Automorphic distributions for $SL(2,\mathbb{R})$

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Abstract. Classical (holomorphic) modular forms on the upper half plane have boundary values along the real axis, as do (non-holomorphic) Maass forms. These boundary values are distributions which transform according to a factor of automorphy under the action of the discrete group which leaves the modular forms invariant. The main result of this paper is a characterization of the degree of regularity of such automorphic distributions. The boundary values of Maass forms, for example, are first derivatives of continuous functions, which are Hölder continuous of index depending on the eigenvalue corresponding to the Maass form in question. The existence of a continuous anti-derivative makes it possible to prove the functional equation for L-functions attached to Maass forms by integration along the *real* axis, rather than the *imaginary* axis, as in the usual argument. This new approach to the functional equation appears likely to generalize to higher rank groups that are beyond the reach of the classical arguments.

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Dedicated to the memory of my friend Moshé Flato

1. Introduction

The automorphic distributions to which the title refers arise as boundary values of classical modular forms and of Maass modular forms. In this paper I shall address the question of how regular these distributions are – in other words, how close they come to being continuous functions. My answer to this question leads to new proofs of certain known results. The new arguments look simpler to me, in some cases, than the classical arguments; more importantly, it appears that they can be generalized to higher dimensional cases which are beyond the reach of the classical arguments.

To explain the main idea, I consider initially an arbitrary real reductive matrix group $G_\mathbb{R}$, a finitely generated, discrete subgroup $\Gamma \subset G_\mathbb{R}$, and an irreducible unitary representation (π, V) of $G_\mathbb{R}$. One calls $v \in V$ a C^∞ vector if $g \mapsto \pi(g)v$ is a C^∞ map from $G_\mathbb{R}$ to V. The space of all C^∞ vectors $V^\infty \subset V$ is dense, and carries a natural Fréchet topology; the representation π restricts to a continuous representation on this Fréchet space. There is also a notion of distribution vector: A continuous linear functional on $V'^{\,\infty}$, the space of C^∞ vectors for the irreducible unitary representation (π', V') dual to (π, V). Dualizing the dense embedding $V'^{\,\infty} \subset V'$,

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one obtains an embedding of V into its space of distribution vectors $V^{-\infty}$. Thus $V^{\infty} \subset V \subset V^{-\infty}$, and dually, $V'^{\infty} \subset V' \subset V'^{-\infty}$.

I now suppose that (π, V) occurs as a discrete summand in $L^2(\Gamma \backslash G_{\mathbb{R}})$. Then (π', V') also occurs as a direct summand, since $L^2(\Gamma \backslash G_{\mathbb{R}})$ is self-dual. The inclusion

$$i : V' \hookrightarrow L^2(\Gamma \backslash G_{\mathbb{R}}) \tag{1.1}$$

sends the subspace $V'^{\infty} \subset V'$ to C^{∞} functions, which can be evaluated at any $g \in G_{\mathbb{R}}$. The evaluation map

$$\tau : \; v' \mapsto i(v')(e) \qquad (v' \in V'^{\infty}) \tag{1.2}$$

is continuous with respect to the topology of V'^{∞}, and thus defines a distribution vector for (π, V) – a Γ-invariant distribution vector since $i(v')$, for $v' \in V'^{\infty}$, is a Γ-invariant function:

$$\tau \in (V^{-\infty})^{\Gamma}. \tag{1.3}$$

The distribution vector τ completely determines the inclusion (1.1). To see this, note that $i(v')(g) =< \tau, \pi'(g)v' >$ for every v' in the dense subspace $V'^{\infty} \subset V'$ and every $g \in G_{\mathbb{R}}$.

The preceeding discussion applies in particular to the group $G_{\mathbb{R}} = SL(2,\mathbb{R})$ and any finitely generated, discrete subgroup Γ. Then Γ acts properly discontinuously on the upper half plane $\mathscr{H}$. One knows that there is a natural bijection between non-zero cuspidal modular forms $F(z)$ of weight k on $\Gamma \backslash \mathscr{H}$ and embeddings of the discrete series representation of lowest weight k,

$$V_k \hookrightarrow L^2(\Gamma \backslash SL(2,\mathbb{R})); \tag{1.4}$$

in effect, the modular form $F(z)$ is obtained by projecting the image of (1.4) onto a lowest $SO(2)$-weight vector and multiplying the resulting function by $(\operatorname{Im} z)^{-\frac{k}{2}}$. In this situation, the limit

$$\tau(x) \;=\; \lim_{y \to 0^+} F(x+iy) \tag{1.5}$$

exists in the sense of distributions, and this distribution τ is a concrete realization of the distribution vector corresponding to the embedding (1.4) via the construction (1.2–3). My main technical result, specialized to the case at hand, asserts:

(1.6)

a) when the weight $k = 2\ell + 1$ is an odd integer, the distribution τ can be expressed as the $(\ell+1)$st derivative $\tau = \phi^{(\ell+1)}$ of a function ϕ which is Hölder continuous of index $\frac{1}{2}$;

b) when the weight $k = 2\ell$ is an even integer, the distribution τ can be expressed as the $(\ell+1)$st derivative $\tau = \phi^{(\ell+1)}$ of a function ϕ which is Hölder continuous of index α, for every $\alpha < 1$.

Recall that a function $\phi(x)$ is Hölder continuous of index α if there exists a constant $C > 0$ such that $|\phi(x) - \phi(y)| \leq C|x-y|^\alpha$ for all x, y; the weaker notion of local Hölder continuity of index α requires only that the constant C can be chosen in a locally uniform manner.

Next I consider a square-integrable Maass modular form on $\Gamma\backslash\mathscr{H}$ – in other words, a non-constant Γ-invariant eigenfunction $F(x,y)$ of the hyperbolic Laplace operator with finite L^2 norm:

$$
\begin{aligned}
y^2\left(\tfrac{\partial^2}{\partial x^2} + \tfrac{\partial^2}{\partial y^2}\right) F(x,y) \ &= \ \tfrac{1}{4}(\lambda^2 - 1)\, F(x,y)\,, \\
\int_{\Gamma\backslash\mathscr{H}} |F(x,y)|^2\, \frac{dx\,dy}{y^2} \ &< \ \infty .
\end{aligned}
\tag{1.7}
$$

Here λ is either purely imaginary or real, $-1 < \lambda < 1$, in both cases determined only up to sign. Square-integrable Maass modular forms correspond bijectively to embeddings

$$
V_{\lambda,+1} \ \hookrightarrow \ L^2(\Gamma\backslash SL(2,\mathbb{R}))
\tag{1.8}
$$

of an even principal series representation (when $\lambda \in i\mathbb{R}$) or a representation of the complementary series (when $\lambda \in \mathbb{R}$, $0 < |\lambda| < 1$); the subscript $+1$ refers to the even parity condition, which simply means that these representations involve only even $SO(2)$-weights. Near the real axis, the Maass modular form $F(x,y)$ has an asymptotic expansion

$$
F(x,y) \ \sim \ y^{\frac{1-\lambda}{2}} \textstyle\sum_{n\geq 0} \tilde{\tau}_{\lambda,n}(x)\, y^n + y^{\frac{1+\lambda}{2}} \textstyle\sum_{n\geq 0} \tilde{\tau}_{-\lambda,n}(x)\, y^n\,,
\tag{1.9}
$$

with distribution coefficients $\tilde{\tau}_{\pm\lambda,n}$. The tilde in the notation $\tilde{\tau}_{\pm\lambda,n}$ has no particular significance for now; it simply makes the notation consistent with that in the later sections. The leading coefficients $\tilde{\tau}_{\pm\lambda,0}$ play the role of τ in the construction (1.2-3) for the embedding (1.8) – both represent the same distribution vector, but in the two different realizations $V_{\pm\lambda,+1}$ of the same abstract representation. In this situation, the main technical result asserts the following regularity property of the distributions $\tilde{\tau}_{\pm\lambda,0}$:

(1.10)

a) when the embedding (1.8) is cuspidal, $\tilde{\tau}_{\pm\lambda,0}$ can be expressed as the first derivative of a function ϕ which is Hölder continuous of index $\frac{1}{2}(1 \pm \mathrm{Re}\lambda)$;

b) when the embedding fails to be cuspidal, $\tilde{\tau}_{\lambda,0}$ with $\lambda > 0$ can be expressed as the first derivative of a function ϕ which is Hölder continuous of index λ, and $\tilde{\tau}_{-\lambda,0}$ as the second derivative of a function which is Hölder continuous of index α, for every $\alpha < 1$.

One calls the embedding (1.8) cuspidal when the $K_{\mathbb{R}}$-finite functions in its image are bounded. This condition holds automatically when λ is imaginary or Γ cocompact, or when Γ contains a principal congruence subgroup of $SL(2,\mathbb{Z})$.

This still leaves the case of the trivial representation, about which there is not much to say, and representations of the odd principal series. The latter always occur cuspidally in $L^2(\Gamma\backslash SL(2,\mathbb{R}))$. There are again two realizations of the distribution τ, just as in the case of the even principal series, and they can be expressed as first derivatives of functions that are Hölder continuous of index $\frac{1}{2}$. My main technical result, theorem 2.18 below, is a statement that uniformly covers all cases.

The boundary values $\tilde{\tau}_{\lambda,0}$ of Maass modular forms turn out to have intriguing properties. According to (1.10), they are first derivatives $\tilde{\tau}_{\lambda,0} = \phi_\lambda'$ of continuous functions ϕ_λ, except in the single case of a non-cuspidal Maass form and the negative choice among the two parameters $\pm\lambda$. The opposite extreme is that of a cuspidal form, with $\lambda > 0$ and $\Gamma \subset SL(2,\mathbb{Z})$ a subgroup of finite index. In this situation, the anti-derivative ϕ_λ is differentiable, with zero derivative, at all rational points; near rational points $\frac{p}{q}$, the difference $\phi_\lambda(x) - \phi_\lambda(\frac{p}{q})$ oscillates, with amplitude proportional to $|qx-p|^{\lambda+1}$ and frequency proportional to $|qx-p|^{-2}$. In particular, when ϕ_λ is real valued, as may be assumed, it assumes an infinite number of local maxima and local minima in every non-empty open interval. A conjecture of Selberg predicts that Maass forms with $\lambda > 0$ do not exist when Γ contains one of the principal congruence subgroups Γ_m, but such forms – even cuspidal Maass forms – do exist for certain other subgroups $\Gamma \subset SL(2,\mathbb{Z})$ of finite index. The wild behavior of the ϕ_λ can be used to establish Selberg's conjecture for small values of m. These matters will be explored in section 3.

Maass [10] attaches L-functions to Maass forms, and proves a functional equation for them. His proof of the functional equation follows the classical pattern, namely taking the Mellin transform along the imaginary axis and exploiting the transformation property under $z \mapsto -\frac{1}{z}$. To get the explicit form of the equation, Maass works with the analogue, involving Bessel functions, of the q-expansion of classical modular forms. The continuity of the anti-derivative ϕ_λ of the boundary distribution $\tilde{\tau}_{\lambda,0}$ can be used to give a different derivation of the functional equation: Instead of integrating the Maass form along the *imaginary* axis, one can take the Mellin transform of ϕ_λ along the *real* axis, and use the integrated transformation law for $\tilde{\tau}_{\lambda,0}$ under the transformation $x \mapsto -\frac{1}{x}$ of the real line. I shall present the details in section 4. In my approach the expansion of the Maass modular form on the upper half plane, in terms of Bessel functions, plays no role. Instead, the argument uses the interplay between Fourier analysis on the real line and the action of the Weyl group. What makes this interesting is the fact that the real axis is the specialization to $SL(2,\mathbb{R})$ of a maximal unipotent subgroup of a reductive group of arbitrary rank, whereas the imaginary axis is the analogue of a maximal split torus. One expects functional equations of automorphic representations to depend on an integration process over a space larger than a maximal split torus – a space of the same dimension, in fact, as a maximal unipotent subgroup. There are some other hints that the strategy of my argument does work in higher rank cases; I shall pursue this idea elsewhere.

The central question of this paper – the regularity of automorphic distributions – was also considered by Bernstein-Reznikov [3, 4], from a somewhat different point of view: They describe the degree of regularity in terms of a Sobolev notion based on the Hilbert space structure of the representation space V. I shall discuss the precise connection between their results and mine following the statement of the main theorem in the next section.

The results of this paper are several years old. I reported on them publicly in the summer of 1997. At that time my results were somewhat less general – they applied only to subgroups $\Gamma \subset SL(2,\mathbb{Z})$ of finite index – and the proofs I knew then were more intricate.

It is a pleasure to acknowledge my indebtedness to several colleagues. Curt McMullen suggested the appropriate setting for my arguments, namely finitely generated discrete subgroups of $SL(2,\mathbb{R})$. Gerald Folland answered my questions about the history of Serge Bernstein's theorem on the growth of Fourier coefficients of functions satisfying a Hölder condition. Peter Sarnak explained to me the existence of cuspidal Maass forms corresponding to small eigenvalues, for subgroups $\Gamma \subset SL(2,\mathbb{Z})$ not covered by Selberg's conjecture. Steve Miller alerted me to the special role played by Γ_2, the principal congruence subgroup of level 2. Dick Gross, finally, introduced me to the literature on functional equations for L-functions attached to automorphic representations of higher rank groups.

2. The main technical result

I begin by establishing notation that will be used throughout the paper. I always denote the ambient group by $G_\mathbb{R}$ and its maximal compact subgroup by $K_\mathbb{R}$:

$$G_\mathbb{R} \;=\; SL(2,\mathbb{R})\,, \qquad\qquad K_\mathbb{R} \;=\; SO(2)\,. \tag{2.1}$$

As in the introduction, I fix a subgroup Γ,

$$\Gamma \subset G_\mathbb{R} \qquad \text{(finitely generated discrete subgroup)}, \tag{2.2}$$

a cocompact discrete subgroup, for example, or an arithmetically defined subgroup. Formally $\Gamma = \{e\}$ is allowed, but the results will be essentially obvious in this extreme case. Other subgroups that will be important:

$$\begin{aligned}
N_\mathbb{R} \;&=\; \{n_x \mid x \in \mathbb{R}\}\,, \qquad && n_x \;=\; \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \\
N^-_\mathbb{R} \;&=\; \{n^-_y \mid y \in \mathbb{R}\}\,, \qquad && n^-_y \;=\; \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \\
A_\mathbb{R} \;&=\; \{a_t \mid t \in \mathbb{R}\}\,, \qquad && a_t \;=\; \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.
\end{aligned} \tag{2.3}$$

The group $A_{\mathbb{R}}$ normalizes both $N_{\mathbb{R}}$ and $N_{\mathbb{R}}^-$, and

$$G_{\mathbb{R}} \;=\; K_{\mathbb{R}} \cdot A_{\mathbb{R}} \cdot N_{\mathbb{R}}^- \tag{2.4}$$

is an Iwasawa decomposition of $G_{\mathbb{R}}$. I denote the Lie algebras of these groups by the corresponding German letters with the subscript $\mathbb{R}$, such as $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{k}_{\mathbb{R}}$, and the complexified Lie algebras by the same letters without the subscript $\mathbb{R}$.

