each $\xi \in \mathfrak{g}$ let ξ_M be the fundamental field $m \mapsto d/dt(exp(t\xi)m)|_{t=0}$, then $\omega^\flat(\xi_M) = d\langle J, \xi\rangle$ and $J(gm) = \mathrm{Ad}^*(g)J(m)$ for all $g \in G, m \in M$. This implies the Lie homomorphism property

$$\{\langle J, \xi\rangle, \langle J, \eta\rangle\} = \langle J, [\xi, \eta]\rangle, \tag{44}$$

for all $\xi, \eta \in \mathfrak{g}$. Recall the Marsden-Weinstein phase space reduction scheme, [28]: Suppose for the rest of this Section that 0 is a regular value of J and that the constraint surface $C := J^{-1}(0)$ is nonempty. Then G acts locally freely on C, and supposing that G acts freely and properly on C the quotient manifold $M_{\mathrm{red}} := C/G$ becomes a symplectic manifold, its symplectic form ω_{red} being determined by the condition that its pull-back to C by the canonical projection equals the restriction of ω to TC. Note that each G-invariant smooth function on C naturally projects to M_{red}.

Now let $*$ be a star-product on M. According to [2] and [33] a formal power series $\boldsymbol{J} = \sum_{r=0}^{\infty} \lambda^r \boldsymbol{J}_r \in C^\infty(M, \mathfrak{g}^*)[[\lambda]]$ will be called a quantum momentum map and the star-product $*$ (quantum) covariant iff $\boldsymbol{J}_0 = J$ and analogously to (44):

$$\langle \boldsymbol{J}, \xi \rangle * \langle \boldsymbol{J}, \eta \rangle - \langle \boldsymbol{J}, \eta \rangle * \langle \boldsymbol{J}, \xi \rangle = \mathbf{i}\lambda \langle \boldsymbol{J}, [\xi, \eta] \rangle, \tag{45}$$

for all $\xi, \eta \in \mathfrak{g}$. We call $(M, *, G, \boldsymbol{J}, C)$ satisfying the previous conditions a *Hamiltonian quantum G-space with regular constraint surface*. According to a Theorem by Fedosov [19, Sect. 5.8] an even stronger condition can be achieved for all such group actions preserving a connection, e.g., proper actions (since they always preserve a Riemannian metric), namely strong invariance:

$$\langle J, \xi \rangle * f - f * \langle J, \xi \rangle = \mathbf{i}\lambda \{ \langle J, \xi \rangle, f \}, \tag{46}$$

which obviously implies (45) setting $\boldsymbol{J} = J$. A simple example is provided by the standard Moyal-Weyl star-product on $\mathbb{R}^{2n}$ together with the Lie algebra of all infinitesimal linear symplectic transformations represented by the space of all quadratic homogeneous polynomials. The problem whether a general classical momentum map can be deformed into a quantum momentum map for a suitable star-product is still an open problem as far as I know.

We are now constructing a BRST complex related to that problem (see for a general introduction the book [24] and our article [8] for more references): Consider the trivial bundle $E := (\mathfrak{g} \oplus \mathfrak{g}^*) \times M$ together with the fiber metric q defined by the natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. Then the superobservable algebra $\mathscr{C}$ (called $\mathscr{A}$) in [8]) of the first Section equals

$$\mathscr{C} = \Lambda \mathfrak{g}^* \otimes \Lambda \mathfrak{g} \otimes C^\infty(M)[[\lambda]]. \tag{47}$$

As a $\mathbb{C}[[\lambda]]$-module this space carries natural $\mathbb{Z}$-gradings, namely the *ghost degree* (form degree in $\Lambda \mathfrak{g}^*$), the *antighost degree* (form degree in $\Lambda \mathfrak{g}$), and the *ghost number* Gh which is defined as the difference of the ghost degree and the antighost degree and which we shall consider as a $\mathbb{C}[[\lambda]]$-linear map $\mathscr{C} \to \mathscr{C}$ with the ghost number integers as eigenvalues. We shall write $\mathscr{C}^{i,j}$ for the submodule of all those elements having ghost degree i and antighost degree j and $\mathscr{C}^{(i)}$ for the submodule of all those elements having ghost number i. We equip $\mathscr{C}$ with a star-product as in Section 2, (43) where the initial star-product on M does not have to be of Fedosov type. Consider now the following three elements of $\mathscr{C}$: $\boldsymbol{J} \in \mathscr{C}^{1,0}$, $\Omega := -1/2[\ ,\ ] \in \mathscr{C}^{2,1}$, and $\gamma :=$ one half of the identity homomorphism of $\mathfrak{g}$, contained in $\mathscr{C}^{1,1}$. Let $\Theta := \boldsymbol{J} + \Omega$, the so-called BRST-charge which is contained in $\mathscr{C}^{(1)}$. Define the *BRST operator Q* by

