Remarks on the zeta-regularized determinant of differential operators

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Abstract. It is shown that the equalities $\det(\not{D}+im)=\det(\not{D}-im)=\sqrt{\det(\not{D}^2+m^2)}$, where $\not{D}$ is the massless Dirac operator, hold if and only if: (i) $\det(\not{D}+im)$ and $\det(\not{D}-im)$ are defined by using the symmetry property of the spectrum of $\not{D}$; (ii) $Z(\not{D}^2+m^2,0)$ is an even integer (the sign of the square root above is positive if $Z(\not{D}^2+m^2,0)$ is divisible by 4, and negative otherwise). On the other hand, the equality $|\det(\not{D}\pm im)|=\sqrt{\det(\not{D}^2+m^2)}$ is always holding. It is also shown, by applying the standard definition of zeta-determinant, that $\det(\not{D}+im)$ is not equal to $\det(\not{D}-im)$ in general. In order to show this fact, a variational formula for $\det(D+m)$ with respect to m is derived for general self-adjoint operators.

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Introduction

Some doubts have been thrown on the validity of the following equalities

$$\det(\not{D}+im)=\det(\not{D}-im)=\sqrt{\det(\not{D}^2+m^2)} \tag{1}$$

(see, e.g. [6]). These doubts were motivated by the presence of the multiplicative anomaly in the zeta-regularized determinant ([3]). In this note, we shall investigate the question of the validity of (1) and, for this purpose, we first establish that

$$\det D=(-1)^{\nu_-}\det D_{(+)}\det D_{(-)},\quad D(\pm)=\frac{|D|\pm D}{2}, \tag{2}$$

where D is a self-adjoint elliptic operator such that its spectral zeta-function

$$Z(D,s)=\mathrm{tr}\left(\frac{1}{2\pi i}\int_\gamma \lambda(\lambda I-D)^{-1}d\lambda\right),\quad \arg(-1)=i\pi, \tag{3}$$

admits an analytic continuation to $s=0$ which is holomorphic at $s=0$. $Z(D(\pm),s)$ are similarly defined and ν_- in (2) stands for $Z(D_{(-)},0)$ (see [1]).

The determinant of D is defined by $\exp(-Z'(D,0))$, where $'$ denotes the derivative with respect to s. More precisely, (2) contains an ambiguity that comes from the many valuedness of the powers of -1. It is shown that the condition $Z(D_{(-)},0)$ being an integer is a necessary and sufficient condition for the uniqueness of the

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definition of $\det D$ (see Corollary 1.6 of Proposition 1.4). It is also shown that $\det D$ is uniquely defined if and only if it takes a real value, and $|\det D|$ is unique although $\det D$ is not unique. If the spectrum of D satisfies the symmetry condition, i.e., $\operatorname{Spec} D = \operatorname{Spec}(-D)$, where $\operatorname{Spec} D$ is the set of eigenvalues of D, then (1) reads

$$\det D = (-1)^{Z(D^2,0)/2}(\det D)^2. \tag{4}$$

If $m \neq 0, D+im$ is not self-adjoint, and we cannot use (1). But since $\lambda + im$ and $-\lambda + im$ are both eigenvalues of $D+im$, and $(\lambda+im)(-\lambda+im) = -(\lambda^2+m^2)$, we define the sym-determinant sym-det$(D+im)$ by

$$\text{sym-det}\,(D+im) = (-1)^{Z(D^2+m^2,0)/2}\det(D^2+m^2). \tag{5}$$

Then we have

$$\text{sym-det}\,(D+im) = \text{sym-det}\,(D-im), \tag{6}$$

$$(\text{sym-det}\,(D+im))(\text{sym-det}\,(D-im)) = (-1)^{Z(D^2+m^2,0)/2}\det(D^2+m^2). \tag{7}$$

From (6) and (7), we can derive a condition for the validity of (1) as it is done in Sections 1 and 2. The sym-determinant may be different from the ordinary determinant. To discuss this problem, we compute $\det(D+im)$ as a function of m, provided D is a self-adjoint first-order operator on a compact manifold (cf. [1]). It is concluded that, due to the existence of residues of $Z(D,s)$, $\det(D+im)$ can be different from $\det(D-im)$ if $m \neq 0$. These are discussed in Sections 3 and 4.