Principal series representations of $G_{\mathbb{R}}$, unitary or not, are parametrized by a complex number and a parity. Specifically, for $\lambda \in \mathbb{C}$ and $\varepsilon \in \{\pm 1\}$, I define

$$\begin{aligned} V^{\infty}_{\lambda,\varepsilon} \;=\; &\text{space of all } f \in C^{\infty}(G_{\mathbb{R}}) \text{ such that} \\ &f((-1_{2\times 2})\, g\, a_t\, n^-) \;=\; \varepsilon\, e^{-(\lambda-1)t} f(g)\,, \\ &\qquad\qquad \text{for all } \; g \in G_{\mathbb{R}},\ t \in \mathbb{R},\ n^- \in N_{\mathbb{R}}^-. \end{aligned} \tag{2.5}$$

The group $G_{\mathbb{R}}$ acts by left translation since $-1_{2\times 2}$ lies in the center of $G_{\mathbb{R}}$:

$$\left(\pi_{\lambda,\varepsilon}(h) f\right)(g) \;=\; f(h^{-1}g)\,, \;\; \text{for } f \in V^{\infty}_{\lambda,\varepsilon}\,,\; g,h \in G_{\mathbb{R}}. \tag{2.6}$$

Because of (2.4), restriction to $K_{\mathbb{R}}$ induces an isomorphism of $K_{\mathbb{R}}$-representations

$$V^{\infty}_{\lambda,\varepsilon} \;\simeq\; \{\, f \in C^{\infty}(K_{\mathbb{R}}) \;\mid\; f((-1_{2\times 2})\, k) \,\equiv\, \varepsilon\, f(k)\,\}. \tag{2.7}$$

Via the usual identification $\mathbb{R} \cup \{\infty\} \simeq G_{\mathbb{R}} / \left((\pm 1_{2\times 2}) A_{\mathbb{R}} N_{\mathbb{R}}^-\right)$, the upper triangular group N acts freely on $\mathbb{R} \simeq N$-orbit through the identity coset. Thus restriction from $G_{\mathbb{R}}$ to $N \simeq \mathbb{R}$ identifies $V^{\infty}_{\lambda,\varepsilon}$ with a subspace of $C^{\infty}(\mathbb{R})$, namely

$$V^{\infty}_{\lambda,\varepsilon} \;\simeq\; \{\, \phi \in C^{\infty}(\mathbb{R}) \;\mid\; (\operatorname{sgn} x)^{\frac{1-\varepsilon}{2}} |x|^{\lambda-1} \phi(\tfrac{1}{x}) \in C^{\infty}(\mathbb{R})\,\}\,; \tag{2.8}$$

to derive the particular form of the condition at infinity, one factors $n_x \in N$ as an Iwasawa product $n_x = k\, a_t\, n^-$ and appeals to the definition (2.5). In terms of the isomorphism (2.8), the action (2.6) can be written as follows:

$$\begin{aligned} \left(\pi_{\lambda,\varepsilon}(g)\phi\right)(x) \;&=\; (\operatorname{sgn}(cx+d))^{\frac{1-\varepsilon}{2}} |cx+d|^{\lambda-1} \phi(\tfrac{ax+b}{cx+d}) \\ \text{when} \quad g^{-1} \;&=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\mathbb{R}}\,. \end{aligned} \tag{2.9}$$

This, too, can be deduced from (2.5) when one factors $g^{-1} n_x$ as an Iwasawa product.

The space $V^{\infty}_{\lambda,\varepsilon}$ coincides with the space of C^{∞} vectors in any Banach realization of the underlying Harish-Chandra module – in effect, this is the specialization to $G_{\mathbb{R}}$ of a general result of Casselman-Wallach [6]. Dually the space of distribution vectors $V^{-\infty}_{\lambda,\varepsilon}$ corresponds, via restriction to $K_{\mathbb{R}}$, to the space of distributions on $K_{\mathbb{R}}$ that satisfy the parity condition specified by ε. I use the word "distribution" to refer to objects *dual to compactly supported smooth measures*, so

(2.10) distributions behave like functions under diffeomorphisms;

in particular, functions are distributions. With this convention, (2.5–9) remain valid when one replaces "C^∞ function" by "distribution" and the superscript ∞ by $-\infty$. Integration over $K_\mathbb{R}$ sets up a $G_\mathbb{R}$-invariant, nondegenerate pairing between $V^\infty_{\lambda,\varepsilon}$ and $V^\infty_{-\lambda,\varepsilon}$, hence

$$(2.11) \qquad V^{-\infty}_{\lambda,\varepsilon} \text{ is the continuous dual of } V^\infty_{-\lambda,\varepsilon}\,.$$

When $V^\infty_{\lambda,\varepsilon}$ is irreducible – equivalently, when $\lambda \notin 2\mathbb{Z} + \frac{1}{2}(\varepsilon+1)$ – the two $G_\mathbb{R}$-modules $V^\infty_{\lambda,\varepsilon}$ and $V^\infty_{-\lambda,\varepsilon}$ are isomorphic via the canonical intertwining operator; in that case, $V^{-\infty}_{\lambda,\varepsilon}$ is also isomorphic to the continuous dual of $V^\infty_{\lambda,\varepsilon}$ as a topological $G_\mathbb{R}$-module. However, the duality between $V^\infty_{\lambda,\varepsilon}$ and $V^{-\infty}_{\lambda,\varepsilon}$ involves the intertwining operator and is *not* given by the integration pairing over $K_\mathbb{R}$.

The statement of the main theorem involves a notion of fractional differentiability. To motivate the definition, I recall a classical theorem of Serge Bernstein [2]; for a modern account, see for example Folland [9]. Let ϕ be a continuous function on $\mathbb{R}/\mathbb{Z}$, with Fourier series

$$(2.12) \qquad \phi(x) \;=\; \textstyle\sum_{-\infty<k<\infty} a_k\, e^{2\pi i k x}\,.$$

When ϕ is Hölder continuous of index α, $0<\alpha\le 1$,

$$(2.13) \qquad \begin{array}{lll} \text{a)}\;\; s<2\alpha & \Longrightarrow & \sum_{k\neq 0} |k|^s\,|a_k|^2 \;<\; \infty \\ \text{b)}\;\; s<\alpha-\frac{1}{2} & \Longrightarrow & \sum_{k\neq 0} |k|^s\,|a_k| \;<\; \infty\,. \end{array}$$

Thus, when α is strictly greater than $\frac{1}{2}$, the Fourier series of ϕ converges absolutely. This latter statement is usually called Bernstein's theorem, but the proof readily extends to give the more general statement above. In view of (2.13a), Hölder continuity can be regarded as the analogue, in the context of continuous functions, of the L^2 notion of Sobolev regularity. The Hölder notion provides a finer measure of regularity than the Sobolev notion: Hölder continuity of any index $\alpha > \frac{1}{2}$, for example, implies finiteness of the Sobolev-$\frac{1}{2}$ norm; but conversely, it requires finiteness of the Sobolev-1 norm – i.e., the existence of an L^2 derivative – to ensure Hölder continuity of index $\frac{1}{2}$.

2.14 Definition. *A distribution τ, defined on an open interval $I\subset\mathbb{R}$ or on the circle $\mathbb{R}/\mathbb{Z}$, is of class C^α, $\alpha\in\mathbb{R}$, provided*
i) for $\alpha=k$ a non-negative integer, τ is a C^k function in the usual sense;
ii) for $\alpha=-k$ a negative integer, $\tau=\phi^{(k)}$ is the k-th derivative of a continuous function ϕ;
iii) for $\alpha=k+\beta$ with k non-negative integral and $0<\beta<1$, τ has k continuous derivatives, and the k-th derivative $\tau^{(k)}$ is locally Hölder continuous of index β;
iv) for $\alpha=-k+\beta$ with k positive integral and $0<\beta<1$, $\tau=\phi^{(k)}$ is the k-th derivative of a function ϕ that is locally Hölder continuous of index β.
By convention, $C^{<\alpha}(I)$ denotes the intersection $\bigcap_{\beta<\alpha} C^\beta(I)$.

In several respects, this notion of fractional differentiability behaves like the usual notion of differentiability; in particular,

(2.15)

a) $C^\alpha(I) \subset C^{<\alpha}(I) \subset C^\beta(I)$ if $\alpha > \beta$;

b) $\tau \in C^\alpha(I) \implies \tau' \in C^{\alpha-1}(I)$;

c) the implications (2.13a,b) remain valid for any $\alpha \in \mathbb{R}$ and any $\phi \in C^\alpha(\mathbb{R}/\mathbb{Z})$;

d) $\phi \in C^\infty(I)\,,\ \tau \in C^\alpha(I) \implies \phi\,\tau \in C^\alpha(I)$;

e) $F : I \xrightarrow{\sim} J$ diffeomorphically, $\tau \in C^\alpha(J) \implies F^*\tau \in C^\alpha(I)$.

Statements a)-c) hold for essentially formal reasons, as do d) and e) when $\alpha \geq 0$. For the general version of d) and e), one can argue by downward induction, using Leibniz' rule.

Now let (π, V) be an irreducible unitary representation of $G_\mathbb{R}$ which occurs as a direct summand in $L^2(\Gamma\backslash G_\mathbb{R})$. I define the Γ-invariant distribution vector $\tau \in (V^{-\infty})^\Gamma$ as in (1.1–3). Like any irreducible unitary representation, one can realize π as a subrepresentation of a not-necessarily-unitary principal series representation $\pi_{\lambda,\varepsilon}$, and correspondingly the dual representation π' as a quotient of the dual principal series representation $\pi_{-\lambda,\varepsilon}$. On the level of distribution vectors, $V^{-\infty} \hookrightarrow V^{-\infty}_{\lambda,\varepsilon}$, for some appropriately chosen $\lambda \in \mathbb{C}$, $\varepsilon \in \{\pm 1\}$. Thus

$$(V^{-\infty})^\Gamma \ni \tau \ \mapsto\ \tau_\lambda \in (V^{-\infty}_{\lambda,\varepsilon})^\Gamma. \tag{2.16}$$

Via the distribution analogue of (2.7), τ now determines a distribution on the circle:

$$\tau_\lambda \in C^{-\infty}(\mathbb{R}/\mathbb{Z})\,. \tag{2.17}$$

2.18 Theorem. *a) If the embedding (1.1) is cuspidal and λ not an odd integer, the distribution τ_λ lies in $C^{\frac{1}{2}(\mathrm{Re}\,\lambda-1)}(\mathbb{R}/\mathbb{Z})$.*
b) If $\lambda \in 2\mathbb{Z}+1$, in which case the embedding is cuspidal, τ_λ lies in $C^{<\frac{1}{2}(\lambda-1)}(\mathbb{R}/\mathbb{Z})$.
c) If the embedding is not cuspidal, τ_λ lies in $C^{\lambda-1}(\mathbb{R}/\mathbb{Z})$ or in $C^{<(-1)}(\mathbb{R}/\mathbb{Z})$, depending on whether $0 < \lambda < 1$ or $-1 < \lambda < 0$.

The distribution vector $\tau \in (V^{-\infty})^\Gamma$ corresponds also to a distribution $\tilde\tau_\lambda \in C^{-\infty}(\mathbb{R})$, via the distribution analogue of the isomorphism (2.8). Going from the "compact realization" (2.7) of $V^\infty_{\lambda,\varepsilon}$ to the "unbounded realization" (2.8) involves the coordinate change $x = \tan\theta$ and multiplication by a C^∞ function. The same process – on the level of distributions – relates $\tilde\tau_\lambda$ to τ_λ. Thus, in view of (2.15), the theorem implies also the analogous statement about $\tilde\tau$:

2.19 Corollary. *The distribution $\tilde{\tau}_\lambda$ corresponding to a cuspidal embedding lies in $C^{\frac{1}{2}(\mathrm{Re}\,\lambda-1)}(\mathbb{R})$ when $\lambda \notin 2\mathbb{Z}+1$, and in $C^{<\frac{1}{2}(\lambda-1)}(\mathbb{R})$ when $\lambda \in 2\mathbb{Z}+1$. If the embedding is non-cuspidal and $0<\lambda<1$, then $\tilde{\tau}_\lambda \in C^{\lambda-1}(\mathbb{R})$ and $\tilde{\tau}_{-\lambda} \in C^{<(-1)}(\mathbb{R})$.*

Bernstein-Reznikov [3, 4] show that the distribution vector $\tau \in V^\infty$ corresponding to a cuspidal embedding $V' \hookrightarrow L^2(\Gamma\backslash G_\mathbb{R})$ lies in the Sobolev-α space for any $\alpha < -\frac{1}{2}$, relative to the Sobolev notion which they define in terms of the unitary structure of the Hilbert space V. When (π, V) belongs to the unitary principal series, realized as in (2.7), Bernstein-Reznikov's Sobolev notion agrees precisely with the usual Sobolev notion on the circle group $K_\mathbb{R}$. In this situation, part a) of theorem 2.18, coupled with Serge Bernstein's result (2.13a), directly implies Bernstein-Reznikov's regularity statement. For other unitary representations, theorem 2.18 and (2.13a) also imply the Bernstein-Reznikov result, but the translation process requires pinning down the relationship of the unitary structure of V and the L^2 structure of $K_\mathbb{R}$, via the embedding $V^{-\infty} \hookrightarrow V^{-\infty}_{\lambda,\varepsilon}$ that relates τ to τ_λ.

