Projectable star-products

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Abstract. We give a contravariant version of Fedosov's construction of star-products on symplectic manifolds. This result is used to provide an alternative construction of star-products on Abelian Poisson Lie groups.

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To the memory of Moshé Flato

1. Introduction

The theory of formal deformation quantization of Poisson manifolds was introduced by Bayen et al. [1]. The main notion of this theory is the concept of a *star-product* on a Poisson manifold N, that is, a particular associative algebraic structure $*$ on the space $C^\infty(N)[[h]]$ of formal power series in h whose coefficients are smooth functions on N. The general question of the existence of such a product for *symplectic manifolds* has been completely solved by several authors, using various techniques [3, 5, 10]. Recently, M. Kontsevich has proved the existence of star-products on arbitrary finite-dimensional Poisson manifolds [8].

In this note, we consider whether a star-product can be constructed on an arbitrary Poisson manifold N by "*projecting*" a star-product from a symplectic manifold M. More precisely, let $*_M$ be a star-product on M, and let $\phi : M \to N$ be a surjective smooth submersion. A star-product $*_M$ on M is said to be projectable by ϕ if $\phi^* C^\infty(N)[[h]]$ is a subalgebra of $(C^\infty(M)[[h]], *_M)$; such star-product induces a star-product in N.

In Fedosov's geometrical construction of star-products on a symplectic manifold M one builds a *formal algebra bundle*, the so-called *Weyl bundle*, and a flat connection on it. One then shows that the algebra of flat sections of this bundle is in bijection with $C^\infty(M)[[h]]$. Therefore, one may attempt to choose the connection in such a way that a natural subalgebra of the algebra of flat sections is identified with $\phi^* C^\infty(N)[[h]]$. Here we shall show that this can be done in the particular case of *Abelian Poisson Lie groups*. In order to do this, we use a "contravariant" version of Fedosov's construction which seems to have some advantages if one wishes to project star-products.

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2. Preliminaries

Let (M,π) be a *Poisson manifold*, let $\Omega_p(M)$ and $\Omega^p(M)$ be the spaces of p-vectors and p-forms on M. Let $\pi^\# : T^*M \to TM$ be the bundle map defined by $\langle \pi^\#(\alpha),\beta\rangle = \pi(\alpha,\beta)$, $\alpha,\beta \in \Omega^1(M)$. For simplicity we write $\pi^\#(\alpha) = \alpha^\#$. A Poisson manifold is called symplectic if the map $\pi^\#$ is an isomorphism, in such a case, we shall denote the inverse map by $\flat$. The existence of a Lie bracket of 1-forms on (M,π) allows to define a coboundary operator $\tilde{\partial}$ (i.e., $\tilde{\partial}^2 = 0$) on $\Omega_*(M) := \oplus_{p=0}\Omega_p(M)$ by setting

$$\begin{aligned}\tilde{\partial}P(\alpha_o,\dots,\alpha_p) = &\sum_{i=0}^{p}(-1)^i\alpha_i^\#(P(\alpha_0,\dots,\hat{\alpha}_i,\dots,\alpha_p))\\ &+\sum_{i<j}(-1)^{i+j}P(\{\alpha_i,\alpha_j\},\alpha_0,\dots,\hat{\alpha}_i,\dots,\hat{\alpha}_j,\dots,\alpha_p).\end{aligned}$$

Definition 1. **([11])** *Let $p : E \to M$ be a vector bundle over (M,π). A* contravariant connection *is a map that assigns to each $\alpha \in \Omega^1(M)$ and $s \in \Gamma(E)$ another smooth section $\nabla_\alpha s \in \Gamma(E)$ such that, for $f \in C^\infty(M)$,*

$$\nabla_\alpha f = \alpha^\# f, \quad \nabla_{f\alpha}s = f\nabla_\alpha s, \quad \textit{and} \quad \nabla_\alpha(fs) = f\nabla_\alpha s + (\alpha^\# f)s.$$