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1. Zeta-determinants of self-adjoint operators

Let D be a non-degenerate elliptic pseudo-differential operator such that the spectrum of D does not meet the ray $R_\theta = \{re^{i\theta} | r \geq 0\}$. Then the complex power D_θ^s and the zeta-function $Z_\theta(D,s)$ of D are defined by

$$D_\theta^s = \frac{1}{2\pi i}\int_\gamma \lambda_\theta^s(\lambda I - D)^{-1}d\lambda, \tag{8}$$

$$Z_\theta(D,s) = \operatorname{tr} D_\theta^{-s}, \tag{9}$$

where $\lambda_\theta^s = |\lambda|\exp(is\arg\lambda)$, $\theta < \arg\lambda < \theta + 2\pi$. Here the contour γ is such that it encloses the spectrum of D (cf. [7, 8]). If D is self-adjoint and admits the spectral decomposition

$$Df = \sum \lambda_n(e_n, f)e_n, \tag{10}$$

we have

$$Z_\theta(D,s) = \sum_{\lambda_n>0}\lambda_n^{-s} + \sum_{\lambda_n<0}(-1)^{-s}|\lambda_n|^{-s}, \tag{11}$$

with $-1 = e^{\pi i}$, if $-\pi < \theta < 0$, and $-1 = e^{-\pi i}$, if $-2\pi < \theta < -\pi$.

If we fix $\arg 1 = 0$, then the only possible choices for $\arg(-1)$ are πi and $-\pi i$. In the remaining, we shall use the notation $Z(D,s)$ if $\arg(-1) = \pi i$, and $Z_{-}(D,s)$ if $\arg(-1) = -\pi i$. We introduce the following operators and zeta-functions related to D:

$$|D|f = \sum |\lambda_n|(e_n, f)e_n, \tag{12}$$

$$D_{(+)} = (|D| + D)/2, \qquad D_{(-)} = (|D| - D)/2, \tag{13}$$

$$Z(D_{(+)}, s) = \sum_{\lambda_n > 0} \lambda_n^{-s}, \qquad Z(D_{(-)}, s) = \sum_{\lambda_n < 0} |\lambda_n|^{-s}. \tag{14}$$

Notice that, by definition, $Z(|D|,s) = Z(D^2, s/2)$. We introduce the eta-function $H(D,s)$ of D, it is defined by $\sum_{\lambda_n} \mathrm{sgn}(\lambda_n)|\lambda_n|^{-s}$. Then we have that $Z(D_{(\pm)},s) = (Z(|D|,s) \pm H(D,s))/2$. We assume that these zeta- and eta-functions are all continued analytically to 0 and are holomorphic at $s = 0$ (cf. [2, 5, 9]). We also denote the projections to the positive and negative eigenspaces of D by $P_{(+)}$ and $P_{(-)}$, respectively. If D is an n-th-order operator on a d-dimensional compact manifold, then the following facts are known ([5], for the case manifold with boundary, cf. [2, 9]).

Fact 1. $H(D,s)$ is continued meromophically to the whole complex plane, with possible poles at most of order 1 located at the points $s = d/n$, $(d-1)/n, \ldots$

Fact 2. $H(D,s)$ is holomorphic at $s = 0$.

Fact 3. $H(D,s)$ takes real value if s is a real number. All the residues of $H(D,s)$ are real numbers.

Fact 1 is a classical result of Seeley and was proved by performing an asymptotic expansion of the heat kernel. Fact 2 is a famous result of Atiyah-Patodi-Singer and Gilkey. Fact 3 follows from Fact 1 and the following Lemma.

Lemma 1.1 *Let f be a meromophic function on the complex plane having poles on the real axis only and taking real values on some interval of the real axis. Then f is real valued on the real axis and the residues of f are real.*

Proof. We may assume that f takes real values on (a,b), where a and b are poles of f (or $a = -\infty$, $b = \infty$). Let the principal part of the Laurent expansion of f be $c_{-k}/(z-a)^k + \cdots$, then c_{-k} is given by $\lim_{x \to a+0}(x-a)^k f(x)$, so c_{-k} is a real number. Hence $f(z) - c_{-k}/(z-a)^k$ takes real values on (a,b). Repeating this, we conclude that $f(z) - \{$principal part of the Laurent expansion of f at $a\}(= g(z))$ takes real values on (a,b). Then the coefficients of the Taylor expansion of $g(z)$ are all real numbers, so $g(z)$ takes real values on (a',b), $a' < a$ and f is holomorphic on (a',a). Hence $f(z)$ takes real values on $(a',a) \cup (a,b)$. By repeating this argument, the lemma follows.

From Fact 3, we have the following proposition.

Proposition 1.2 *If D is a positive elliptic operator on a compact manifold, then* $\det D$ *is a positive real number.*

Note. If we express $1 = e^{2m\pi i}$, where $m \neq 0$ is an integer, $Z(D,s)$ changes to $e^{-2ms\pi i}Z(D,s)$. Then, since $(e^{-2ms\pi i}Z(D,s))' = e^{-2ms\pi i}(-2m\pi i Z(D,s) + Z'(D,s))$, $\det D$ changes to $e^{2mZ(D,0)\pi i}\det D$. Proposition 1.2 can be formulated in a different way.