To what extent does theorem 2.18 have a converse? If $\pi_{\lambda,\varepsilon}$ is a unitary principal series representation and f_v the function on $\mathbb{R}/\mathbb{Z}$ corresponding to a vector $v \in V_{\lambda,\varepsilon}$ in the (unitary) representation space via (2.7), then $f_v \in L^2(\mathbb{R}/\mathbb{Z})$. The indefinite integral of an L^2 function is Hölder continuous of index $\frac{1}{2}$, as follows readily from the Schwarz inequality. This makes f_v a distribution of class $C^{\frac{1}{2}(\mathrm{Re}\,\lambda-1)}$, since $\lambda \in i\mathbb{R}$ by assumption. No principal series representation occurs discretely in $L^2(G_\mathbb{R})$, so the converse of theorem 2.18 fails for $\Gamma = \{e\}$ – but not by much, since $\pi_{\lambda,\varepsilon}$ does occur continuously. The failure reflects the fact that the trivial subgroup is "too small". To make this precise, I consider an arbitrary irreducible unitary representation (π, V) and a non-zero, Γ-invariant distribution vector $\tau \in (V^{-\infty})^\Gamma$. As before, I embed V^∞ into the space of C^∞ vectors of a (not necessarily unitary) principal series representation $V_{\lambda,\varepsilon}$, and let $\tau_\lambda \in C^{-\infty}(\mathbb{R}/\mathbb{Z})$ denote the image of τ via $V^{-\infty} \hookrightarrow V^{-\infty}_{\lambda,\varepsilon}$ and the distribution analogue of the isomorphism (2.7).

2.20 Theorem. *Let Γ be a finitely generated, discrete subgroup of finite co-volume.*
a) If $\tau_\lambda \in C^{<\frac{1}{2}(\mathrm{Re}\,\lambda-1)}(\mathbb{R}/\mathbb{Z})$, the distribution vector τ arises from a cuspidal embedding $V' \hookrightarrow L^2(\Gamma\backslash G_\mathbb{R})$ as in (1.1–3).
b) For $0<\lambda<1$, τ corresponds to a possibly non-cuspidal embedding $V' \hookrightarrow L^2(\Gamma\backslash G_\mathbb{R})$ provided $\tau_\lambda \in C^{<(\lambda-1)}(\mathbb{R}/\mathbb{Z})$ or $\tau_{-\lambda} \in C^{<(-1)}(\mathbb{R}/\mathbb{Z})$.

Note that $\tau_\lambda \in C^{<\alpha}(\mathbb{R}/\mathbb{Z})$ is a weaker condition than $\tau_\lambda \in C^{\alpha}(\mathbb{R}/\mathbb{Z})$; in fact, the proof will show that there is even more room to spare between the conclusion of theorem 2.18 and the hypotheses of theorem 2.20.

I shall take up a number of consequences of theorem 2.18, before turning to its proof in section 5, along with that of its converse 2.20.

3. Maass forms

Any non-zero, square-integrable Maass form with eigenvalue $\frac{1}{4}(1-\lambda^2)$ corresponds to an embedding of the even principal series or complementary series representation $V_{\lambda,+1}$ into $L^2(\Gamma\backslash G_{\mathbb{R}})$, and thus to a Γ-invariant distribution vector $\tau_\lambda \in V^{-\infty}_{\lambda,+1}$. I let $\tilde{\tau}_\lambda \in C^{-\infty}(\mathbb{R})$ denote the distribution obtained from τ_λ via the distribution analogue of the isomorphism (2.8). The proof of theorem 2.18 will show that $\tilde{\tau}_\lambda$ coincides with the distribution $\tilde{\tau}_{\lambda,0}$ in (1.9). I can assume

$$\text{(3.1)} \qquad \begin{aligned} &\text{either} \quad \text{a) the embedding } V_{\lambda,+1} \hookrightarrow L^2(\Gamma\backslash G_{\mathbb{R}}) \text{ is cuspidal and } |\operatorname{Re}\lambda| < 1 \\ &\qquad\text{or} \quad \text{b) } \lambda \in \mathbb{R} \text{ and } 0 < \lambda < 1, \end{aligned}$$

without essential loss of generality: Because $V_{\lambda,+1} \cong V_{-\lambda,+1}$, there are two distribution vectors τ_λ and $\tau_{-\lambda}$ – except when $\lambda = 0$ – but they are related by the standard intertwining operator; in the non-cuspidal case, λ is necessarily real, of absolute value strictly between 0 and 1, so b) amounts to a definite choice between τ_λ and $\tau_{-\lambda}$. In any case, for representations of the unitary principal series or complementary series, $\operatorname{Re}\lambda$ is strictly bounded by 1. The reason for the assumption (3.1) is that it implies the existence of a continuous first anti-derivative for $\tilde{\tau}_\lambda$,

$$\text{(3.2)} \qquad \tilde{\tau}_\lambda = \phi'_\tau, \qquad \phi_\tau \in C^0(\mathbb{R}),$$

whether or not the embedding is cuspidal – see corollary 2.19. The function ϕ_τ is determined only up to an additive constant; however, in the most important special cases, there is a canonical choice, as we shall see.

The transformation law (2.9) for the action of $G_{\mathbb{R}}$ on the even principal series representation $V_{\lambda,+1}$ extends naturally to

$$\text{(3.3)} \qquad G^{\pm}_{\mathbb{R}} = SL^{\pm}(2,\mathbb{R}) = \{ g \in GL(2,\mathbb{R}) \mid \det g = \pm 1 \},$$

which normalizes $G_{\mathbb{R}}$. In fact, $PGL(2,\mathbb{R}) \cong SL^{\pm}(2,\mathbb{R})/\{\pm 1_{2\times 2}\}$ acts on $V_{\lambda,+1}$, but many of the interesting discrete groups Γ lie in $SL^{\pm}(2,\mathbb{R})$ rather than $PGL(2,\mathbb{R})$. The transformation law applies in particular to the distribution $\tilde{\tau}_\lambda$, except at the singularity of the factor of automorphy. To simplify the notation, I write the action of $G^{\pm}_{\mathbb{R}}$ on $V_{\lambda,+1}$ by juxtaposition. With this convention,

$$\text{(3.4)} \qquad \begin{aligned} &(\widetilde{g\tau})_\lambda(x) = |cx+d|^{\lambda-1}\,\tilde{\tau}_\lambda\!\left(\tfrac{ax+b}{cx+d}\right) \qquad \left(x \neq -\tfrac{d}{c}\right) \\ &\qquad\qquad\qquad \text{when } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G^{\pm}_{\mathbb{R}}. \end{aligned}$$

The distribution vector $g\tau \in V^{-\infty}_{\lambda,+1}$ corresponds to an embedding of $V_{\lambda,+1}$ into $L^2(\Gamma^g\backslash G_{\mathbb{R}})$, with $\Gamma^g = g\Gamma g^{-1}$. It is therefore of the same type as τ, so

$$\text{(3.5)} \qquad g\tilde{\tau}_\lambda = \phi'_{g\tau}, \qquad \phi_{g\tau} \in C^0(\mathbb{R}).$$

When the transformation law (3.4) is integrated, it takes the form

$$\begin{aligned}\phi_{g\tau}(x) \;&=\; \det g\,|cx+d|^{\lambda+1}\,\phi_\tau\!\left(\tfrac{ax+b}{cx+d}\right) \;+ \\ &+\; (\lambda+1)\; c\,\mathrm{sgn}(cx+d)\,\det g \int^{\frac{ax+b}{cx+d}} |a-ct|^{-\lambda-2}\,\phi_\tau(t)\,dt \;+\; \text{const.}\,;\end{aligned} \tag{3.6}$$

here $g \in G^{\pm}_{\mathbb{R}}$ is the same matrix as in (3.4). What matters here is the fact that the coordinate change $t = \frac{ax+b}{cx+d}$ has derivative $\det g\,(cx+d)^{-2} = \det g\,(a-ct)^2$.

The equation (3.6) can be made more explicit for certain groups Γ, in particular for subgroups of finite index in $SL(2,\mathbb{Z})$. Any such subgroup contains another subgroup of finite index which is normal in $SL^{\pm}(2,\mathbb{Z})$. Thus, for the remainder of this section, I shall assume

$$\Gamma \subset SL^{\pm}(2,\mathbb{Z}) \quad \text{is a normal subgroup of finite index}. \tag{3.7}$$

The quotient group $SL^{\pm}(2,\mathbb{Z})/\Gamma$ acts on the spaces

$$\begin{aligned}&\{\,\tau \in (V^{-\infty}_{\lambda,+1})^{\Gamma} \mid \tau_\lambda \in C^{\frac12(\mathrm{Re}\lambda-1)}(\mathbb{R}/\mathbb{Z})\,\} \qquad &&(\,\lambda \text{ arbitrary}\,)\\ &\{\,\tau \in (V^{-\infty}_{\lambda,+1})^{\Gamma} \mid \tau_\lambda \in C^{\lambda}(\mathbb{R}/\mathbb{Z})\,\} &&(\,0<\lambda<1\,)\,;\end{aligned} \tag{3.8}$$

cf. (3.1). According to theorem 2.20, every non-zero τ in either of these two spaces corresponds to an embedding $V_{\lambda,+1} \hookrightarrow L^2(\Gamma\backslash G^{\pm}_{\mathbb{R}})$.

The cyclic group $\{n_k \mid k \in \mathbb{Z}\}$, in the notation of (2.3), maps to a – necessarily finite – subgroup of $SL^{\pm}(2,\mathbb{Z})/\Gamma$, so there exists a least positive integer m such that $n_m \in SL^{\pm}(2,\mathbb{Z})$. This makes the distribution $\tilde\tau_\lambda$ periodic of period m. It therefore has a Fourier series

$$\tilde\tau_\lambda(x) \;=\; \textstyle\sum_{-\infty<k<\infty} a_k\, e^{2\pi i \frac{k}{m} x}\,, \tag{3.9}$$

which can be integrated term-by-term:

$$\phi_\tau(x) \;=\; a_0\, x \;+\; \textstyle\sum_{k\neq 0} \frac{m}{2\pi i k}\, a_k\, e^{2\pi i \frac{k}{m} x}\,. \tag{3.10}$$

The coefficient $a_0 = a_{\tau,0}$ has special significance:

$$\begin{gathered}\text{the embedding } V_{\lambda,+1} \hookrightarrow L^2(\Gamma\backslash G^{\pm}_{\mathbb{R}}) \text{ defined by } \tau \text{ is cuspidal}\\ \text{if and only if } a_{\gamma\tau,0} = 0\,, \text{ for all } \gamma \in SL(2,\mathbb{Z}).\end{gathered} \tag{3.11}$$

Indeed, because of (3.7), all the cusps of Γ occur at rational points or infinity – in other words, at points of the $SL(2,\mathbb{Z})$-orbit of infinity. We shall see that $a_{\tau,0} = 0$ is the cuspidality condition at ∞; see lemma 5.52 and the proof of lemma 5.23, in particular, (5.29). Thus $a_{\gamma,0} = 0$ for all $\gamma \in SL(2,\mathbb{Z})$ precisely when the cuspidality condition is satisfied at every cusp.

I shall take a more detailed look at the cuspidal case. The antiderivatives $\phi_{\gamma\tau}$ of the various $SL^{\pm}(2,\mathbb{Z})$-translates $\tilde{\gamma}\tau$ are then periodic with vanishing constant term,

$$\phi_{\gamma\tau}(x) \;=\; \textstyle\sum_{k\neq 0} \frac{m}{2\pi i k}\, a_{\gamma\tau,k}\, e^{2\pi i \frac{k}{m} x} \qquad (\,\gamma \in SL^{\pm}(2,\mathbb{Z})\,)\,. \tag{3.12}$$

By the very definition of $V_{\lambda,+1}$, $\{\pm 1_{2\times 2}\}$= center of $G^{\pm}_{\mathbb{R}}$ acts trivially. Since the square of the matrix

$$w \;=_{\text{def}}\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.13}$$

is the negative of the identity, $w^2\tau$ equals τ. Thus τ and $w\tau$ play symmetric roles. I claim

$$\phi_\tau(x) \,-\, \phi_\tau(0) \;=\; \phi_{w\tau}(-\tfrac{1}{x})\,|x|^{\lambda+1} \,-\, (\lambda+1)\int_{\frac{1}{|x|}}^{+\infty} t^{-\lambda-2}\,\phi_{w\tau}(-\operatorname{sgn}(x)\,t)\,dt\,. \tag{3.14}$$

To verify this identity, note that the integral converges and that both sides coincide when $x=0$. One can then appeal to (3.6), or directly to (3.4) by differentiating both sides. At first glance, the two terms on the right have the same order of magnitude. However,

$$(\lambda+1)\int_{\frac{1}{|x|}}^{+\infty} t^{-\lambda-2}\,\phi_{w\tau}(-\operatorname{sgn}(x)\,t)\,dt \;=\; O(|x|^{\operatorname{Re}\lambda+2}) \qquad \text{as } x\to 0\,, \tag{3.15}$$

because $\phi_{w\tau}$ is periodic with zero constant term. The leading term on the right oscillates, with amplitude and frequency proportional to $|x|^{\operatorname{Re}\lambda+1}$ and $|x|^{-2}$, respectively. This implies

$$\limsup_{a,b\to 0,a\neq b}\left(|\phi_\tau(a)-\phi_\tau(b)|\,|a-b|^{-\frac{\operatorname{Re}\lambda+1}{2}}\right) \;>\; 0\,. \tag{3.16}$$

In particular, ϕ_τ cannot be Hölder continuous of index strictly greater than $\frac{\operatorname{Re}\lambda+1}{2}$, showing that (2.18a) is sharp.