We use the contravariant connection to define a *contravariant differential* of E-valued vectors, i.e., a map $\tilde{\partial}^\nabla : \Gamma(\Omega_p(M)\otimes E) \to \Gamma(\Omega_{p+1}(M)\otimes E)$, $p \geq 0$, by means of the formula

$$\begin{aligned}\tilde{\partial}^\nabla s(\alpha_o,\dots,\alpha_p) = &\sum_{i=0}^{p}(-1)^i\nabla_{\alpha_i}(s(\alpha_0,\dots,\hat{\alpha}_i,\dots,\alpha_p))\\ &+\sum_{i<j}^{p}(-1)^{i+j}s(\{\alpha_i,\alpha_j\},\alpha_0,\dots,\hat{\alpha}_i,\dots,\hat{\alpha}_j,\dots,\alpha_p).\end{aligned}$$

3. Contravariant construction of star-products

Definition 2. **([1])** *A $*$-product on (M,π) is a bilinear map $C^\infty(M)\times C^\infty(M) \to C^\infty(M)[[h]]$ defined by*

$$(f,g) \mapsto f*g = \sum_{n=0}^{\infty} h^n C_n(f,g),$$

where $C_0(f,g) = fg$, $C_1(f,g) = \pi(df,dg)$, and for $n \geq 2$, C_n is a bilinear map with values in $C^\infty(M)$, which vanishes on the constants and satisfy the following relation

$$\sum_{r+s=k} C_r(C_s(f,g),h) = \sum_{r+s=k} C_r(f,C_s(g,h)), \quad k \geq 0.$$

Let π be the Poisson bivector on $\mathbb{R}^{2n}$ associated to the *canonical symplectic form* ω. Let us consider the coordinates $(y^1,\dots,y^{2n})$ of $\mathbb{R}^{2n}$ as elements of $\mathbb{R}^{2n\,*}$, let h be a formal parameter, and let $\pi^{ij}=\pi(y^i,y^j)$. We assign the degree 1 to each y^i and degree 2 to the parameter h.

Definition 3 *The formal Weyl algebra W associated to $(\mathbb{R}^{2n},\pi)$ is the space of formal series*

$$W=\Big\{a(y,h)=\sum_{m=0}^{\infty}\sum_{2k+p=m}a_{k,i_1,\dots,i_p}h^k y^{i_1}\cdots y^{i_p}\,\Big|\; a_{k,i_1,\dots,i_p}\in\mathbb{R}\text{ is symmetric in } i_1,\dots,i_p\Big\}$$

with the associative product (the Weyl product)

$$a\circ b=\sum_{k=0}^{\infty}\left(\frac{h}{2}\right)^k\frac{1}{k!}\pi^{i_1j_1}\cdots\pi^{i_kj_k}\frac{\partial^k a}{\partial y^{i_1}\cdots\partial y^{i_k}}\frac{\partial^k b}{\partial y^{j_1}\cdots\partial y^{j_k}}.$$

The contragredient action of the natural action of the *symplectic group* $Sp(2n,\mathbb{R})$ on $\mathbb{R}^{2n}$ extends naturally to an action ρ on W, which respects the Weyl product, that is to say, $Sp(2n,\mathbb{R})$ acts by automorphisms of W.

Definition 4 *Let (M,π) be a symplectic manifold, and let $F(M)$ be its symplectic frame bundle. The associated bundle $\mathscr{W}:=F(M)\times_{Sp(n,\mathbb{R}),\rho}W$ is called the (formal)* Weyl bundle.

The space of sections $\Gamma(\mathscr{W})$ with respect to pointwise multiplication forms an associative algebra. Furthermore, one defines the product of $X\otimes a$, $X'\otimes a'\in\Gamma(\Omega_*(M)\otimes\mathscr{W})$ by setting $(X\otimes a)\circ(X'\otimes a')=X\wedge X'\otimes a\circ a'$, and a graded Lie algebra structure on $\Gamma(\Omega_*(M)\otimes\mathscr{W})$ by setting

$$[s_1,s_2]=s_1\circ s_2-(-1)^{q_1q_2}s_2\circ s_1,\qquad s_i\in\Gamma(\Omega_{q_i}(M)\otimes\mathscr{W}).$$