Proposition 1.3 *If D is a positive elliptic operator on a compact manifold, then* $\det D$ *does not depend on the expression of 1 (as a power of e), if and only if $Z(D,0)$ is an integer.*

From (11), we have

$$Z'(D,s) = \sum_{\lambda_n > 0} \Big((-\log\lambda_n)\lambda_n^{-s} + e^{-s\pi i}(-\log\lambda_n - \pi i)|\lambda_n|^{-s} \Big). \tag{15}$$

Hence using the above notations, we have

$$\begin{aligned} &\exp(-Z'(D,s)) = \\ &\quad \exp(iZ(D(-),0)\exp(-Z'(D(+),s))\exp(-Z'(D(-),s))\exp(e^{-s\pi i}). \end{aligned} \tag{16}$$

Proposition 1.4 *The zeta-determinant of a self-adjoint operator D is given by*

$$\det D = (-1)^{Z(D(-),0)}\det D(+)\det D(-). \tag{17}$$

Corollary 1.5 *Let* $\det D = \exp(-Z'(D,s))$. $\det D$ *is equal to* $\overline{\det D}$, *the complex conjugate of* $\det D$. *Consequently, we have* $|\det D| = \det|D|$ *(which is also equal to* $\det D(+)\det D(-)$*).*

Corollary 1.6 *The following statements are equivalent.*

i) $\det D$ *is a real number.*

ii) $\det D$ *does not depend on the expression of* -1.

iii) $Z(D(-),0)$ *is an integer.*

In general, $\det D$ depends on the expression of -1. The notations $\det D$ when $\arg(-1) = \pi i$ and $\det_- D$ when $\arg(-1) = -\pi i$ shall be used in the rest of the paper.

Note. As for $\det(-D)$, we have that $\det(-D) = (-1)^{H(D,0)}\det D$. In general, the right-hand side differs from $(-1)^{Z(|D|,0)}\det D$. But if $Z(D(-),0)$ is an integer, we have $(-1)^{H(D,0)} = (-1)^{Z(|D|,0)}$. That is, $\det(-D) = (-1)^{Z(|D|,0)}\det D$ if and only if $Z(D(-),0)$ is an integer. In Proposition 1.4, we assume $\arg\lambda_n = 0$ if $\lambda_n > 0$. If we release this assumption, we need to assume that $Z(D(+),0)$ is an integer to keep the uniqueness of $\det D$. Hence, together with Corollary 1.6, we need to assume that $H(D,0)$ and $Z(|D|,0)$ are integers, and mod 2 congruence of $H(D,0)$ and $Z(|D|,0)$.

2. Sym-determinant

In this section, we assume that the spectrum of D is symmetric, that is, if λ is an eigenvalue of D, then $-\lambda$ is also an eigenvalue (with same multiplicity). Then we have

$$Z(D_{(+)},s) = Z(D_{(-)},s) = Z(D,s/2)/2. \tag{18}$$

By combining (18) and Proposition 1.4, we obtain a simple consequence.

Proposition 2.1 *If the spectrum of D is symmetric, then*

$$\det D = (-1)^{Z(D_{(-)},0)}(\det D_{(+)})^2, \tag{19}$$

and $\det D$ *is equal to* $\det\_ D$ *if and only if* $Z(D,0)$ *is an even integer.*

Corollary 2.2 $(\det D)^2 = \det(D^2)$ *if and only if* $Z(D^2,0)$ *is an even integer. More precisely, we have*

$$\begin{aligned} \det D &= \sqrt{\det(D^2)}, \quad &&\text{if } Z(D,0) \equiv 0 \mod 4,\\ \det D &= -\sqrt{\det(D^2)}, \quad &&\text{if } Z(D,0) \equiv 2 \mod 4. \end{aligned} \tag{20}$$

We shall apply this result to the determinant of $D+im$. Usually, $\det(D+im)$ is defined by using the symmetry property of the spectrum and involves the following change of the order of multiplication (cf. [3])

$$(\lambda_n + im)(-\lambda_n + im) = (-\lambda_n - im)(\lambda_n - im)) = -(\lambda_n^2 + m^2). \tag{21}$$

If we accept this reordering of factors, we must have $\det(D+im)$=$\det(D-im)$. But since we are working with regularized infinite product, the change of the order of multiplication must have some effect on the answer. So we shall use the following notation.