$$Q(\phi) := \frac{1}{\mathbf{i}\lambda} \big(\Theta * \phi - (-1)^{a+b} \phi * \Theta \big), \quad \forall a, b \in \mathbb{Z}, \, \forall \phi \in \mathscr{C}^{a,b}. \tag{48}$$

Then we have the following Theorem:

Theorem 3.1 *Let $(M,*,G,\boldsymbol{J},C)$ be a Hamiltonian quantum G-space with regular constraint surface. Then*

1. *The Ghost number operator* Gh *is equal to* $\phi \mapsto \frac{1}{\mathrm{i}\lambda}(\gamma * \phi - \phi * \gamma)$ *and therefore is a derivation of* $(\mathscr{C},*)$ *which thus becomes a* $\mathbb{Z}$*-graded associative algebra.*
2. $\Theta * \Theta = 0$.
3. *The BRST operator has square zero,* $Q^2 = 0$, *and is a superderivation of ghost number one of* $(\mathscr{C},*)$.

The proof of this statement is a rather straightforward consequence of Equation (45). For more details see [8].

Define the *quantum BRST cohomology* by $\mathrm{Ker}Q/\mathrm{Im}Q =: \boldsymbol{H}_{\mathrm{BRST}}(\mathscr{C}[[\lambda]])$. Then we have the following

Theorem 3.2 *Let $(M,*,G,\boldsymbol{J},C)$ be a Hamiltonian quantum G-space with regular constraint surface. Then*

1. $\boldsymbol{H}_{\mathrm{BRST}}(\mathscr{C}[[\lambda]])$ *becomes a* $\mathbb{Z}$*-graded associative algebra in a canonical way.*
2. *There is a representation* $\boldsymbol{\varrho}_C$ *of the Lie algebra* $\mathfrak{g}$ *on the* $\mathbb{C}[[\lambda]]$*-module* $C^\infty(C)[[\lambda]]$ *deforming the representation* ϱ_C *induced by the restriction of the fundamental fields to C such that the quantum BRST cohomology is isomorphic to the Chevalley-Eilenberg cohomology of* $\mathfrak{g}$ *with values in* $C^\infty(C)[[\lambda]]$ *with respect to* $\boldsymbol{\varrho}_C$.
3. *In particular, the component of ghost number zero of the quantum BRST cohomology is isomorphic to the submodule of all those elements in* $C^\infty(C)[[\lambda]]$ *which are invariant under* $\boldsymbol{\varrho}_C$.

See again [8] for a detailed proof.

In case the Hamiltonian action of the connected Lie group G on M is proper and the reduced phase space exists, we can choose a strongly invariant star-product on M (see (46)). Under these circumstances we have the stronger Theorem:

Theorem 3.3 *With the assumption of the previous Theorem and the above additional assumptions we have:*

1. *The quantum BRST-cohomology is isomorphic to the Chevalley-Eilenberg cohomology of* $\mathfrak{g}$ *with values in* $C^\infty(C)[[\lambda]]$ *with respect to the undeformed representation* ϱ_C.
2. *In particular, the component of ghost number zero of the quantum BRST cohomology is isomorphic to the submodule of all those elements in* $C^\infty(C)[[\lambda]]$ *which are invariant under* ϱ_C. *This space being isomorphic to* $C^\infty(M_{\mathrm{red}})[[\lambda]]$ *the algebra structure on the cohomology induces a star-product on the reduced space* M_{red}.