For $j>0$, let $\phi_\tau^{(-j)}$ denote the j-th anti-derivative of ϕ_τ, normalized by the requirement that all the $\phi_\tau^{(-j)}$ shall be periodic with zero constant term. Repeated integration by parts in (3.14) results in the identity

(3.17)
$$\begin{aligned} \phi_\tau(x) \,-\, \phi_\tau(0) \;=\; &\textstyle\sum_{j=0}^{n}(-\operatorname{sgn}(x))^j \left(\prod_{1\le i\le j}(\lambda+i)\right) \phi_{w\tau}^{(-j)}(-\tfrac{1}{x})\,|x|^{\lambda+j+1} \\ &-\,(-\operatorname{sgn}(x))^n \left(\textstyle\prod_{0\le i\le n}(\lambda+i+1)\right)\int_{\frac{1}{|x|}}^{+\infty} t^{-\lambda-n-2}\,\phi_{w\tau}^{(-n)}(-\operatorname{sgn}(x)\,t)\,dt\,. \end{aligned}$$

Here, too, the last term has strictly lower order of magnitude than the others. In effect, (3.17) can be regarded as an asymptotic expansion of $\phi_\tau(x)-\phi_\tau(0)$ as $x\to 0$, with coefficients that oscillate.

The function ϕ_τ has the same qualitative behavior near all rational points – not surprisingly, since all the rational points correspond to cusps. I consider a particular rational number $\frac{p}{q}$, with p and q relatively prime and $q > 0$. Then there exists a matrix

$$\gamma = \begin{pmatrix} r & -s \\ -q & p \end{pmatrix} \in SL(2,\mathbb{Z}), \tag{3.18}$$

which sends the point $\frac{p}{q}$ to ∞. With this notation,
(3.19)

$$\begin{aligned} \phi_\tau(x) - \phi_\tau(\tfrac{p}{q}) &= \phi_{\gamma\tau}(\gamma x)\, q^{\lambda+1}\, |x - \tfrac{p}{q}|^{\lambda+1} \\ &\quad - q(\lambda+1) \int_{\operatorname{sgn}(p-qx)\gamma x}^{+\infty} (qt + r\operatorname{sgn}(p-qx))^{-\lambda-2}\, \phi_{\gamma\tau}(\operatorname{sgn}(p-qx)\,t)\, dt\,. \end{aligned}$$

The verification is essentially the same as that of (3.14); one needs to observe that the expression $(qt + r\operatorname{sgn}(p-qx))$ takes the value $|qx-p|$ at the left endpoint of the interval of integration, and is therefore strictly positive over the entire interval – recall that $q > 0$. The second term on the right has strictly lower order of magnitude as $x \to \frac{p}{q}$, just as in the case of (3.14). Since

$$\gamma x = -\tfrac{1}{q^2}\left(1 - rq\,(x - \tfrac{p}{q})\right)\left(x - \tfrac{p}{q}\right)^{-1} \sim -\tfrac{1}{q^2}\left(x - \tfrac{p}{q}\right)^{-1} \quad \text{as } x \to \tfrac{p}{q}, \tag{3.20}$$

the dominating first term oscillates near $\frac{p}{q}$, with increasing frequency and decreasing amplitude as x approaches $\frac{p}{q}$. More precisely,
(3.21)

as x approaches $\frac{p}{q}$, $\phi_\tau(x) - \phi_\tau(\frac{p}{q})$ oscillates, with frequency proportional to $|qx-p|^{-2}$ and amplitude proportional to $|qx-p|^{\operatorname{Re}\lambda+1}$.

Though the oscillations have larger amplitude in the small as the denominator q increases, they also die out more quickly for larger q, as x moves away from $\frac{p}{q}$. The asymptotic expansion (3.17) has an analogue in the present setting, but we shall not pursue this further.

The identity (3.14) can be used to give elementary proofs of certain known facts. I consider an $SL^{\pm}(2,\mathbb{Z})$-invariant subspace

$$U \subset \{\, \tau \in (V_{\lambda,+1}^{-\infty})^{\Gamma} \mid \tau_\lambda \in C^{\frac{1}{2}(\operatorname{Re}\lambda-1)}(\mathbb{R}/\mathbb{Z}) \,\}, \tag{3.22}$$

the entire ambient space, for example, or an $SL^{\pm}(2,\mathbb{Z})$-irreducible subspace. The ambient space is known to be finite dimensional – the discrete part of $L^2(\Gamma\backslash G_{\mathbb{R}})$ breaks up with finite multiplicities – but this will become clear momentarily. Thus, to keep the argument logically self-contained, I expressly assume $\dim U < \infty$. Since $SL^{\pm}(2,\mathbb{Z})$ acts on U via a finite quotient group, I can introduce an invariant inner product. Let $\tau_1, \tau_2, \dots, \tau_n$ be an orthonormal basis of U, and let $\tilde{\tau}_j \in C^{-\infty}(\mathbb{R})$ denote the image of τ_j under the distribution analogue of the isomorphism (2.8). I write $\Phi(x)$, $\Phi_w(x)$ for the column vectors with entries $\phi_{\tau_j}(x)$ and $\phi_{w\tau_j}(x)$, respectively:

$$(3.23)\qquad \Phi(x) \;=\; \begin{pmatrix} \phi_{\tau_1}(x) \\ \phi_{\tau_2}(x) \\ \cdot \\ \cdot \\ \cdot \\ \phi_{\tau_n}(x) \end{pmatrix}, \qquad \Phi_w(x) \;=\; \begin{pmatrix} \phi_{w\tau_1}(x) \\ \phi_{w\tau_2}(x) \\ \cdot \\ \cdot \\ \cdot \\ \phi_{w\tau_n}(x) \end{pmatrix}.$$

Translation commutes with the integration process which produces the ϕ_{τ_j} from the $\tilde{\tau}_j$. Since $(n_1\tilde{\tau}_j)(x) = \tilde{\tau}_j(x-1)$, $\Phi(x-1) \;=\; A(n_1)\,\Phi(x)$, where $A(n_1)$ is the unitary matrix which represents the action of $n_1 \in SL^{\pm}(2,\mathbb{Z})$ on U, relative to the basis $\{\tau_1,\ldots,\tau_n\}$. The reflection $x \mapsto -x$ reverses the sign when one takes the anti-derivative of $\tilde{\tau}_j$, so $\Phi(-x) = -A(r)\Phi(x)$, where $A(r)$ represents reflection about the origin. Similarly $\tilde{\Phi}(x) = A(w)\Phi(x)$, with $A(w)$ representing the action of w. For future reference,

$$(3.24)\qquad \begin{gathered} \Phi(x-1) \,=\, A(n_1)\Phi(x)\,, \quad \Phi(-x) \,=\, -A(r)\Phi(x)\,, \quad \Phi_w(x) \,=\, A(w)\Phi(x)\,, \\ \text{with } A(n_1)\,,\, A(r)\,,\, A(w) \in U(m) \quad \text{representing} \\ n_1 \;=\; \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad r \;=\; \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad w \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{gathered}$$

The pointwise norm $\|\Phi(x)\|$ of the column vector $\Phi(x)$ is therefore invariant under reflection and translation by one, and is equal to $\|\Phi_w(x)\|$. In particular, it assumes its maximum at some point $x_{\max}$, $0 \le x_{\max} \le \frac{1}{2}$. Thus, using (3.14) component-by-component, I find

$$(3.25)\qquad \begin{aligned} \sup\|\Phi(x)\| \;&=\; \|\Phi(x_{\max})\| \\ &\le\; \|\Phi(0)\| + \sup\|\Phi(x)\| \left(x_{\max}^{\operatorname{Re}\lambda+1} + |1+\lambda| \int_{\frac{1}{x_{\max}}}^{+\infty} t^{-\operatorname{Re}\lambda-2}\,dt \right) \\ &\le\; \|\Phi(0)\| + x_{\max}^{\operatorname{Re}\lambda+1}\left(1+\tfrac{|1+\lambda|}{|1+\operatorname{Re}\lambda|}\right)\sup\|\Phi(x)\|\,. \end{aligned}$$

When λ is positive and $\Phi(0) = 0$, this cannot be – even for $\lambda = 0$, since the oscillatory behavior makes the inequality strict unless $x_{\max} = 0$. This implies the crude bound

$$(3.26)\qquad \dim\{\, \tau \in (V_{\lambda,+1}^{-\infty})^{\Gamma} \mid \tau_\lambda \in C^{\frac{1}{2}(\operatorname{Re}\lambda-1)}(\mathbb{R}/\mathbb{Z})\,\} \;\le\; \#(SL^{\pm}(2,\mathbb{Z})/\Gamma) \quad \text{if } \lambda \in \mathbb{R}.$$

Indeed, if the bound on the dimension were to fail, one could construct an $SL^{\pm}(2,\mathbb{Z})$-invariant subspace U consisting of distribution vectors τ such that $\phi_\tau(0) = 0$. One can also assume $\lambda \ge 0$ without loss of generality, in which case

(3.25) gives the desired contradiction. For imaginary λ, one can argue similarly, using (3.19) instead of (3.14), but the resulting bounds are even less precise.

A well known conjecture of Selberg asserts that representations of the complementary series do not occur as summands in $L^2(\Gamma\backslash G_{\mathbb{R}})$ when Γ is one of the principal congruence subgroups of $SL(2,\mathbb{R})$. Non-cuspidal summands, if they were to occur, would correspond to residues of Eisenstein series. In the case of principal congruence subgroups, Eisenstein series are known not to have poles in the relevant region. Thus Selberg's conjecture is equivalent to the vanishing of the space

$$\{\, \tau \in (V_{\lambda,+1}^{-\infty})^{\Gamma} \mid \tau_\lambda \in C^{\frac{1}{2}(\mathrm{Re}\lambda - 1)}(\mathbb{R}/\mathbb{Z}) \,\} \qquad 0 < \lambda < 1\,; \tag{3.27}$$

cf. theorem 2.20. I shall give a quick argument for the vanishing of this space when $\Gamma = \Gamma_2$, the principal congruence subgroup of level 2, even including $\lambda = 0$. Not that this is new – I merely wish to illustrate further how (3.14) can be used.

The reflection r, which does not lie in $SL(2,\mathbb{Z})$, commutes with w. Also $rn_1r^{-1}n_1^{-1} = n_{-2} \in \Gamma_2$, so r commutes with n_1 modulo Γ_2. Since w and n_1 generate $SL(2,\mathbb{Z})$, I conclude that the image of r in $SL^{\pm}(2,\mathbb{Z})/\Gamma_2$ commutes with

$$SL(2,\mathbb{Z})/\Gamma_2 \;=\; SL(2,\mathbb{Z}/2\mathbb{Z}) \;\cong\; S_3\,. \tag{3.28}$$

Thus, if the space (3.27) is non-zero, there exists an $SL^{\pm}(2,\mathbb{Z})$-irreducible subspace U such that r acts on all of U either trivially or as multiplication by -1. Since $r\tau = \tau$ implies $\phi_\tau(x) = -\phi_\tau(-x)$ and hence $\phi(0) = 0$, the case of trivial r-action is dealt with already by (3.25). Let me assume, then, that $r\tau = -\tau$ for all $\tau \in U$, which makes all the functions ϕ_τ even. The group (3.28) has three distinct irreducible representations. For the trivial representation, I use a variant of (3.14),

$$\begin{aligned} \phi_\tau(x) - \phi_\tau(1) \;&=\; \phi_{w\tau}(-\tfrac{1}{x})\,|x|^{\lambda+1} \;-\; \phi_{w\tau}(-1) \\ &\qquad +\; (\lambda+1)\int_1^{\frac{1}{|x|}} t^{-\lambda-2}\,\phi_{w\tau}(-\operatorname{sgn}(x)\,t)\,dt\,, \end{aligned} \tag{3.29}$$

which is obtained by evaluating (3.14) at $x = 1$ and subtracting the result from (3.14). Since $\phi_\tau(x) = \phi_{w\tau}(x) = \phi_\tau(-x)$, the terms $\phi_\tau(1)$ and $\phi_{w\tau}(-1)$ cancel. For $|x| < 1$, this implies

$$|\phi_\tau(x)| \;<\; \sup|\phi_\tau(x)| \left(|x|^{\lambda+1} \;+\; (\lambda+1)\int_1^{\frac{1}{|x|}} t^{-\lambda-2}\,dt \right) \;=\; \sup|\phi_\tau(x)|\;; \tag{3.30}$$

the inequality is strict since ϕ_τ cannot be constant for non-zero τ. Contradiction: ϕ_τ is periodic of period one. Next the non-trivial one dimensional representation of S_3. Now $\phi_\tau = -\phi_{n_1\tau} = -\phi_{w\tau}$, hence $\phi_\tau(x) = -\phi_\tau(x-1) = -\phi_{w\tau}(x) = \phi_\tau(-x)$.

This allows me to re-write (3.29) as follows:

(3.31)

$$\begin{aligned}\phi_\tau(x) - \phi_\tau(1) \ &= \ -\phi_\tau(\tfrac{1}{x})\,|x|^{\lambda+1} \ + \ \phi_\tau(1)\\ &\quad - \ (\lambda+1)\int_1^{\frac{1}{|x|}} t^{-\lambda-2}\,(\phi_\tau(t)-\phi_\tau(1))\,dt \ - \ \phi_\tau(1)(1-|x|^{\lambda+1})\\ &= \ -(\phi_\tau(\tfrac{1}{x})-\phi_\tau(1))\,|x|^{\lambda+1} \ - \ (\lambda+1)\int_1^{\frac{1}{|x|}} t^{-\lambda-2}\,(\phi_\tau(t)-\phi_\tau(1))\,dt\,.\end{aligned}$$

Now, arguing as in (3.30), I find $|\phi_\tau(x)-\phi_\tau(1)| < \sup|\phi_\tau(x)-\phi_\tau(1)|$ for $|x|<1$. Impossible if $\tau \neq 0$ since ϕ_τ is periodic of period two. Finally the two dimensional irreducible representation. There exists a non-zero vector $\tau \in U$ that is fixed by w, and one can argue as in (3.30), since periodicity of period 2 suffices to derive a contradiction; the oscillatory behavior rules out a maximum at $x = \pm 1$.

It should be remarked that the vanishing of ϕ_τ directly implies the vanishing of $\tilde{\tau}_\lambda$ only away from the point at infinity. In the arguments above this is not a problem because τ is invariant under Γ, whose orbit through ∞ contains finite points. Alternatively, one can argue that a non-zero distribution supported at a single point cannot be of class C^α for $\alpha \geq -1$.