Definition 2.3 *We define* sym-det $(D+im)$ *by*

$$\text{sym-det}\,(D+im) = (-1)^{Z(D_{(+)}^2+m^2,0)}\det(D_{(+)}^2+m^2). \tag{22}$$

This definition is motivated by the fact that the rearrangement (21) regards "det"$(D+im)$ to be $\det\big(-(D_{(+)}^2+m^2)P_{(+)}\big)$. In the rest, we use the notation $D_{(+)}^2+m^2$, instead of $(D_{(+)}^2+m^2)P_{(+)}$ for the sake of simplicity. Since $\det(D_{(+)}^2+m^2) = \det(D_{(-)}^2+m^2)$ and $\det(D_{(+)}^2) = (\det D_{(+)})^2$, we have

$$\text{sym-det}\,(\not{D} + im) = \text{sym-det}\,(\not{D} - im), \tag{23}$$

$$\text{sym-det}\,\not{D} = \det\not{D}. \tag{24}$$

By definition, $Z(\not{D}^2+m^2)$ is equal to $2Z(\not{D}_{(+)}^2+m^2)$ $(= 2Z(\not{D}_{(-)}^2+m^2))$. Hence we have

$$\det(\not{D}^2+m^2) = (\det(\not{D}_{(+)}^2+m^2))^2. \tag{25}$$

By (25) and the equality $Z(\not{D}_{(+)}^2+m^2,0) = Z(\not{D}^2+m^2,0)/2$, we obtain

Theorem 2.4 *The following equalities are holding.*

$$\text{sym-det}\,(\not{D}+im)=\sqrt{\det(\not{D}^2+m)},\qquad \text{iff } Z(\not{D}^2+m^2,0)\equiv 0 \mod 4, \tag{26}$$

$$\text{sym-det}\,(\not{D}+im)=-\sqrt{\det(\not{D}^2+m^2)},\quad \text{iff } Z(\not{D}^2+m^2,0)\equiv 2 \mod 4. \tag{27}$$

We also have

$$|\text{sym-det}\,(D+im)|=|\sqrt{\det(D^2+m^2)}|, \tag{28}$$

if D is defined on a compact manifold.

Note. The sym-determinant is also defined for $D+m$ and the computation of sym-det $(D+m)=\det(-D_{(+)}{}^2-m^2)$ is not simple, as $-(D_{(+)}{}^2-m^2)$ is not a positive operator. As for the relation between $\det(-(D_{(+)}{}^2-m^2))$ and $\det(-(D^2-m^2))$, set $c=\#\{\lambda|\lambda$ is an eigenvalue value of $D_{(+)}$ such that $\lambda<m\}$, we have $\det(-(D_{(+)}{}^2-m^2))=(-1)^c\{\det(-(D_{(+)}{}^2-m^2))\}$. So we need to take into account the contribution from c. Theorem 2.4 and (23) give an answer to the queries stated in the Introduction. But there still remains the problem to know whether sym-det$(\not{D}+im)=\det(\not{D}+im)$ or not if $m\neq 0$. To give an answer to this problem, we calculate the variation of $\det(D+m)$ with respect to m.

Note. As for the operator $iD+m$, sym-det $(iD+m)$ is defined to be $\det(D_{(+)}{}^2+m^2)$, which is equal to $\sqrt{\det(D^2+m^2)}$ if we fix $\arg\lambda_n=0$ (cf. [10]). But if the choice $\arg\lambda_n=2m\pi i$, $m\neq 0$, is allowed, $Z(D^2+m^2,0)/2$ is still an integer.

3. Variation of $Z(D+m,0)$ with respect to m

We assume that D is a first-order operator defined on a d-dimensional compact manifold. Then $H(D,s)$ and $Z(|D|,s)=H(D,s/2)$ are continued meromorphically to the whole complex plane with possible poles at $s=d,d-1,\dots,$ of order 1, and are holomorphic at $s=0$ ([5], for the operators on manifold with boundary, cf. [2, 9]). In the remaining, we shall use only these properties of $H(D,s)$ and $Z(|D|,s)$. Since $Z(D,s)_{(\pm)}=(Z(|D|,s)\pm H(D,s))/2$, these functions have the same properties as $H(D,s)$ and $Z(|D|,s)$. Moreover we have

$$\mathrm{Res}_{s=m}\,Z(|D|,s)=\mathrm{Res}_{s=m}\,Z(D_{(+)},s)+\mathrm{Res}_{s=m}\,Z(D_{(-)},s), \tag{29}$$

$$\mathrm{Res}_{s=m}\,H(D,s)=\mathrm{Res}_{s=m}\,Z(D_{(+)},s)-\mathrm{Res}_{s=m}\,Z(D_{(-)},s). \tag{30}$$

For a positive number N, we set

$$Z(D_{(+)},N,s)=\sum_{\lambda_n>N}\lambda_n^{-s}=Z(D_{(+)},s)-\sum_{0<\lambda_n\le N}\lambda_n^{-s}, \tag{31}$$

$$Z(D_{(-)},N,s)=\sum_{\lambda_n<-N}|\lambda_n|^{-s}=Z(D_{(-)},s)-\sum_{0>\lambda_n\ge -N}|\lambda_n|^{-s}. \tag{32}$$