For a proof see [8].
Remarks:

1. The proofs of the last two theorems are rather technical. They heavily rely on one side on purely geometric considerations, namely the existence of tubular neighborhoods (which can be chosen G-invariant for proper G-actions) and the triviality of the normal bundle of C in M (since 0 is a regular value of J), which leads to the construction of an acyclic Koszul complex (first on the submodule of $\mathscr{C}$ of ghost degree zero which is in a standard way extended to all of $\mathscr{C}$), and a rather explicit chain homotopy for that complex analogous to the one used in the proof of Poincaré's Lemma. Secondly, we have used a purely tensorial, explicit equivalence transformation which modifies the Clifford part of the multiplication in $\mathscr{C}$ in such a way that Q splits into a boundary operator lowering the antighost degree by 1 and leaving invariant the ghost degree (which turns out to be a deformation of the aforementioned Koszul boundary operator) and a coboundary operator raising the ghost degree by one and leaving invariant the antighost degree (which turns out to be equal to a certain Chevalley-Eilenberg operator of $\mathfrak{g}$). Hence $\mathscr{C}$ becomes a double complex where one differential is acyclic. This fact has been known in the classical situation, but miraculously remains true in this deformed situation. Thirdly, to relate the total cohomology to the data on the constraint surface C we use an augmentation of this complex consisting in a *deformation of the restriction map* by a formal series of differential operators which can be constructed out of Q and the classical chain homotopies. Finally, in the case of a proper group action the resulting star-product on the reduced space can be related to the one on M essentially by means of the deformed restriction map.

2. For quantum covariant, but *not* strongly invariant star-products it can happen that the above mentioned ghost number zero part of the cohomology, the space of 'quantum G-invariant functions on C', can be too small in the sense that it is no longer a deformation of the whole space of classical G-invariant functions, but of a subspace of the latter, which is quite an anomaly. In a simple example (see [8], Section 7) we have seen that the reduced algebra can ultimately become commutative which does no longer seem to resemble a reasonable reduction of quantization, but which — with a little bad luck — in principle is possible as the example shows.

4. Classical reducible BRST without ghosts of ghosts

The results of this Section have been obtained in collaboration with Hans-Christian Herbig in [7].

Let C be an arbitrary closed coisotropic submanifold of a symplectic manifold (M,ω) of codimension n, i.e., the ω-orthogonal space to each tangent space of C is contained in that tangent space. Physicists would speak of C as a 'first class constraint surface'. Let TC^ω be the ω-orthogonal bundle to TC. This is known to be an integrable subbundle of TC and gives rise to a local foliation thanks to Frobenius' Theorem. If this foliation allows for a smooth quotient manifold M_{red} it becomes a symplectic manifold in a canonical way, see, e.g., [1], p. 417–418. Fix a subbundle N of $TM|_C$ such that $TM|_C = N \oplus TC$ (e.g., as the normal bundle to TC with respect to some Riemannian metric). The symplectic form provides an identification of N with the dual of TC^ω via $v \mapsto (w \mapsto \omega(v,w))$ where $c \in C$, $v \in N_c$ (the fiber of N over c) and $w \in T_cC^\omega$ and an identification of TC^ω with the conormal bundle of TC, i.e., the subbundle TC^{ann} of $T^*M|_C$ of all those cotangent vectors annihilating TC via $v \mapsto (w \mapsto \omega(v,w))$ where $c \in C$, $v \in T_cC^\omega$ and $w \in T_cM$, whence

$$N^* \cong TC^\omega \cong TC^{\mathrm{ann}}. \tag{49}$$

The nontriviality of the bundle N (and hence of the two others in the above equation) is related to the physicists' 'reducible case': Here the submanifold C is given as the zero locus of a finite set of in general not functionally independent smooth real valued functions.

Next, choose a tubular neighborhood around C, i.e., an open neighborhood U of the zero-section of N together with a diffeomorphism Φ of U onto an open neighborhood V of C in M such that $\Phi(c) = c$ for all $c \in C$ (where we identify C with the zero-section in N). Hence U becomes a symplectic manifold with the pulled-back form $\Phi^*\omega$. Denoting the bundle projection $N \to C$ by p we consider the pulled-back bundle p^*N over U. We shall denote the dual bundle of p^*N by F, whence p^*N can be identified with F^*. We have made this choice of notation to have an analogy $M \times \mathfrak{g} \cong F$ and $M \times \mathfrak{g}^* \cong F^*$ with the previous section.