The group Γ_2 plays a special role – this is an observation that has been attributed to Mark Stern. As was argued above, if U is a finite-dimensional $SL^\pm(2,\mathbb{Z})/\Gamma_2$-module, then the eigenspaces of r are $SL(2,\mathbb{Z})$-invariant. This property characterizes Γ_2 among all normal subgroups $\Gamma \subset SL(2,\mathbb{Z})$. Indeed, if Γ is such a subgroup, and if r acts as multiplication by a scalar on an $SL^\pm(2,\mathbb{Z})/\Gamma$-module U, then the action of n_1 on U agrees with that of $n_1^{-1} = n_{-1} = rn_1r^{-1}$, and hence n_2 acts trivially. Similarly wn_2w^{-1} acts trivially. The two elements n_2 and wn_2w^{-1} of $SL(2,\mathbb{Z})$ generate Γ_2, so Γ_2 acts trivially on U. For emphasis,

3.32 Observation. *Let $\Gamma \subset SL(2,\mathbb{Z})$ be a normal subgroup, and U an $SL^\pm(2,\mathbb{Z})/\Gamma$-module. If r acts on U as multiplication by a scalar, then Γ_2 acts trivially.*

Coupled with the statement about Γ_2 that was proved earlier, this implies: If U is a non-zero, $SL(2,\mathbb{Z})$-invariant subspace as in (3.22), with $\Gamma \subset SL(2,\mathbb{Z})$ normal of finite index and $-1 < \lambda < 1$, then U contains both r-invariant and anti-invariant distributions. Equivalently, any non-zero, $SL(2,\mathbb{Z})$-invariant space of cuspidal Maass forms corresponding to an eigenvalue $\leq \frac{1}{4}$ must contain both even and odd forms. Since Γ_2 does not admit non-cuspidal, square-integrable Maass forms, the statement is correct even for spaces of square-integrable, but not necessarily cuspidal, Maass forms.

I finish this section with the formula which generalizes (3.14) to the non-cuspidal case. The parameter λ is then necessarily real and non-zero. I may and shall assume $\lambda > 0$ – cf. (3.1) – to ensure that $\tilde{\tau}_\lambda$ has a continuous anti-derivative

ϕ_τ. The Fourier series (3.9) can now have a non-zero constant term $a_{\tau,0}$. I write

$$\begin{aligned}\phi_\tau(x) \;&=\; a_{\tau,0}\,x \;+\; \psi_\tau(x)\,,\\ \text{where}\quad \psi_\tau(x) \;&=\; \textstyle\sum_{k\neq 0}\frac{m}{2\pi i k}\,a_k\,e^{2\pi i \frac{k}{m}x}\,,\end{aligned}\tag{3.33}$$

so that $\tilde{\tau}_\lambda = a_{\tau,0} + \frac{d}{dx}\psi_\tau$, with ψ_τ periodic and having zero constant term. In this situation,

$$\begin{aligned}\psi_\tau(x) \,-\, \psi_\tau(0) \;=\; &\lambda^{-1} a_{w\tau,0}\,\mathrm{sgn}(x)\,|x|^\lambda \,-\, a_{\tau,0}\,x + \psi_{w\tau}(-\tfrac{1}{x})\,|x|^{\lambda+1}\\ &-(\lambda+1)\int_{\frac{1}{|x|}}^{+\infty} t^{-\lambda-2}\;\psi_{w\tau}(-\mathrm{sgn}(x)t\,)\;dt\,.\end{aligned}\tag{3.34}$$

The proof is the same as that of (3.14): The two sides agree at $x = 0$, and the first derivatives are related by the transformation law for $\tilde{\tau}_\lambda$. When the term $|x|^\lambda$ occurs with non-zero coefficient, as it may, ϕ_τ cannot be Hölder continuous of index greater than λ. Thus theorem 2.18 is sharp in this case as well.

4. L-functions and functional equation

The L-functions attached to modular forms satisfy a functional equation, analogous to that satisfied by the Riemann zeta function. The functional equation is usually proved by taking the Mellin transform of the modular form along the imaginary axis and then applying the transformation law under the inversion $z \mapsto -\frac{1}{z}$. Maass [10] showed that the same strategy works in the context of Maass forms. The "q-expansion" of modular forms has an analogue in the Maass case, with coefficients that are Bessel functions, rather than exponentials. The Mellin transform of the relevant Bessel function is essentially a product of two Gamma functions, and that accounts for the particular form of the functional equation.

In this section I shall show that the functional equation can also be derived by integration along the *real axis*. For groups of higher rank, a maximally split torus and a maximal unipotent subgroup are the analogues of, respectively, the imaginary axis and the real axis. Formally at least, a maximal unipotent subgroup has the "right dimension" for the derivation of a functional equation attached to automorphic representations, whereas the maximally split torus is "too small". The argument I am about to present suggests that functional equations for L-functions in the higher rank case should be provable by integration along a maximal unipotent subgroup – they should reflect the interplay between harmonic analysis on $N_{\mathbb{Z}}/(N_{\mathbb{Z}}\cap G_{\mathbb{Z}})$ and the action of the Weyl group.

For classical – holomorphic – modular forms it is not so surprising, perhaps, that the contour of integration can be moved from one axis to the other. I shall therefore concentrate on the Maass case. The notation and hypotheses are the same as in the previous section. In particular, $\Gamma \subset SL(2,\mathbb{Z})$ denotes a normal subgroup of finite index. I consider a Γ-invariant distribution vector $\tau \in (V_{\lambda,+1}^{-\infty})^\Gamma$ which corresponds to a cuspidal embedding $V_{\lambda,+1}^{-\infty} \hookrightarrow L^2(\Gamma, G_{\mathbb{R}})$. I express its image $\tilde{\tau}_\lambda \in C^{-\infty}(\mathbb{R})$ as a Fourier series

$$\tilde{\tau}_\lambda(x) \;=\; \sum_{k\neq 0} a_k\, e^{2\pi i \frac{k}{m} x}\,, \tag{4.1}$$

as in (3.9); the cuspidality assumption eliminates the constant term. One can write τ as the sum of an even and an odd part. Thus, without essential loss of generality, I may assume

$$\tilde{\tau}_\lambda(-x) \;=\; \eta\, \tilde{\tau}_\lambda(x)\,, \quad \text{with } \eta \in \{\pm 1\}\,. \tag{4.2}$$

This has the advantage of making the positive Fourier coefficients determine the negative ones as well. Consequently

$$L(\tau, s) \;=\; \sum_{k>0} a_k\, k^{\frac{\lambda}{2}-s}\,, \tag{4.3}$$

the Maass L-function of the embedding $V_{\lambda,+1}^{-\infty} \hookrightarrow L^2(\Gamma, G_{\mathbb{R}})$ defined by τ, completely determines $\tilde{\tau}_\lambda$, and hence τ itself – recall that the behavior of $\tilde{\tau}_\lambda$ at finite points determines the behavior of the Γ-invariant distribution τ also at infinity.

The canonical intertwining operator $V_{\lambda,+1} \cong V_{-\lambda,+1}$ relates $\tilde{\tau}_\lambda$ to another Γ-invariant distribution vector $\tilde{\tau}_{-\lambda}$, which represents the same embedding. One can show that the intertwining operator sends $e^{2\pi i \frac{k}{m} x}$, $k \neq 0$, to $|k|^\lambda e^{2\pi i \frac{k}{m} x}$, up to some universal constant which reflects the normalization of the operator. The L-function is therefore independent of the choice of λ over $-\lambda$. The same conclusion can be drawn from the fact that (4.3) really agrees with Maass' definition, up to a non-zero constant. To see this, one needs to go into the details of the proof of theorem 2.18 – specifically the series expansion (5.29) and lemma 5.65. Maass' explicit version of (5.29), in terms of Bessel functions, then makes the connection between the coefficients a_k in (4.1) and the coefficients used by Maass to define his L-function.

Theorem 2.18 and (2.15c) give an estimate for the abscissa of absolute convergence of the L-function: If $\delta > 0$,

(4.4) the series (4.3) converges uniformly and absolutely for $\operatorname{Re} s \geq 1 + \delta$.

Maass' functional equation asserts that $L(\tau, s)$ extends meromorphically to the entire complex plane, and

$$\begin{aligned} &\left(\tfrac{m}{\pi}\right)^s \Gamma\!\left(\tfrac{s}{2} + \tfrac{\lambda}{4} + \tfrac{1-\eta}{4}\right) \Gamma\!\left(\tfrac{s}{2} - \tfrac{\lambda}{4} + \tfrac{1-\eta}{4}\right) L(\tau, s) \\ &\qquad = \;\eta\, \left(\tfrac{m}{\pi}\right)^{1-s} \Gamma\!\left(\tfrac{1-s}{2} + \tfrac{\lambda}{4} + \tfrac{1-\eta}{4}\right) \Gamma\!\left(\tfrac{1-s}{2} - \tfrac{\lambda}{4} + \tfrac{1-\eta}{4}\right) L(w\tau, 1-s) \end{aligned} \tag{4.5}$$

[10]. This version of the functional equation clearly displays the symmetry between λ and $-\lambda$. It is not the simplest way of writing the formula, however.

Recall the definition (3.13) of the matrix w and the definition (3.10) of the anti-derivative ϕ_τ of $\tilde{\tau}_\lambda$. At first glance, the integral defining the Mellin transform

$$s \longmapsto M(\phi_\tau, s) \;=\; \int_0^{+\infty} \phi_\tau(x)\, x^{s-1}\, dx \tag{4.6}$$

converges in the two disjoint regions $\mathrm{Re}\, s > 0$ and $\mathrm{Re}\, s < 0$ for x near 0 and ∞, respectively. However, the fact that the continuous function ϕ_τ is periodic, with zero constant term, makes $M(\phi_\tau, s)$ well-defined and holomorphic at least in the region $0 < \mathrm{Re}\, s < 1$.

4.7 Lemma. *The Mellin transform $M(\phi_\tau, s)$ extends meromorphically to the region $-2 - \mathrm{Re}\,\lambda < \mathrm{Re}\, s < 1$, where it has no singularities except for a first order pole at $s = 0$, with residue $\phi_\tau(0)$. On this region, $sM(\phi_\tau, s) = -\eta(s+\lambda+1)M(\phi_{w\tau}, -s-\lambda-1)$.*

When one applies integration by parts purely formally, the assertion of the lemma gets transformed into the equation $M(\tau, s) = \eta\, M(w\tau, 1-s-\lambda)$ which "follows", again purely formally, from the transformation law (2.9). Before turning to the proof, I shall show that the lemma implies Maass' formula (4.5).

Let $\phi_\tau^{(-1)}$ denote the first anti-derivative of ϕ_τ, normalized so as to have zero constant term. By definition, $M(\phi_\tau^{(-1)}, s)$ is holomorphic on the strip $0 < \mathrm{Re}\, s < 1$, but the identity

$$\begin{aligned} \int_0^{+\infty} \phi_\tau^{(-1)}(x)\, x^{s-1}\, dx \ &= \ s^{-1}\, \phi_\tau^{(-1)}(0) \ + \\ &+ \int_0^1 (\phi_\tau^{(-1)}(x) - \phi_\tau^{(-1)}(0))\, x^{s-1}\, dx + \int_1^{+\infty} \phi_\tau^{(-1)}(x)\, x^{s-1}\, dx \end{aligned} \tag{4.8}$$

shows how to extend $M(\phi_\tau^{(-1)}, s)$ meromorphically to $-1 < \mathrm{Re}\, s < 1$, with a first order pole at $s = 0$, where the residue equals $\phi_\tau^{(-1)}(0)$. I claim:

$$M(\phi_\tau, s) \ = \ -(s-1)\, M(\phi_\tau^{(-1)}, s-1) \qquad (\, 0 < \mathrm{Re}\, s < 1\,)\,. \tag{4.9}$$

When $\phi_\tau^{(-1)}(0) = 0$, this follows from integration by parts; the vanishing of $\phi_\tau^{(-1)}(0)$ makes the boundary term drop out. The assertion is known for the function $e^{2\pi i x}$ – see (4.11) below; the general case follows by taking a suitable linear combination of $\phi_\tau^{(-1)}$ and $e^{2\pi i x}$.

The Fourier series for $\phi_\tau^{(-1)}$ converges uniformly and absolutely, as follows from theorem 2.18 and (2.15c). Thus, in the notation of (4.1–2), for $0 < \mathrm{Re}\, s < 1$,

$$\begin{aligned} M(\phi_\tau^{(-1)}, s) \ &= \ -\tfrac{m^2}{4\pi^2} \textstyle\sum_{k \neq 0} \dfrac{a_k}{k^2} \displaystyle\int_0^{+\infty} e^{2\pi i \frac{k}{m} x}\, x^{s-1}\, dx \\ &= \ -\tfrac{m^2}{4\pi^2} \textstyle\sum_{k > 0} \dfrac{a_k}{k^2} \displaystyle\int_0^{+\infty} \left(e^{2\pi i \frac{k}{m} x} + \eta\, e^{-2\pi i \frac{k}{m} x} \right) x^{s-1}\, dx\,. \end{aligned} \tag{4.10}$$

A standard Gamma identity asserts

$$\int_0^{+\infty} e^{iax}\, x^{s-1}\, dx \ = \ a^{-s}\, \Gamma(s)\, e^{i\frac{\pi}{2}s} \qquad (\, 0 < a,\ 0 < \mathrm{Re}\, s < 1\,)\,. \tag{4.11}$$

When combined with (4.3) and (4.10), this leads to the identity

$$
\begin{aligned}
M(\phi_\tau^{(-1)},s) \;&=\; -m^{2+s}(2\pi)^{-2-s}\,\Gamma(s)\left(e^{i\frac{\pi}{2}s}+\eta\,e^{-i\frac{\pi}{2}s}\right)\textstyle\sum_{k>0}a_k\,k^{-2-s}\\
&=\; -m^{2+s}(2\pi)^{-2-s}\,\Gamma(s)\left(e^{i\frac{\pi}{2}s}+\eta\,e^{-i\frac{\pi}{2}s}\right)L(\tau,s+2+\tfrac{\lambda}{2})\,.
\end{aligned}
\tag{4.12}
$$