Following [5], we define two linear operators $\delta:\Gamma(\Omega_q(M)\otimes\mathscr{W})\to\Gamma(\Omega_{q+1}(M)\otimes\mathscr{W})$ and $\delta^*:\Gamma(\Omega_q(M)\otimes\mathscr{W})\to\Gamma(\Omega_{q-1}(M)\otimes\mathscr{W})$, $q\geq 1$, as follows. Let $\hat{\pi}:=\sum_{i=1}^{2n}\frac{\partial}{\partial x^i}\otimes y^i$

$$\delta a=\frac{1}{h}[\hat{\pi},a],\qquad \delta^*(a)=\sum_{k=1}^{2n}y^k i_{(\frac{\partial}{\partial x^k})^\flat}a,\quad a\in\Gamma(\Omega_q(M)\otimes\mathscr{W}),\tag{1}$$

where i stands for the interior product of a 1-form and a p-vector. If $q=0$, we set $\delta^*a=0$.

Lemma 1 *The operators δ and δ^* do not depend on the choice of local coordinates, and satisfy $\delta^2=\delta^{*2}=0$. Furthermore, the operator δ is a graded derivation.*

Note that any $a \in \Gamma(\Omega_*(M) \otimes \mathscr{W})$ can be written (locally) as

$$a = \sum_{2k+p\geq 0, q\geq 0} h^k a_{k,j_1,\ldots,j_q,i_1,\ldots,i_p} \frac{\partial}{\partial x^{j_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{j_q}} \otimes y^{i_1} \cdots y^{i_p} =: \sum_{q\geq 0, p\geq 0} a_{qp}, \quad (2)$$

where the coefficients are mixed tensors symmetric in $i_1,\ldots,i_p$ and antisymmetric in $j_1,\ldots,j_q$. In particular, $a_{00} = \sum_{k\geq 0} h^k a_k$, and so if we define the operator δ^{-1} acting on a monomial a_{qp} by $\delta^{-1}(a_{00}) = 0$, and $\delta^{-1}(a_{qp}) = \frac{1}{q+p}\delta^*(a)$ if $q+p>0$, then we get the decomposition

$$(\delta\delta^{-1} + \delta^{-1}\delta)(a) = a - a_{00}, \quad a \in \Gamma(\Omega_*(M) \otimes \mathscr{W}). \quad (3)$$

Flat connections on the Weyl bundle

An *almost symplectic connection* ∇' on a symplectic manifold (M,π) induces a contravariant connection on TM by setting $\nabla_\alpha X := \nabla'_{\alpha^\#} X$. Moreover, using the contravariant differentiation of the coefficients in (2), we define a contravariant differential on $\mathscr{W}$ as an operator $\partial : \Gamma(\mathscr{W}) \to \Gamma(\Omega_1(M) \otimes \mathscr{W})$ by setting $\partial a = \sum_i \frac{\partial}{\partial x^i} \otimes \nabla_{dx^i} a$. A calculation yields that in a *Darboux coordinate system*, ∂ is given by

$$\partial a = \tilde{\partial} a + \frac{1}{2h}\Big[\sum_{i,j,r=1}^{2n} \pi^{ir}\Gamma^r_j \otimes y^i y^j, a\Big], \quad a \in \Gamma(\mathscr{W}), \quad (4)$$

where Γ^r_j is the connection matrix of 1-vectors (of the connection ∇). A simple calculation shows that the curvature of ∂ is given by $\partial^2 a = \frac{1}{h}[\bar{R}, a]$, where $\bar{R} = \frac{1}{2}\sum_{r,l,k=1}^{2n} \pi^{rl} R^l_k \otimes y^r y^k$ and R^i_j is the curvature matrix of 2-vectors.

Now we look for a flat connection D of the type $D = \partial - \delta + \frac{1}{h}[r,\cdot]$, where r is a $\mathscr{W}$-valued 1-vector. The condition for D to be flat is that the element $\pi + \partial\hat{\pi} + \bar{R} + \partial r - \delta r + \frac{1}{2h}[r,r]$ has to be a central element of $\Gamma(\Omega_* \otimes \mathscr{W})$.