The main idea which will make the construction work is the fact that the bundle $F^*(= p^*N)$ admits the tautological section J which maps each point u of U to the same point in the fiber over $p(u)$. J can be seen as a generalization of the momentum map of the previous section. I had been inspired by a similar construction in Connes's book [13], p. 210, used for the computation of the Hochschild cohomology of the algebra of all complex-valued C^∞-functions on a given manifold M.

Choosing an arbitrary covariant derivative ∇^F in F (inducing a covariant derivative ∇^{F^*} in F^* in the standard way) we set

$$E := F \oplus F^*; \quad \nabla^E := \nabla^F + \nabla^{F^*} \tag{50}$$

and choose the natural pairing between F and F^* as fiber metric q. It is clear that the above ∇^E preserves q.

Consider now $\mathscr{C}_0 := \Gamma(\Lambda F^* \otimes \Lambda F)$ together with the Rothstein super-Poisson bracket $\{\ ,\ \}_R$ constructed out of the above data. Define the ghost degree, antighost degree, and ghost number maps in the same way as in the previous section. Then we have the following

Theorem 4.1 *We use the above-made assumptions. Then*

1. *The ghost number map* Gh *is a derivation of the super-Poisson algebra* $\mathscr{C}_0$ *which thus becomes* $\mathbb{Z}$*-graded.*

2. *There is an element* $\Theta := \sum_{i=0}^{n} \Theta_i \in \mathscr{C}_0$*, the so-called classical BRST charge, such that* $\mathsf{Gh}(\Theta) = 1$*,* $\Theta_0 = J$*, the antighost degree of* Θ_i *is i, and, most importantly,* $\{\Theta, \Theta\}_R = 0$*.*

3. *The classical BRST operator* $Q := \{\Theta, \ \}_R$ *has square zero, increases the ghost-number by one, and its classical BRST-cohomology* $\mathrm{Ker}Q/\mathrm{Im}Q$ *carries a canonical* $\mathbb{Z}$*-graded super-Poisson algebra structure induced by the one on* $\mathscr{C}_0$*.*

In order to compute the above cohomology we consider the space of *vertical differential forms on C*, i.e., the space of sections $\Omega_v(C) := \Gamma(\Lambda(TC^{\omega})^*)$ together with the vertical exterior derivative d_v which is defined by the same formula as the standard exterior derivative but restricted to vertical vector fields, i.e., sections of the integrable subbundle TC^{ω}. Then we have the following result (which should be known by other methods):

Theorem 4.2 *We use the above-made assumptions and notations. Then the classical BRST-cohomology is isomorphic to the vertical de Rahm cohomology, i.e., the cohomology of the complex* $(\Omega_v(C), d_v)$*. This latter space thus carries the structure of a* $\mathbb{Z}$*-graded super-Poisson bracket. Moreover, the sector of the classical BRST-cohomology having vanishing ghost number exactly corresponds to the space of all complex-valued* C^∞*-functions on C which are constant on the connected leaves of the foliation defined by* TC^{ω}*. In case the reduced space* M_{red} *exists this last space is equal to the space of all complex-valued* C^∞*-functions on* M_{red}*.*

For details of the proof, see [7]. The main tool is the fact that J defines a Koszul boundary operator on the space $\Gamma(\Lambda F)$ in the same way as has been remarked in the previous Section, that the resulting complex is acyclic allowing for an augmentation map consisting of the restriction to C, and that the component of $\{J, J\}_R$ having vanishing antighost degree vanishes when restricted to C thanks to the fact that C is coisotropic and to the chosen connection ∇^E. In the irreducible case where C is given as the zero locus of $n := \mathit{codim} C$ functionally independent functions the method of deforming J is well known, see, e.g., [24]

The advantage of the above construction is that it is contained in a simple, geometrically defined BRST-complex $\mathscr{C}_0$ with only a finite number of nonzero $\mathscr{C}^{i,j}$ (although it may become difficult to explicitly compute the tubular neighborhoods) in contrast to the more elaborate multi-step ghosts-of-ghosts methods based on spectral sequence techniques, [21]. It is tempting to try a quantization of this complex by means of the Fedosov-type star-product constructed in the first section, but this would require a more sophisticated analysis of the (affine) geometry of the vicinity of C to solve the obvious problem whether the component of antighost degree zero of $J * J$ vanishes when restricted to C.

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