At this point, analytic continuation, (4.9), and the identity $s\Gamma(s)=\Gamma(s+1)$ imply

$$
M(\phi_\tau,s) \;=\; \tfrac{1}{i}\,m^{1+s}(2\pi)^{-1-s}\,\Gamma(s)\left(e^{i\frac{\pi}{2}s}-\eta\,e^{-i\frac{\pi}{2}s}\right)L(\tau,s+1+\tfrac{\lambda}{2})\,,
\tag{4.13}
$$

for $0<\operatorname{Re}s<1$. Lemma 4.7 asserts that $sM(\phi_\tau,s)$ is holomorphic on $-2-\operatorname{Re}\lambda<\operatorname{Re}s<1$ and invariant under the simultaneous substitutions $s\mapsto -s-\lambda-1$, $\tau\mapsto w\tau$, up to a factor η. But $L(\tau,s)$ is known to be holomorphic for $\operatorname{Re}s>1$, so the symmetry of $sM(\phi_\tau,s)$ continues $L(\tau,s)$ meromorphically to the entire plane, and implies

(4.14)

$$
\begin{aligned}
&\left(\tfrac{m}{2\pi}\right)^{1+s}\Gamma(s+1)\left(e^{i\frac{\pi}{2}s}-\eta\,e^{-i\frac{\pi}{2}s}\right)L(\tau,s+1+\tfrac{\lambda}{2})\\
&\qquad=\;\eta\,\left(\tfrac{m}{2\pi}\right)^{-s-\lambda}\Gamma(-s-\lambda)\left(e^{-i\frac{\pi}{2}(s+\lambda+1)}-\eta\,e^{i\frac{\pi}{2}(s+\lambda+1)}\right)L(w\tau,-s-\tfrac{\lambda}{2})\,.
\end{aligned}
$$

Next, I substitute $s-1-\frac{\lambda}{2}$ for s, multiply both sides by $i\,m^{\frac{\lambda}{2}}(2\pi)^{-\frac{\lambda}{2}}$, and re-write the exponential terms:

$$
\begin{aligned}
&i\,\left(\tfrac{m}{2\pi}\right)^{s}\Gamma(s-\tfrac{\lambda}{2})\left(e^{i\frac{\pi}{2}(s-\frac{\lambda}{2})}+\eta\,e^{-i\frac{\pi}{2}(s-\frac{\lambda}{2})}\right)L(\tau,s)\\
&\qquad=\;\left(\tfrac{m}{2\pi}\right)^{1-s}\Gamma(1-s-\tfrac{\lambda}{2})\left(e^{i\frac{\pi}{2}(s+\frac{\lambda}{2})}-\eta\,e^{-i\frac{\pi}{2}(s+\frac{\lambda}{2})}\right)L(w\tau,1-s)\,.
\end{aligned}
\tag{4.15}
$$

I must show that this implies the functional equation (4.5).

Continuing the derivation, I observe that $e^{i\frac{\pi}{2}z}+\eta\,e^{-i\frac{\pi}{2}z}=2\,e^{i\frac{\pi}{4}(1-\eta)}\sin(\frac{\pi}{2}(z+\frac{1+\eta}{2}))$ and similarly for $-\eta$ in place of η, and $e^{i\frac{\pi}{2}\eta}=i\eta$. I also use the argument doubling formula

$$
\Gamma(\tfrac{z}{2})\Gamma(\tfrac{z}{2}+\tfrac{1}{2})=2^{1-z}\pi^{\frac{1}{2}}\Gamma(z)
$$

and cancel various like terms, to put (4.15) into the equivalent form

(4.16)

$$
\begin{aligned}
&\left(\tfrac{m}{\pi}\right)^{s}\Gamma(\tfrac{s}{2}-\tfrac{\lambda}{4})\,\Gamma(\tfrac{s}{2}-\tfrac{\lambda}{4}+\tfrac{1}{2})\,\sin\left(\pi(\tfrac{s}{2}-\tfrac{\lambda}{4}+\tfrac{1+\eta}{4})\right)L(\tau,s)\\
&=\;\eta\,\left(\tfrac{m}{\pi}\right)^{1-s}\Gamma(\tfrac{1}{2}-\tfrac{s}{2}-\tfrac{\lambda}{4})\,\Gamma(1-\tfrac{s}{2}-\tfrac{\lambda}{4})\,\sin\left(\pi(\tfrac{s}{2}+\tfrac{\lambda}{4}+\tfrac{1-\eta}{4})\right)L(w\tau,1-s)\,.
\end{aligned}
$$

For the last step, I recall that $\Gamma(z)\Gamma(1-z)\sin(\pi z)=\pi$, hence

$$
\begin{aligned}
&\left(\tfrac{m}{\pi}\right)^{s}\frac{\Gamma(\frac{s}{2}-\frac{\lambda}{4})\,\Gamma(\frac{s}{2}-\frac{\lambda}{4}+\frac{1}{2})}{\Gamma(\frac{s}{2}-\frac{\lambda}{4}+\frac{1+\eta}{4})\,\Gamma(1-\frac{s}{2}+\frac{\lambda}{4}-\frac{1+\eta}{4})}\,L(\tau,s)\\
&=\;\eta\,\left(\tfrac{m}{\pi}\right)^{1-s}\frac{\Gamma(\frac{1}{2}-\frac{s}{2}-\frac{\lambda}{4})\,\Gamma(1-\frac{s}{2}-\frac{\lambda}{4})}{\Gamma(\frac{s}{2}+\frac{\lambda}{4}+\frac{1-\eta}{4})\,\Gamma(1-\frac{s}{2}-\frac{\lambda}{4}-\frac{1-\eta}{4})}\,L(w\tau,1-s)\,.
\end{aligned}
\tag{4.17}
$$

For both choices of η, i.e., $\eta = 1$ and $\eta = -1$, the appropriate terms cancel, and the equation becomes equivalent to (4.5). Thus lemma 4.7 does imply Maass' functional equation.

Proof of lemma 4.7. The identity (4.2) is equivalent to $(r\,\tau)(x) = \eta\,\tau(x)$; cf. (3.24). Since r and w commute, $w\tau$ also satisfies the symmetry relation (4.2). The passage from $\tilde{\tau}_\lambda$ to ϕ_τ reverses parity with respect to $x \mapsto -x$, so

$$\phi_\tau(-x) \;=\; -\eta\,\phi_\tau(x)\,, \qquad \phi_{w\tau}(-x) \;=\; -\eta\,\phi_{w\tau}(x)\,. \tag{4.18}$$

I now re-interpret the integral (4.6) as a sum of two integrals, over the intervals $(0,1]$ and $[1,+\infty)$. As was argued already,

$$s \longmapsto \int_1^{+\infty} \phi_\tau(x)\,x^{s-1}\,dx \quad \text{is holomorphic on } \{\operatorname{Re} s < 1\}\,. \tag{4.19}$$

On the other subinterval, I use (3.14): For $0 < \operatorname{Re} s < 1$,

$$\begin{aligned}
&\int_0^1 \phi_\tau(x)\,x^{s-1}\,dx \;-\; s^{-1}\,\phi_\tau(0) \;=\; \int_0^1 (\phi_\tau(x) - \phi_\tau(0))\,x^{s-1}\,dx \\
&\quad= -\eta \int_0^1 \left(\phi_{w\tau}(\tfrac{1}{x})\,x^{\lambda+1} - (\lambda+1)\int_{\frac{1}{x}}^{+\infty} t^{-\lambda-2}\,\phi_{w\tau}(t)\,dt \right) x^{s-1}\,dx \\
&\quad= -\eta \int_1^{+\infty} \left(\phi_{w\tau}(x)\,x^{-\lambda-1} - (\lambda+1)\int_x^{+\infty} t^{-\lambda-2}\,\phi_{w\tau}(t)\,dt \right) x^{-s-1}\,dx \\
&\quad= -\eta \int_1^{+\infty} \phi_{w\tau}(x)\,x^{-\lambda-s-2}\,dx \\
&\qquad\qquad + \;\eta\,(\lambda+1)\int_{t=1}^{+\infty}\int_{x=1}^{t} t^{-\lambda-2}\,x^{-s-1}\,\phi_{w\tau}(t)\,dx\,dt \\
&\quad= -\eta\,(1+\tfrac{\lambda+1}{s})\int_1^{+\infty} \phi_{w\tau}(x)\,x^{-\lambda-s-2}\,dx \;+\; \eta\,\tfrac{\lambda+1}{s}\int_1^{+\infty} x^{-\lambda-2}\,\phi_{w\tau}(x)\,dx\,.
\end{aligned} \tag{4.20}$$

At the middle step, I have applied the change of variables $x \mapsto \frac{1}{x}$; the penultimate step involves a switch in order of integration, and the final step simply rearranges the terms. I now multiply through by s and use (3.14) at $x = 1$ to express the last integral on the right in terms of $\phi_\tau(0)$, $\phi_\tau(1)$, and $\phi_{w\tau}(1)$,

$$\begin{aligned}
&s\int_0^1 \phi_\tau(x)\,x^{s-1}\,dx \;-\; \phi_\tau(1) \;= \\
&\qquad\qquad -\,\eta\left((\lambda+s+1)\int_1^{+\infty} \phi_{w\tau}(x)\,x^{-\lambda-s-2}\,dx \;-\; \phi_{w\tau}(1) \right).
\end{aligned} \tag{4.21}$$

The right hand side represents a holomorphic function on the region $\operatorname{Re} s > -\operatorname{Re}\lambda - 2$, in complete analogy to (4.19). This, coupled with (4.19), implies the first assertion of the lemma, except possibly the identification of the residue at

$x = 0$. The fact that the residue is $\phi_\tau(0)$ already follows from the very first equality in (4.20), since $\phi_\tau(x) - \phi_\tau(0)$ has order of magnitude $|x|^{\mathrm{Re}\,\lambda+1}$ as $x \to 0$. I now multiply the equation (4.21) by η, interchange the roles of ϕ_τ and $\phi_{w\tau}$, and replace s by $-s-\lambda-1$. Adding the resulting equation to (4.21), I find

$$\text{(4.22)} \qquad s\int_0^{+\infty} \phi_\tau(x)\, x^{s-1}\, dx \;=\; -\,\eta\,(\lambda+s+1)\int_0^{+\infty} \phi_{w\tau}(x)\, x^{-\lambda-s-2}\, dx\,,$$

with both sides holomorphic at least on the strip $-\mathrm{Re}\,\lambda - 2 < \mathrm{Re}\,s < 1$. This completes the proof of the lemma. □

5. Proof of theorems 2.18 and 2.20

I use the notation and conventions of §2. In particular, (π, V) is an irreducible unitary representation, i an embedding of the dual $G_{\mathbb{R}}$-module V' into $L^2(\Gamma\backslash G_{\mathbb{R}})$, and τ the Γ-invariant distribution vector defined by (1.1–3). For any $v' \in V'^{\infty}$,

$$\text{(5.1)} \qquad G_{\mathbb{R}} \ni g \;\mapsto\; \langle\, \tau\,,\, \pi'(g)v'\,\rangle$$

is the function $i(v') \in L^2(\Gamma\backslash G_{\mathbb{R}}) \cap C^\infty(G_{\mathbb{R}})$. Indeed, for $g = e$, this amounts to the definition of τ, and the general case then follows, with $\pi'(g)v'$ in place of v'.

When v' is a $K_{\mathbb{R}}$-isotypic vector, (5.1) can be regarded as function on the upper half plane

$$\text{(5.2)} \qquad \mathscr{H} \;=\; \{\, z = x + iy \in \mathbb{C} \mid y > 0 \,\} \;\simeq\; G_{\mathbb{R}}/K_{\mathbb{R}}\,.$$

For this purpose, I parametrize the groups $A_{\mathbb{R}}$, $N_{\mathbb{R}}$ as in (2.3) and set

$$\text{(5.3)} \qquad k_\theta \;=\; \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

so that $K_{\mathbb{R}} = \{k_\theta \mid \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$. Every $g \in G_{\mathbb{R}}$ can then be expressed as product

$$\text{(5.4)} \qquad g \;=\; n_x\, a_t\, k_\theta\,,$$

with $x, t \in \mathbb{R}$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ uniquely determined. Now let $v' \in V'$ be a non-zero vector such that

$$\text{(5.5)} \qquad \pi'(k_\theta)v' \;=\; e^{ir\theta}\, v'\,,$$

for some $r \in \mathbb{Z}$. In this situation, the function $F_{\tau,v'} : \mathscr{H} \to \mathbb{C}$,

$$\text{(5.6)} \qquad F_{\tau,v'}(x,y) \;=\; \langle\, \tau\,,\, \pi'(n_x a_t)v'\,\rangle\,, \qquad \text{with } e^{2t} = y\,,$$

determines the function (5.1) and is determined by it.

The function $F_{\tau,v'}$ satisfies a second-order differential equation, which reflects the fact that the Casimir operator acts as a scalar in the irreducible unitary representation π'. In terms of the basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.7}$$

of the Lie algebra $\mathfrak{g}_{\mathbb{R}}$, the Casimir operator – suitably normalized – can be expressed as

$$\begin{aligned} \Omega &= 2ef + 2fe + h^2 = 4ef + h^2 - 2h \\ &= -4e(e-f) + 4e^2 + h^2 - 2h. \end{aligned} \tag{5.8}$$

I choose $\lambda \in \mathbb{C}$ so that Ω acts on V' by the constant $\lambda^2 - 1$; in particular, then,

$$\pi'(\Omega)v' = (\lambda^2 - 1)v'. \tag{5.9}$$

Note that λ is determined only up to sign.

For the statement of the next lemma, I temporarily drop the assumption that τ arise from an embedding of (π', V') into $L^2(\Gamma\backslash G_{\mathbb{R}})$. To signal this change in hypotheses, I let $\sigma \in V^{-\infty}$ denote an arbitrary distribution vector, and write $F_{\sigma,v'}$ for the function defined by (5.6), but with σ in place of τ.