If $|m|<|\lambda_1|$, where λ_1 the smallest eigenvalue of $|D|$, we have

$$Z(D_{(+)}+m,s)=\sum(\lambda_n+m)^{-s}=\sum\lambda_n^{-s}(1+m/\lambda_n)^{-s}. \tag{33}$$

Since $(1+m/\lambda_n)^{-s} = 1 - s(m/\lambda_n) + \cdots + (-1)^k \big(s\cdots(s+k-1)/k!\big)(m/\lambda_n)^k + \cdots$, we have by (33)

$$\begin{aligned} Z(D(+)+m,s) =& Z(D(+),s) - sZ(D(+),s+1)m + \cdots \\ &\cdots + (-1)^k (s\cdots(s+k-1))/k!\, Z(D(+),s+k)m^k + \cdots . \end{aligned} \tag{34}$$

By (34), we obtain for $|m| < |\lambda_1|$,

$$\begin{aligned} Z(D(+)+m,0) =& Z(D(+),0) - \mathrm{Res}_{s=1}\, Z(D(+),s)m + \cdots \\ &\cdots + (-1)^k \mathrm{Res}_{s=k}\, Z(D(+),s)(m^k/k) + \cdots \\ &\cdots + (-1)^d \mathrm{Res}_{s=d}\, Z(D(+),s)(m^d/d). \end{aligned} \tag{35}$$

A similar formula holds for $D(-)$,

$$\begin{aligned} Z(D(-)+m,0) =& Z(D(-),0) + \mathrm{Res}_{s=1}\, Z(D(-),s)m + \cdots \\ &\cdots + \mathrm{Res}_{s=k}\, Z(D(-),s)(m^k/k) + \cdots \\ &\cdots + \mathrm{Res}_{s=d}\, Z(D(-),s)(m^d/d). \end{aligned} \tag{36}$$

If m is real and $|m| < |\lambda_1|$, we have

$$\begin{aligned} Z(|D+m|,s) &= Z(D(+)+m,s) + Z(D(-)-m,s), \\ H(D+m,s) &= Z(D(+)+m,s) - Z(D(-)-m,s). \end{aligned} \tag{37}$$

Hence we obtain

$$\begin{aligned} Z(|D+m|,0) =& Z(|D|,0) + \mathrm{Res}_{s=1}\, H(D,s)m \\ &+ \mathrm{Res}_{s=2}\, Z(|D|,s)(m/2) + \cdots + (-1)a(d)(m^d/d), \end{aligned} \tag{38}$$

$$a(d) = \begin{cases} \mathrm{Res}_{s=d} Z(|D|,s), & \text{if } d \text{ is even}, \\ \mathrm{Res}_{s=d} H(D,s), & \text{if } d \text{ is odd}. \end{cases} \tag{39}$$

$$\begin{aligned} H(D+m,0) =& H(D,0) + \mathrm{Res}_{s=1}\, Z(|D|,s)m \\ &+ \mathrm{Res}_{s=2}\, H(D,s)(m/2) + \cdots + (-1)b(d)(m^d/d), \end{aligned} \tag{40}$$

$$b(d) = \begin{cases} \mathrm{Res}_{s=d} H(D,s), & \text{if } d \text{ is even}, \\ \mathrm{Res}_{s=d} Z(|D|,s), & \text{if } d \text{ is odd}. \end{cases} \tag{41}$$

Equations (38) and (40) are proved by assuming that m is real, but their right-hand sides are polynomials in m, hence holomorphic in m. On the other hand, taking the contours in the definitions of the complex powers of $|D+m|$, etc., to be $\{x+ai \mid -\infty < x < \infty\} \cup \{x-ai \mid \infty > x > -\infty\}$, the right-hand sides of (38) and (40) are holomorphic in m if $|\mathrm{Im}\, m| < a$. Therefore (38) and (40) are valid though m is a complex number provided $|m| < |\lambda_1|$. Before computing $Z(|D+m|,0)$, etc., for large $|m|$, let us note that for $j = 1, \ldots, d$,

$$\begin{aligned} \mathrm{Res}_{s=j}\, Z(D(+),N,s) &= \mathrm{Res}_{s=j}\, Z(D(+),s), \\ \mathrm{Res}_{s=j}\, Z(D(-),N,s) &= \mathrm{Res}_{s=j}\, Z(D(-),s). \end{aligned} \tag{42}$$