Note that there is a filtration in the Weyl algebra $W \supset W_1 \supset W_2 \supset \cdots$ with respect to the total degree $2k+p$ of the terms of the series (2): $W_k =$ {elements of degree $\geq k$}. The proof of the following lemma follows exactly the same pattern as the one given in [5].

Lemma 2 *If $r_0 \in \Gamma(\Omega_1(M) \otimes W_k)$, $k \geq 2$, then the equation $r = r_0 + \delta^{-1}(\partial r + \frac{1}{2h}[r,r])$ has a unique solution belonging to $\Gamma(\Omega_1(M) \otimes W_k)$.*

The contravariant version of Theorem 5.2.2 in [5] is as follows.

Theorem 1 *Let $\bar{\pi} = \sum_{i\geq 1} h^i \pi_i$ be a series of σ-closed 2-vectors on M. The equation*

$$\delta r = -\partial\hat{\pi} + \bar{R} + \partial r + \frac{1}{2h}[r,r] - \bar{\pi}, \quad (5)$$

has a unique solution $r \in \Gamma(\Omega_1(M) \otimes \mathscr{W})$ satisfying the normalization condition $\delta^{-1} r = 0$, and such that the $\mathscr{W}$-degree of the leading term of r is at least 2.

Proof. Let $r_0 = \delta^{-1}(-\partial\hat{\pi} + \bar{R} - \bar{\pi})$, and observe that deg $r_0 \geq 2$, so from Lemma 2 we deduce the existence of a unique solution r of the equation $r = \delta^{-1}(-\partial\hat{\pi} + \bar{R} - \bar{\pi}) + \delta^{-1}\partial r + \frac{1}{2h}\delta^{-1}[r,r]$. Moreover, since $(\delta^{-1})^2 = 0$ it follows that $\delta^{-1}r = 0$.

In order to prove that r is a solution of Equation (5), we set $K = \delta r + \partial\hat{\pi} - \bar{R} - \partial r - \frac{1}{h}r^2 + \bar{\pi}$. Using the decomposition (3) and the fact that r is a 1-vector (so $r_{00} = 0$), we get $\delta^{-1}K = 0$. Furthermore,

$$DK = \frac{1}{h}[\partial\hat{\pi}, r] + \frac{1}{h}[r, \partial\hat{\pi}] + \partial^2\hat{\pi} - \partial\bar{R} + \delta\bar{R} - \delta\partial\hat{\pi} = -\delta\partial\hat{\pi} = \frac{1}{2h}\partial[\hat{\pi}, \hat{\pi}] = 0.$$

Hence it follows that K can be written as $K = \delta^{-1}(\partial K + \frac{1}{h}[r,K])$. Applying the iteration method to the above equation we can conclude that $K = 0$ and the theorem follows. ■

Since D acts as a derivation the space $\Gamma_D(\mathscr{W}) := \{a \in \Gamma(\mathscr{W}) \mid Da = 0\}$ is a subalgebra of $\Gamma(\mathscr{W})$, and Theorem 5.2.4 in [5] becomes:

Theorem 2 *Given a flat contravariant connection D, for any $a_0 \in C^\infty(M)[[h]]$ there is a unique $a \in \Gamma_D(\mathscr{W})$ such that $a(x,0,h) = a_0(x,h)$.*

Proof. The proof of the theorem follows exactly the same pattern as the one given in [5]. ■

Finally, using the *symbol map* $\sigma^{-1} : \Gamma_D(\mathscr{W}) \to C^\infty(M)[[h]]$, $\sigma^{-1}(a) = a(x,0,h)$, and the above theorem one deduces the existence of star-product on M by setting

$$a * b = \sigma^{-1}(\sigma(a) \circ \sigma(b)), \qquad a,b \in C^\infty(M)[[h]]. \tag{6}$$

4. Star-products on Abelian Poisson Lie Groups

Definition 5. ([4]) *A Lie group G is called a* Poisson Lie group *if it is also a Poisson manifold such that the multiplication map $m : G \times G \to G$ is also a Poisson map, where $G \times G$ is equipped with the product Poisson structure.*

In what follows we shall assume that G is a real connected Lie group with Lie algebra $\mathbf{g}$. Let R_g be the right translation of G by g. We denote by $\pi_r : G \to \wedge^2\mathbf{g}$ the map defined by $\pi_r(g) = dR_{g^{-1}}\pi(g)$ and by $d_e\pi_r$ its differential at the identity $e \in G$.