5.10 Lemma. $\left(y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) - iry\frac{\partial}{\partial x} - \frac{1}{4}(\lambda^2 - 1)\right) F_{\sigma,v'}(x,y) = 0.$

Proof. I combine the definition (5.6) of $F_{\sigma,v'}$ with (5.8–9):

$$\begin{aligned} (\lambda^2 - 1) F_{\sigma,v'}(x,y) &= -4\langle \tau, \pi'(n_x a_t)\, \pi'(e(e-f))\, v' \rangle \\ &\quad + 4\langle \tau, \pi'(n_x a_t)\, \pi'(e^2)\, v' \rangle \\ &\quad + \langle \tau, \pi'(n_x a_t)\, \pi'(h^2 - 2h)\, v' \rangle. \end{aligned} \tag{5.11}$$

Here, as before, y equals e^{2t}. Since $e - f$ generates the one parameter subgroup $\{k_\theta \mid \theta \in \mathbb{R}\}$, (5.5) is equivalent to

$$\pi'(e-f)v' = \frac{d}{d\theta}\left(\pi'(k_\theta)v'\right)\Big|_{\theta=0} = irv'. \tag{5.12}$$

For similar reasons,

$$\begin{aligned} \pi'(n_x a_t)\, \pi'(h)\, v' &= \tfrac{\partial}{\partial t}\left(\pi'(n_x a_t)\, v'\right) \\ &= 2y\tfrac{\partial}{\partial y}\left(\pi'(n_x a_t)\, v'\big|_{e^{2t}=y}\right) \end{aligned} \tag{5.13}$$

and

$$\begin{aligned} \pi'(n_x a_t)\, \pi'(e)\, v' &= \pi'(n_x)\, \pi'((\mathrm{Ad}\, a_t)e)\, \pi'(a_t)\, \pi'(e)\, v' \\ &= y\, \pi'(n_x)\, \pi'(e)\, \pi'(a_t)\, \pi'(e)\, v' \\ &= y\tfrac{\partial}{\partial x}\left(\pi'(n_x a_t)\, v'\right). \end{aligned} \tag{5.14}$$

In (5.13–14), which do not depend on (5.12), v' can be replaced by $\pi(h)v'$ and $\pi(e)v'$, respectively, so that

$$(5.15)\qquad \begin{aligned} \pi'(n_x a_t)\,\pi'(h^2)\,v' \;&=\; 4\,(y\tfrac{\partial}{\partial y})^2 F_{\sigma,v'}(x,y)\,,\\ \pi'(n_x a_t)\,\pi'(e^2)\,v' \;&=\; (y\tfrac{\partial}{\partial x})^2 F_{\sigma,v'}(x,y)\,. \end{aligned}$$

At this point, (5.11–15) imply the lemma. □

One can formally re-write the differential equation (5.10) as first order system with respect to the variable y. Since

$$(5.16)\qquad \left(y\tfrac{\partial}{\partial y} - \tfrac{1-\lambda}{2}\right)\left(y\tfrac{\partial}{\partial y} - \tfrac{1+\lambda}{2}\right) \;=\; \left(y^2\tfrac{\partial^2}{\partial y^2} + \tfrac{1-\lambda^2}{4}\right),$$

the equation can be put into the equivalent form

$$(5.17)\qquad \left(\tfrac{\partial}{\partial y} - \tfrac{1-\lambda}{2y}\right)\left(\tfrac{1+\lambda}{2} - y\tfrac{\partial}{\partial y}\right)F_{\sigma,v'} \;=\; \left(y\tfrac{\partial^2}{\partial^2 x} - ir\tfrac{\partial}{\partial x}\right)F_{\sigma,v'}\,.$$

At this point, integration with respect to y gives

5.18 Corollary. $y^{\frac{\lambda-1}{2}}\left(\tfrac{1+\lambda}{2} - y\tfrac{\partial}{\partial y}\right)F_{\sigma,v'} \;=\; \int y^{\frac{\lambda-1}{2}}\left(y\tfrac{\partial^2}{\partial^2 x} - ir\tfrac{\partial}{\partial x}\right)F_{\sigma,v'}\,dy.$

Note that the derivation of corollary 5.18 involves λ^2 rather than λ itself. Thus the corollary remains valid with $-\lambda$ in place of λ.

When π belongs to the discrete series or limits of the discrete series, the function $F_{\sigma,v'}$ – with v' appropriately chosen – satisfies a first order differential equation. To state it, I suppose that v' is a $K_{\mathbb{R}}$-weight vector corresponding to the extreme weight. In other words, if r is the weight as in (5.5), the other $K_{\mathbb{R}}$-weights are $r+2\ell$, $\ell \in \mathbb{N}$ (lowest weight case), or $r-2\ell$, $\ell \in \mathbb{N}$ (highest weight case). The extreme weight r determines the action of the Casimir operator: The two choices $\lambda = \pm(1-|r|)$ of the constant λ satisfy the equation (5.9).

5.19 Lemma. *Under the hypotheses just stated, with $\lambda = 1-|r|$, and with $\eta = +1$ or $\eta = -1$ depending on whether r is the highest or lowest weight for π',*

$$y^{\frac{\lambda-1}{2}} F_{\sigma,v'} \;=\; -\int y^{\frac{\lambda-1}{2}}\, i\,\eta\,\tfrac{\partial}{\partial x}F_{\sigma,v'}\,dy\,.$$

Proof. A short computation shows

$$(5.20)\qquad \mathrm{Ad}\,k_\theta(h+i\eta(e+f)) = 2\eta(h+i\eta(e+f))\,,$$

which implies that $\pi'(h+i\eta(e+f))v'$ has $K_{\mathbb{R}}$-weight $r+2\eta$. According to the hypotheses, $r+2\eta$ is not a weight, so

$$(5.21)\qquad \begin{aligned} 0 \;&=\; \pi'(h+i\eta(e+f))v' \;=\; \pi'(h-i\eta(e-f)+2i\eta e)v'\\ &=\; \pi'(h+2i\eta e)v' + \eta r v' \;=\; \pi'(h+2i\eta e)v' - (1-\lambda)v'\,; \end{aligned}$$

at the last two steps, I have used (5.12), the equality $\lambda = 1-|r|$, and the fact that the lowest weight is positive, the highest weight negative. I now argue as in the proof of lemma 5.10, to find

$$\left(\tfrac{\partial}{\partial y} + i\eta\tfrac{\partial}{\partial x}\right)F_{\sigma,v'} \;=\; \tfrac{1-\lambda}{2y}F_{\sigma,v'}\,, \tag{5.22}$$

the Cauchy-Riemann equations in disguise. The lemma follows by integration. □

I return to the original definition of the function (5.6), with $\tau \in V^{-\infty}$ corresponding to an embedding of (π', V') into $L^2(\Gamma\backslash G_{\mathbb{R}})$. The parameter λ is related to π' by (5.9), and thus is determined only up to sign.

5.23 Lemma. *The functions $F_{\tau,v'}(x,y)$, $y\frac{\partial}{\partial x}F_{\tau,v'}$, and $y\frac{\partial}{\partial y}F_{\tau,v'}$ are bounded, provided the embedding $V' \hookrightarrow L^2(\Gamma\backslash G_{\mathbb{R}})$ is cuspidal. In the non-cuspidal situation – in which case $\lambda\in\mathbb{R}$, $0<|\lambda|<1$ – these three functions are bounded by a constant multiple of* $\max\{y^{\frac{1-|\lambda|}{2}}, y^{\frac{|\lambda|-1}{2}}\}$.

Proof. In more concrete, but equivalent form this is certainly known. However, I shall sketch the argument to make the point that one need not use properties of Bessel functions and other very specific information.

For the first part of the proof, it is better to interpret the function (5.6) more invariantly, as a section of a $G_{\mathbb{R}}$-equivariant line bundle on $G_{\mathbb{R}}/K_{\mathbb{R}} \cong \mathscr{H}$. To simplify the notation, I drop the subscripts τ, v', and define

$$F : G_{\mathbb{R}} \longrightarrow \mathbb{C}, \qquad F(g) \;=\; \langle\, \tau\,,\pi'(g)v'\,\rangle\,. \tag{5.24}$$

The $K_{\mathbb{R}}$-isotopy condition (5.5) implies

$$F(g\,k_\theta) \;=\; e^{ir\theta}\,F(g)\,, \tag{5.25}$$

which allows me to regard F as section of the $G_{\mathbb{R}}$-equivariant line bundle

$$\mathbf{L}_{-r} \;\to\; G_{\mathbb{R}}/K_{\mathbb{R}} \cong \mathscr{H}\,, \qquad \mathbf{L}_{-r} \;=\; G_{\mathbb{R}}\times_{K_{\mathbb{R}}}\mathbb{C}_{-r}\,, \tag{5.26}$$

(fiber product), where $\mathbb{C}_{-r}$ denotes the one dimensional $K_{\mathbb{R}}$-module dual to $\mathbb{C}v'$. By construction, the section is Γ-invariant and satisfies the $G_{\mathbb{R}}$-invariant, second order, elliptic differential equation

$$\Omega F \;=\; (\lambda^2-1)F\,; \tag{5.27}$$

cf. (5.8). Lastly, F is square-integrable over the quotient of $G_{\mathbb{R}}/K_{\mathbb{R}} \cong \mathscr{H}$ by the action of Γ,

$$\int_{\Gamma\backslash\mathscr{H}} |F|^2\,dm \;<\; \infty\,, \tag{5.28}$$

relative to the hyperbolic volume form dm.

The hypotheses on Γ – discrete in $G_{\mathbb{R}}$ and finitely generated – imply that there at most finitely many elliptic conjugacy classes. These correspond to points in $\Gamma\backslash\mathscr{H}$ near which the quotient looks like the quotient of a disc centered at the origin in $\mathbb{C}$, modulo the action of a finite rotation group. There also may be finitely many cusps, i.e., Γ-orbits of parabolic fixed points on the boundary $\mathbb{R}\cup\{\infty\}$ of $\mathscr{H}$. For a discussion of the nature of $\Gamma\backslash\mathscr{H}$ see, for example, [1].

Now let $U_1, U_2, \ldots, U_m$ be punctured neighborhoods of the cusps, and $\tilde{U}$ the inverse image of their union in $\mathscr{H}$. Then there exists $\delta > 0$ with the following property: For any $z \in \mathscr{H} - \tilde{U}$, the disc (in the hyperbolic metric) $\Delta(z,\delta)$ centered at z, of radius δ maps to its image in $\Gamma\backslash\mathscr{H}$ with fibers of bounded cardinality – namely, the maximal branch index at the elliptic fixed points. Standard elliptic estimates then bound the value of F at z, and also of $y\frac{\partial F}{\partial x}$ and $y\frac{\partial F}{\partial y}$ at z, in terms of δ, the global L^2-norm of F, and the maximal branch index – the assertion about the derivatives uses the fact that the vector fields $y\frac{\partial}{\partial x}$, $y\frac{\partial}{\partial y}$ have constant length.

What has been said so far proves the assertion of the lemma away from the cusps. Since the assertion is invariant under translation by real numbers and under the inversion $z \mapsto -\frac{1}{z}$, it suffices to establish the bound for $y \gg 1$ in case there is a cusp at infinity. Let me suppose then that Γ contains an infinite cyclic group of translations fixing the point at infinity. I may as well suppose that $z \mapsto z+1$ generates this group. I now regard F again as function of the variables x, y, as in (5.6), and express $F(x,y)$ as a Fourier series in the variable x,

$$F(x,y) \;=\; \textstyle\sum_{-\infty<k<\infty} a_k(y)\, e^{2k\pi i x}\,. \tag{5.29}$$

For y_0 sufficiently large, the half-strip $y \geq y_0$, $0 \leq x < 1$ maps to $\Gamma\backslash\mathscr{H}$ in one-to-one fashion, so the square-integrability condition (5.28) implies

$$\textstyle\sum_{-\infty<k<\infty} \displaystyle\int_{y_0}^{\infty} y^{-2}\,|a_k(y)|^2\,dy \;<\; \infty\,. \tag{5.30}$$

The differential equation (5.10) is equivalent to an ordinary differential equations for the Fourier coefficients,

$$a_k''(y) \;=\; \left(4\pi^2k^2 - 2\pi r k y^{-1} + \tfrac{1}{4}(\lambda^2-1)y^{-2}\right) a_k(y)\,. \tag{5.31}$$

Changing variables from y to $|k|\,y$, one finds that

$$\begin{gathered} a_k(y) \;=\; \begin{cases} c_k\, b_+(ky) & \text{if } k>0 \\ c_k\, b_-(-ky) & \text{if } k<0 \end{cases} \\ \text{where } b_+(y) \text{ and } b_-(y) \text{ are solutions of the equation} \\ b_\pm''(y) \;=\; \left(4\pi^2 \mp 2\pi r y^{-1} + \tfrac{1}{4}(\lambda^2-1)y^{-2}\right) b_\pm(y) \end{gathered} \tag{5.32}$$

and the c_k are complex constants. The equation satisfied by $b_\pm(y)$ can be transformed into the confluent hypergeometric equation, whose solutions are known.

However, one can argue qualitatively, as follows. The theory of asymptotic expansions – see [8], for example – produces two solutions, one asymptotic to $y^{\mp\frac{r}{2}}e^{2\pi y}$ as $y \to +\infty$, the other asymptotic to $y^{\pm\frac{r}{2}}e^{-2\pi y}$. Only exponential decay is compatible with the finiteness condition (5.30), so

$$(5.33) \qquad b_{\pm}(y) \;\sim\; y^{\pm\frac{r}{2}}\,e^{-2\pi y} \quad \text{and} \quad b_{\pm}'(y) \;\sim\; -2\pi y^{\pm\frac{r}{2}}\,e^{-2\pi y}\,, \quad \text{as } \; y \to +\infty\,.$$

Comparing (5.32–33) and noticing that the series (5.29) converges absolutely for any particular y, one finds that

$$(5.34) \qquad F(x,y) \;\sim\; a_0(y)\,, \quad y\tfrac{\partial}{\partial y}F(x,y) \;\sim\; y\,a_0'(y)\,, \quad y\tfrac{\partial}{\partial x}F(x,y) \;\sim\; 0$$

uniformly in x, as $y \to +\infty$.

The embedding $V' \hookrightarrow L^2(\Gamma\backslash G_{\mathbb{R}})$ corresponding to τ is said to be cuspidal at ∞ if $a_0(y) \equiv 0$, and cuspidal without qualification if the analogous condition holds at every cusp. At first glance, the notion of cuspidality appears to depend on the choice of the $K_{\mathbb{R}}$-isotypic vector (5.5) used to define F. One can show directly that the notion is well-defined – the functions F corresponding to different choices of v' are related by the so-called "Maass operators", certain very simple differential operators – but this will also become obvious as a consequence of the proof of theorem 2.18. In any case, (5.34) implies the lemma when the embedding is cuspidal.