Because $\sum_{0<\lambda_n\le N}\lambda_n^{-s}$, etc., are entire functions of s. By (35) and (42), to set $Z(D(+)+m,\text{“}N\text{”},s)=\sum_{\lambda_n>N}(\lambda_n+m)$, etc., we have for $|m|<N$,

$$\begin{aligned} Z(D(+)+m,\text{“}N\text{”},0) = & Z(D(+),N,0)-\operatorname{Res}_{s=1} Z(D(+),s)m+\cdots \\ & \cdots+(-1)^d\operatorname{Res}_{s=d} Z(D(+),s)(m^d/d). \end{aligned} \tag{43}$$

On the other hand, by the definition of $Z(D(+)+m,\text{“}N\text{”},s)$, etc., we have

$$Z(D(+)+m,\text{“}N\text{”},0)=Z(D(+)+m,0)-\#\{\lambda_n|0<\lambda_n<N\}, \tag{44}$$

$$Z(D(+),0)=Z(D(+),0)-\#\{\lambda_1|0<\lambda_n<N\}. \tag{45}$$

Hence (35) is valid for any m. As for (38) and (40), answers may be different if $|m|$ is large. But the differences must be even numbers. So we have ([1])

$$\begin{aligned} Z(|D+m|,0) \equiv & Z(|D|,0)-\operatorname{Res}_{s=1}H(D,s)m+\cdots+(-1)^d a(d)(m^d/d) \mod 2, \\ H(D+m,0) \equiv & H(D,0)-\operatorname{Res}_{s=1}Z(|D|,s)m+\cdots+(-1)^d b(d)(m^d/d) \mod 2. \end{aligned}$$

By the same calculation, we get the following formula, which is valid for any m, and useful to the application of Theorem 2.4.

$$\begin{aligned} Z(\not{D}^2+m^2,0) = & Z(\not{D}^2,0)-\operatorname{Res}_{s=1}Z(\not{D}^2,s)m^2+\cdots \\ & \cdots+(-1)^{2[d/2]}\operatorname{Res}_{s=2[d/2]}Z(D,s)(m^{2[d/2]}[d/2]). \end{aligned} \tag{46}$$

Here $[d/2]$ stands for Gauss' symbol. Although $Z(\not{D}^2,s)$ has poles at $s=d/2$, $(d-1)/2,\ldots$, (46) shows that residues at $s=j/2,j,\ldots$ are odd numbers and they do not contribute to the value of $Z(\not{D}^2+m^2,0)$ (cf. [3, 4]). The values of m for which the second equality of (1) is determined by $\not{D}$. If $d=1$, $Z(\not{D}^2+m^2)$ is constant in m, so the validity of (1) is determined by $\not{D}$. If $d=2$ or 3, $Z(\not{D}^2+m^2)$ is linear in m. If $d=2k$, $\operatorname{Res}_{s=k}Z(\not{D}^2,s)>0$, so the coefficient of m is negative if $d=2$. If $d=4$ or 5, $Z(\not{D}^2,0)$ is a quadratic form in m. The coefficient of m^4 is positive if $d=4$. Since the equation $Z(\not{D}^2+m^2,0)=4n$ (or $4n+2$), n is an integer, should have real positive solutions, these remarks have meanings.

4. Variation of $\det(D+m)$ with respect to m

In this section, we shall compute $\det(D+m)$. Since the orders of the poles of $Z(D,s)$ etc., are at most 1, we can make the expansion:

$$Z(D(+),s+j)=a(j,+)_{-1}s^{-1}+a(j,+)_0+a(j,+)_1 s+\cdots. \tag{47}$$

By (47), we have

$$\frac{d}{ds}(sZ(D(+),s+1))|_{s=0}=a(1,+)_0. \tag{48}$$

$$\begin{aligned} & \frac{d}{ds}(s(s+1)\cdots(s+j-1)/j!\,Z(D(+),s+j))|_{s=0} \\ & \quad =\frac{1}{j}(1+\cdots+1/(j-1))a(j,+)_{-1}+(1/j)a(j,+)_0, \quad j>2. \end{aligned} \tag{49}$$

By (35), (49), and (49), we obtain

$$
\begin{aligned}
Z'(D(+)+m,0) =& Z'(D(+),0) + \sum_{j=2}^{d}(1/j!)(1+\cdots+1/(j-1))a(j,+)_{-1}m^j \\
&+ \sum_{j=1}^{\infty}(-1)^j a(j,+)/jm^j.
\end{aligned} \tag{50}
$$

Since $Z(D(+),N,s+j) = Z(D(+),s+j) - \sum_{0<\lambda_n\le N}\lambda_n^{-(s+j)}$, we have

$$
\begin{aligned}
Z'(D(+)+m,\cdots,0) =& Z'(D(+),\text{“}N\text{”},0) + A(+,m) \\
&+ \sum_{j=1}^{\infty}(-1)^j(1/j)(a(j,+)_0 - \sum_{0<\lambda_n\le N}\lambda_n^{-j})m^j,
\end{aligned} \tag{51}
$$

$$
\begin{aligned}
Z'(D(-)+m,\cdots,0) =& Z'(D(-),\text{“}N\text{”},0) + A(-,m) \\
&+ \sum_{j=1}^{\infty}(-1)^j(1/j)(a(j,-)_0 - \sum_{0>\lambda_n\ge N}|\lambda_n|^{-j})m^j.
\end{aligned} \tag{52}
$$