Theorem 3. ([4, 9]) *The following conditions are equivalent:*

1. (G, π) is a Poisson-Lie group.

2. The Poisson structure π satisfies $\pi(e) = 0$, and $L_X\pi$ is right invariant, whenever X is a right invariant vector field, where $L_X\pi$ denotes the Lie derivative of π with respect to X.

3. The dual map $(d_e\pi_r)^* : \wedge^2\mathbf{g}^* \to \mathbf{g}^*$ of $d_e\pi_r$ defines a Lie algebra structure on $\mathbf{g}^*$.

Let G^* be the *dual group* of G, i.e., the connected and simply connected Lie group whose Lie algebra is $\mathbf{g}^*$. Recall that there exists [9] a unique Poisson structure, say π^* on G^* which induces the Lie algebra structure of $\mathbf{g}$.

Next we shall endow the space $G \times G^*$ with a Poisson structure so that we can search for projectable $*$-products under the projection $p_1 : G \times G^* \to G$, $p_1(g, g^*) = g$.

Let $\tilde{X}_i$, $1 \leq i \leq n$, be n right-invariant vector fields on G forming a global frame on it, let $\tilde{X}_{n+i}$ be the right-invariant vector fields on G^* corresponding to the dual basis of $\tilde{X}_i(e)$, and let $p_2 : G \times G^* \to G^*$ be the projection onto the second factor. We denote by X_i, X_{n+i} the $2n$ vector fields on $G \times G^*$ such that $dp_1(X_i) = \tilde{X}_i$, $dp_2(X_i) = 0$, $dp_1(X_{n+i}) = 0$, $dp_2(X_{n+i}) = \tilde{X}_{n+i}$, and by C_{ij}^k (resp. D_k^{ij}) the structure constants of $\mathbf{g}$ (resp. $\mathbf{g}^*$) relative to the basis $\tilde{X}_i(e)$ (resp. $\tilde{X}_{n+i}(e^*)$). The following result was first considered in [7].

Proposition 1 *Let $(G, -\pi)$ be a Lie Poisson group, let $-\pi + \pi^*$ be the product Poisson structure on $G \times G^*$, and let $\psi = \sum_i X_i \wedge X_{n+i}$. Then, the 2-vector $\Lambda = \psi - \pi + \pi^*$ determines a Poisson structure on $G \times G^*$.*

Proof. It is enough to prove that $[\Lambda, \Lambda]_s = 0$. By linearity of the Schouten bracket one gets

$$[\psi - \pi + \pi^*, \psi - \pi + \pi^*]_s = [\psi, \psi]_s - 2[\psi, \pi]_s + 2[\psi, \pi^*]_s,$$

However, writing the right-hand side of the equality relative to the 3-vectors $X_i \wedge X_j \wedge X_k$. $1 \leq i, j, k \leq 2n$, and using condition 2 of Theorem 3 it becomes

$$(C_{ia}^s - L_{X_{n+s}} {\pi^*}_{ia}(e^*))X_s \wedge X_{n+i} \wedge X_{n+a} + (D_s^{ia} - L_{X_s} \pi^{ia}(e))X_i \wedge X_a \wedge X_{n+s}\,,$$

which vanishes by virtue of condition 3 of Theorem 3. ∎

Remark 1 *The Poisson structure Λ is symplectic in an open neighborhood U of $G \times e^*$ on $G \times G^*$.*