Here $A(+,m)$ and $A(-,m)$ are given by

$$
A(\pm,m) = \sum_{j=2}^{d}(1/j)(1+\cdots+1/(j-1))a(j,\pm)_{-1}m^j. \tag{53}
$$

We note that $a(1,+)_{-1}$ do not affect to the values of $A(+,m)$ and $Z'(D(+)+m,0)$. We also note that since $|a(j,+)_0 - \sum_{0<\lambda_n\le N}\lambda_n^{-j}| = O(N^j)$ if $j>d$, the infinite sums in (51) and (52) converge if $|m|<N$. On the other hand, we have

$$
Z'(D(+),N,0) = Z'(D(+),0) - \sum_{0<\lambda_n\le N}\log\lambda_n, \tag{54}
$$

$$
Z'(D(+)+m,\text{“}N\text{”},0) = Z'(D(+)+m,0) - \sum_{0<\lambda_n\le N}\log(\lambda_n+m). \tag{55}
$$

Hence we obtain for $|m|<N$,

$$
\begin{aligned}
Z'(D(+)+m,0) =& Z'(D(+),0) + \sum_{0<\lambda_n\le N}\log(1+(m/\lambda_n)) + \\
&+ \sum_{j=1}^{\infty}(-1)^j(1/j)(a(j,+)_0 - \sum_{0<\lambda_n\le N}\lambda_n^{-j})m^j + A(+,m),
\end{aligned} \tag{56}
$$

$$
\begin{aligned}
Z'(D(-)-m,0) =& Z'(D(-),0) + \sum_{0>\lambda_n\ge -N}\log(1-(m/|\lambda_n|)) + \\
&+ \sum_{j=1}^{\infty}(1/j)(a(j,-)_0 - \sum_{0>\lambda_n\ge N}|\lambda_n|^{-j})m^j + A(-,m).
\end{aligned} \tag{57}
$$

By (17), $\det(D+m) = (-1)^{(Z(|D-m|,0)-H(D-m,0))/2}\det(D(+)+m)\det(D(-)-m)$. Hence by (56) and (57), we have

$$\det(D+m) =$$
$$(-1)^{(Z(|D-m|,0)-H(D-m,0))/2}\det D\prod_{|\lambda_n|\le N}(1+(m/\lambda_n))\exp(A(+,m)+A(-,-m))$$
$$\exp\Big(\sum_{j=1}^{d}((-1)^j/j)\big(a(j,+)_0+(-1)a(j,-)_0-\sum_{|\lambda_n|\le N}|\lambda_n|^{-j}\big)m^j\Big). \tag{58}$$

Here we have

$$\begin{aligned}
&(Z(|D-m|,0)-H(D-m,0))/2\\
&=(Z(|D|,0)-H(D,0))/2-\big((\mathrm{Res}_{s=1}Z(|D|,s)-\mathrm{Res}_{s=1}H(D,s))/2\big)m+\cdots\\
&\qquad\cdots+(-1)^d\big(\mathrm{Res}_{s=d}Z(|D|,s)-\mathrm{Res}_{s=d}H(D,s)\big)/2(m^d/d)\\
&=Z(D(-),0)\cdots\mathrm{Res}_{s=1}Z(D(-),s)m+\cdots+(-1)^d\mathrm{Res}_{s=d}Z(D(-),s)(m^d/d),
\end{aligned}$$

$$\begin{aligned}
&A(+,m)+A(-,-m)\\
&=\sum_{j=2}^{d}(1/j)(1+\cdots+1/(j-1))(a(j,+)_{-1}+(-1)a(j,-)_{-1})m\\
&=(1/2)\mathrm{Res}_{s=2}Z(|D|,s)m^2+\cdots+(-1)^d(1/d)(1+\cdots+1/(d-1))b(d)m^d.
\end{aligned}$$

Equation (58) is valid only for $|m|<N$. But since

$$\begin{aligned}
\sum_{j=1}^{\infty}(-1)^{-1}(1/j)(\sum_{\lambda_n\le N}|\lambda_n|^{-j})m^j &= \sum_{\lambda_n\le N}(\sum_{j=1}^{\infty}(-1)^j(1/j)(m/|\lambda_n|)^j\\
&=-\sum_{\lambda_n\le N}\log(1+m/|\lambda_n|),
\end{aligned}$$

if $|m|<\min\{\lambda_n|\lambda_n>0\}$, we have on the same domain

$$\begin{aligned}
Z'(D(+)+m,0)=&Z'(D(+),0)-\sum_{\lambda_n\le N}\log(1+m/|\lambda_n|)\\
&+A(+,m)+\sum_{j=1}^{\infty}(-1)^j(1/j)a(j,+)_0m^j+\log(1+m/|\lambda_n|)\\
=&Z'(D(+),0)+A(+,m)+\sum_{j=1}^{\infty}(-1)^j(1/j)a(j,+)_jm^j.
\end{aligned}$$

Hence $\det(D(+)+m)$ is continued holomorphically to the whole complex plane. Similarly, $\det(D(-)+m)$ is continued holomorphically to the whole complex plane. Therefore the function

$$\text{“det”}(D+m) = (-1)^{\{Z(|D-m|,0)-H(D-m,0)\}/2}\det(D(+)+m)\det(D(-)-m) \tag{59}$$

is an entire function of m. "det"$(D+m)$ may differ by a sign from $\det(D+m)$ if $|m|$ is large, because the difference between $\det(D+m)$ and "det"$(D+m)$ comes from the relative sign between $\lambda_n - m$ and $m-\lambda_n$. So we may use "det"$(D+m)$ instead of $\det(D+m)$ ([1]). As a function of m, (59) is written

$$\text{“det”}(D+m) = (-1)^{Z(D(-)-m,0)} \prod_{|\lambda_n|\leq N} (1+m/\lambda_n)\exp(A(+,m)+A(-,m))\times$$
$$\times \exp(\sum_{j=1}^{\infty}(-1)^j(1/j)(a(j,+)_0+(-1)^j a(j,-)_0) - \sum_{|\lambda_n|\leq N}|\lambda_n|^{-j})m^j. \tag{60}$$

If the spectrum of D is symmetric, we have $\prod_{|\lambda_n|\leq N}(1+m/\lambda_n) = (1-m^2/\lambda_n^2)$ and $a(j,+)_0 = a(j,-)_0$. Hence we have

$$\begin{aligned} &Z(D(-)-m,0)\\ &= Z(D(-),0) - \mathrm{Res}_{s=1}Z(D(-),s)m + \cdots + (-1)^d \mathrm{Res}_{s=d}Z(D(-),s)(m^d/d)\\ &= \frac{1}{2}\big(Z(|D|,0) - \mathrm{Res}_{s=1}Z(|D|,s)m + \cdots + (-1)^d\mathrm{Res}_{s=d}Z(|D|,s)(m^d/d)\big), \end{aligned} \tag{61}$$

$$A(+,m)+A(-,m) = \sum_{j=2}^{d}\mathrm{Res}_{s=j}Z(|D|,s)(1/2j)(1+\cdots+1/(2j-1))m^j. \tag{62}$$

Since $Z(|D|,s) = Z(D^2,s/2)$, we have $\mathrm{Res}_{s=j}(|D|,s) = 2\mathrm{Res}_{s=j/2}(D^2,s)$, so Equations (61) and (62) become

$$\begin{aligned} Z(D(-)-m,0) = Z(D^2,0)/2 - \mathrm{Res}_{s=1/2}Z(D^2,s)m\\ \cdots + (-1)^d\mathrm{Res}_{s=d/2}Z(D^2,s)(m^2/d), \end{aligned}$$
$$A(+,m)+A(-,m) = \sum_{j=2}^{d}\mathrm{Res}_{s=d/2}Z(D^2,s)(1/j)(1+\cdots+1/(2j-1))m^j.$$

Hence (60) becomes

$$\begin{aligned} &\text{“det”}(D+m) =\\ &\exp\Big(\big(Z(D^2,0)/2 - \mathrm{Res}Z(D^2,s)m + \cdots + (-1)^d\mathrm{Res}Z(D^2,s)(m^d/d)\big)\pi i\Big)\times\\ &\times \prod_{|\lambda_n|\leq N}(1-m^2/\lambda_n^2)\exp\Big(\sum_{j=2}^{d}\mathrm{Res}_{s=d/2}Z(D^2,s)(1/j)(1+\cdots+1/(2j-1))m^j\Big)\times\\ &\times \exp\Big(\sum_{j=1}^{\infty}((-1)^j/j)(1+(-1)^j)a(j,+)_0 - 2\sum_{0<\lambda_n\leq N}\lambda_n^{-j})m^j\Big). \end{aligned}$$

Since this formula gives $\det(D+m)$ if $|m|$ is small, $\det(D+im)$ may different from $\det(D-im)$, by virtue of the presence of residues of $Z(D,s)$. It also shows that sym-det$(D+im)$ may be different from $\det(D+im)$ if $m\neq 0$.

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