Ideally, we would like to prove the existence of star-products on (G, π), by means of projectable (by p_1) star-products on $G \times G^*$ or U. Unfortunately, we do not know in general if this is possible. However, if we assume that (G, π) is an Abelian Poisson Lie group, then there is a $*$-product on $(G \times G^*, \Lambda)$ such that $(p_1^* C^\infty(G)[[h]], *)$ is a subalgebra of $(C^\infty(G \times G^*)[[h]], *)$. Observe that under the Abelian hypothesis Λ can be written as $\Lambda = \sum_i e_i \wedge e_{n+i}$, where $e_i = X_i$ and $e_{n+i} = X_{n+i} - \frac{1}{2}\sum_j \pi^{ij} X_j$, and so we can define a symplectic contravariant connection by setting

$$\nabla_{\theta^i} e_j = \nabla_{\theta^{n+i}} e_j = 0, \quad \nabla_{\theta^i} e_{n+j} = \nabla_{\theta^{n+i}} e_{n+j} = 0, \tag{7}$$

where $\{\theta^i, \theta^{n+i}\}$ is the dual basis of $\{e_i, e_{n+i}\}$.

Let $\mathscr{W}$ be the Weyl bundle over $G \times G^*$, and let ∂ be the connection on $\mathscr{W}$ induced by ∇. One easily verifies that the local expression of $r = \delta^{-1}(-\partial\hat{\pi}) + \delta^{-1}(\partial r + \frac{1}{h}r^2)$, $\hat{\pi} = e_i \otimes y^i$, in the above basis is of the type

$$r = \sum_{2k+p\geq 2} r_{k,j_1,i_1,\ldots,i_p} h^k e_{j_1} \otimes y^{i_1} \cdot \cdot y^{i_p}, \qquad r_{k,j_1,i_1,\ldots,i_p} \in p_1^* C^\infty(G). \tag{8}$$

Lemma 3 *Let $\lambda^i = \theta^i - \frac{1}{2}\sum_j \pi^{ij}\theta^{n+j}$ for $1 \leq i \leq n$ and let $D = \partial - \delta + \frac{1}{h}[r,\cdot]$. If $a_0 \in p_1^* C^\infty(G)[[h]]$, then $\partial a_0(\lambda^i) = 0$. Furthermore, since ∂ acts as a derivation, the space $A = \{a \in \Gamma_D(\mathscr{W}) \mid \partial a(\lambda^i) = 0,\ 1 \leq i \leq n\}$ is a subalgebra of $\Gamma_D(\mathscr{W})$.*

Theorem 4 *The elements of the subalgebra A are parameterized by $C^\infty(G)[[h]]$.*

Proof. Let $a_0 \in p_1^* C^\infty(G)[[h]]$, and let a be the flat section corresponding to it. Since a is obtained recursively from the equation $a = \delta^{-1}(\partial a + \frac{1}{h}[r,a]) + a_0$, it follows from (8) and the above lemma that $a \in A$. The converse is easily deduced from the fact that $(\lambda^i)^\# = X_{n+i}$. ■

Using a different method, S. Gutt [6] has shown that the star-product constructed on the cotangent bundle of a Lie group G in [2] is projectable to the dual of the Lie algebra of G. Therefore, one can consider Theorem 4 as a generalization of such a result.

Example 1 Let G be the dual of the Heisenberg Lie algebra endowed with the standard Lie-Poisson structure π. Let G^* be its dual group. We define a flat contravariant connection on $G \times G^*$ as in (7) and so a connection ∂ on $\mathscr{W}$. Let (x^1, x^2, x^3) be the coordinate system on G such that $\pi = x^3\partial_1 \wedge \partial_2$. Let r and r' be the respective solutions of the equation of Lemma 2 with $r_0 = -\partial\hat{\pi}$ and $r'_0 = -\partial\hat{\pi} + h\partial_1 \wedge \partial_3$, and let D, D' be the corresponding flat connections. Then one verifies that the two star-products defined on $(G \times G^*, \Lambda)$ corresponding to D and D' are projectable (with respect to the projection onto the first factor) and that the star-products induced on (G,π) are not *equivalent*.

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