The parameter λ in (5.9) can be purely imaginary, or real with $0 < |\lambda| < 1$, or a non-zero integer; in the first situation π' belongs to the unitary principal series or the limits of the discrete series, in the second situation π' belongs to the complementary series, and in the third π' is a discrete series representation or possibly the trivial one dimensional representation. We may disregard the case of the trivial representation, since then F is a constant function. Solving (5.31) for $a_0(y)$ gives

$$(5.35) \qquad a_0(y) \;=\; \begin{cases} c_+\,y^{\frac{1}{2}(1+\lambda)} + c_-\,y^{\frac{1}{2}(1-\lambda)} & \text{if } \lambda \neq 0 \\ c_+\,y^{\frac{1}{2}}\,\log y + c_-\,y^{\frac{1}{2}} & \text{if } \lambda = 0\,. \end{cases}$$

For $\lambda \in i\mathbb{R}$, only $a_0(y) \equiv 0$ is compatible with (5.30), and the embedding is cuspidal at ∞. If $\lambda \in \mathbb{R}$ and $0 < |\lambda| < 1$, square-integrability forces $c_+ = 0$ or $c_- = 0$ depending on the sign of λ; this is the possibly non-cuspidal case referred to in the lemma, and the assertion of the lemma then follows from (5.34–35). In the case of the discrete series, the first order differential equation implicit in lemma 5.19 can be satisfied only when $c_+ = 0$. However, in the setting of lemma 5.19, λ is a strictly negative integer, so c_- must also vanish if F is to be square-integrable. Thus, in this remaining case, the embedding must again be cuspidal at ∞. $\square$

I shall need an analogue of lemma 5.23 that applies to certain anti-derivatives of the function $F_{\tau,v'}$. The hypotheses are the same as those for lemma 5.23. As usual,

$[a]$ shall denote the largest integer less than or equal to $a \in \mathbb{R}$. Note that $[\frac{|\mathrm{Re}\,\lambda|+1}{2}] = 0$ when the embedding τ fails to be cuspidal, since then $\lambda \in \mathbb{R},\ 0 < |\lambda| < 1$.

5.36 Lemma. *There exist C^∞ functions $G_k(x,y)$, $k = 1,2,\dots,[\frac{|\mathrm{Re}\,\lambda|+1}{2}]$ such that $\frac{\partial^k}{\partial x^k} G_k(x,y) = F_{\tau,v'}(x,y)$ and, for $0 < y \le \frac{1}{2}$,*

$$|G_k(x,y)|\,,\ |y\tfrac{\partial}{\partial x}G_k(x,y)|\,,\ |y\tfrac{\partial}{\partial y}G_k(x,y)| \ \le\ \begin{cases} C\,y^k & \text{if } k < \frac{|\mathrm{Re}\,\lambda|+1}{2} \\ C\,y^k\,|\log y| & \text{if } k = \frac{|\mathrm{Re}\,\lambda|+1}{2} \end{cases},$$

with a positive constant $C = C(x,k,\tau,v')$ that can be chosen locally uniformly in the variable x. If the embedding τ is non-cuspidal, there exists a C^∞ function $G_1(x,y)$, such that $\frac{\partial}{\partial x}G_1(x,y) = F_{\tau,v'}(x,y)$ and, for $0 < y \le \frac{1}{2}$,

$$|G_1(x,y)|\,,\ |y\tfrac{\partial}{\partial x}G_1(x,y)|\,,\ |y\tfrac{\partial}{\partial y}G_1(x,y)| \ \le\ C\,y^{\frac{1+|\lambda|}{2}}\,|\log y|\,,$$

again with $C = C(x,\tau,v') > 0$ depending locally uniformly on x.

Proof. Since π' is irreducible unitary, the parameter λ must be real or purely imaginary. The statement holds vacuously if $\mathrm{Re}\,\lambda = 0$, and becomes obvious if π is the trivial representation. Also, the assertion of the lemma remains unchanged when λ is replaced by $-\lambda$. I can therefore assume

$$\lambda \in \mathbb{R},\ \ \lambda > 0\,, \quad \pi \neq \text{ trivial representation}\,. \tag{5.37}$$

Let $\mathfrak{n} = \mathbb{C}e$ denote the one dimensional Lie algebra spanned by e. Then $\mathfrak{n}$ acts on the space of C^∞ vectors V'^∞. It is known, and can also be checked by direct computation, that the (Lie algebra) $\mathfrak{n}$-homology of V'^∞ – for any infinite dimensional, irreducible representation π' – is finite dimensional and concentrated in degree zero. In other words, the action map $\mathfrak{n} \otimes V'^\infty \to V'^\infty$ has zero kernel and finite dimensional cokernel. Dually,

$$\pi(e) : V^{-\infty} \to V^{-\infty} \quad \text{is surjective}. \tag{5.38}$$

This makes it possible to inductively choose $\sigma_k \in V^{-\infty}$, $k = 0,1,2,\dots$, so that

$$\sigma_0 \ =\ \tau\,, \quad \text{and} \quad \pi(e)\sigma_{k+1} \ =\ -\,\sigma_k \quad \text{for all } k\,. \tag{5.39}$$

I define $G_k = F_{\sigma_k,v'}$; here, as in (5.10), $F_{\sigma_k,v'}$ denotes the function (5.6), but with σ_k taking the place of τ. Then $G_0 = F_{\tau,v'}$ and

$$\begin{aligned} \tfrac{\partial}{\partial x}G_k(x,y) \ &=\ \tfrac{\partial}{\partial u}\langle\, \sigma_k\,,\, \pi'(n_{x+u}\,a_t)v'\,\rangle\Big|_{u=0} \qquad\qquad (\,y = e^{2t}\,) \\ &=\ \langle\, \sigma_k\,,\, \pi'(e)\pi'(n_x\,a_t)v'\,\rangle \\ &=\ \langle\, \pi(-e)\sigma_k\,,\, \pi'(n_x\,a_t)v'\,\rangle \ =\ G_{k-1}(x,y)\,. \end{aligned} \tag{5.40}$$

I first consider the cuspidal case. In view of (5.37), the assertion of the lemma is then vacuous unless $\lambda \geq 1$. This forces π to belong to the discrete series or the limits of the discrete series.

I prove the bounds on $|G_k(x,y)|$, $|y\frac{\partial}{\partial x}G_k(x,y)|$, and $|y\frac{\partial}{\partial y}G_k(x,y)|$ by induction on k. Lemma 5.23 establishes the bounds for $k=0$. For the induction step, I fix the integer k, $0<k\leq\frac{\lambda+1}{2}$, and suppose that the bounds are known at the previous step. The bound for $|y\frac{\partial}{\partial x}G_k(x,y)|$ follows immediately from (5.40) and the inductive hypothesis. Until now the choice of the (non-zero) $K_\mathbb{R}$-isotypic vector v' has been arbitrary. Taking finite linear combinations, one can draw the following conclusion from the induction hypothesis: If $0\leq\ell<k$,

$$(5.41)\qquad |\langle\, \sigma_\ell\,,\, \pi'(n_x a_t)w'\,\rangle| \;\leq\; C(x,\sigma_\ell,w')\, e^{2\ell t} \quad \text{for } t\ll -1 \text{ and } w'\in V'_0\,;$$

here V'_0 denotes the space of $K_\mathbb{R}$-finite vectors in V' – in other words, the linear span of the $K_\mathbb{R}$-isotypic vectors. Similarly, the bound for $|G_k(x,y)|$ would follow from

$$(5.42)\qquad |\langle\, \sigma_\ell\,,\, \pi'(n_x a_t)w'\,\rangle| \;\leq\; \begin{cases} C(x,\sigma_\ell,w')\, e^{2kt} & \text{if } k<\frac{\lambda+1}{2} \\ C(x,\sigma_\ell,w')|t|\, e^{2kt} & \text{if } k=\frac{\lambda+1}{2}\,, \end{cases}$$

in both cases for $t\ll-1$ and arbitrary $w'\in V'_0$. Since

$$(5.43)\qquad \begin{aligned} y\tfrac{\partial}{\partial y}G_k(x,y) \;&=\; \tfrac{1}{2}\tfrac{\partial}{\partial s}\langle\, \sigma_k\,,\, \pi'(n_x\,,a_{t+s})v'\,\rangle\Big|_{s=0} \\ &=\; \tfrac{1}{2}\,\langle\, \sigma_k\,,\, \pi'(n_x a_t)\pi'(h)v'\,\rangle\,, \end{aligned}$$

and since V'_0 is $\pi'(\mathfrak{g})$-invariant, (5.42) also implies the bound for $|y\frac{\partial}{\partial y}G_k(x,y)|$. I claim:

$$(5.44)\qquad \text{the assertion (5.42) is correct when } w'\in\pi'(\mathfrak{n})V'_0\,.$$

Indeed, this follows from the bound for $|y\frac{\partial}{\partial x}G_k(x,y)|$, which has already been established, and the identity

$$(5.45)\qquad \begin{aligned} y\tfrac{\partial}{\partial x}G_k(x,y) \;&=\; e^{2t}\,\tfrac{\partial}{\partial u}\langle\, \sigma_k\,,\, \pi'(n_{x+u}\,a_t)v'\,\rangle\Big|_{u=0} \\ &=\; e^{2t}\,\langle\, \sigma_k\,,\, \pi'(n_x)\pi'(e)\pi'(a_t)v'\,\rangle \\ &=\; \langle\, \sigma_k\,,\, \pi'(n_x a_t)\pi'(e)v'\,\rangle\,. \end{aligned}$$

It now suffices to establish (5.42) for a collection of vectors w' whose images span $V'_0/\pi'(\mathfrak{n})=H_0(\mathfrak{n},V'_0)$. Since $\mathfrak{a}=\mathbb{C}h$ normalizes $\mathfrak{n}$, it acts on $V'_0/\pi'(\mathfrak{n})$. In the current situation, it was remarked, π' belongs to the discrete series or limits of the discrete series. For representations of this type,

(5.46)

a) $\dim\big(V'_0/\pi'(\mathfrak{n})\big) \;=\; 1\,;$

b) $\pi'(h)$ acts on $V'_0/\pi'(\mathfrak{n})$ as multiplication by $\lambda-1\,;$

c) the image of any non-zero $K_\mathbb{R}$-isotypic vector spans $V'_0/\pi'(\mathfrak{n})\,.$

These statements are known, and also follow from the discussion of related matters in the proof of lemma 5.65 below. I conclude that it suffices to verify (5.42) for the single vector $w' = \frac{1}{2}(\pi'(h) + 1 - \lambda)v'$, with v' as in (5.5). Now

$$(5.47) \qquad \langle\, \sigma_k\,, \pi'(n_x\, a_t)(\pi'(h) + 1 - \lambda)v' \,\rangle \;=\; -2\left(\tfrac{1-\lambda}{2} - y\,\tfrac{\partial}{\partial y}\right) G_k(x,y)\,,$$

so the bounds (5.42), for all $w' \in V_0'$, follow from the single bound

$$(5.48) \qquad \left|\left(\tfrac{1-\lambda}{2} - y\,\tfrac{\partial}{\partial y}\right) G_k(x,y)\right| \;\le\; \begin{cases} C(x,k,\tau,v')\, y^k & \text{if } k < \frac{\lambda+1}{2} \\ C(x,k,\tau,v')\, y^k\, |\log y| & \text{if } k = \frac{\lambda+1}{2}\,, \end{cases}$$

for $0 < y \le \frac{1}{2}$. Recall that $G_k = F_{\sigma_k, v'}$. Corollary (5.18) remains valid with $-\lambda$ in place of λ, as was remarked below the statement of the corollary. Hence, for $0 < y \le \frac{1}{2}$,

$$(5.49) \qquad \begin{aligned} &\left| y^{-\frac{\lambda+1}{2}} \left(\tfrac{1-\lambda}{2} - y\,\tfrac{\partial}{\partial y}\right) G_k(x,y)\right| - \left| 2^{\frac{\lambda+1}{2}} \left(\tfrac{1-\lambda}{2} - \tfrac{1}{2}\,\tfrac{\partial}{\partial y}\right) G_k(x,\tfrac{1}{2})\right| \\ &\quad\le\; \left| y^{-\frac{\lambda+1}{2}} \left(\tfrac{1-\lambda}{2} - y\,\tfrac{\partial}{\partial y}\right) G_k(x,y) \;-\; 2^{\frac{\lambda+1}{2}} \left(\tfrac{1-\lambda}{2} - \tfrac{1}{2}\,\tfrac{\partial}{\partial y}\right) G_k(x,\tfrac{1}{2})\right| \\ &\quad=\; \left| \int_y^{\frac{1}{2}} s^{-\frac{\lambda+1}{2}} \left(s\,\tfrac{\partial^2}{\partial x^2} - ir\tfrac{\partial}{\partial x}\right) G_k(x,s)\, ds \right| \\ &\quad=\; \left| \int_y^{\frac{1}{2}} s^{-\frac{\lambda+1}{2}} \left(s\,\tfrac{\partial}{\partial x} - ir\right) G_{k-1}(x,s)\, ds \right| \\ &\quad\le\; D \int_y^{\frac{1}{2}} s^{k-1-\frac{\lambda+1}{2}}\, ds \;\le\; \begin{cases} D\,\frac{2}{\lambda+1-2k}\, y^{k-\frac{\lambda+1}{2}} & \text{if } k < \frac{\lambda+1}{2} \\ D\,|\log y| & \text{if } k = \frac{\lambda+1}{2}\,, \end{cases} \end{aligned}$$

with $D = (1+|r|)C(x,k-1,\tau,v')$. The inequality (5.48) now follows when I bring the second term on the left all the way to the other side of the inequality and multiply through by $y^{\frac{\lambda+1}{2}}$. This implies the lemma in the cuspidal case.

Turning to the non-cuspidal case, I continue with the hypotheses (5.37) and the definition (5.39). In the current situation, $0 < \lambda < 1$. Lemma 5.24 provides bounds

$$(5.50) \qquad |G_0(x,y)|\,,\; |y\tfrac{\partial}{\partial x} G_0(x,y)|\,,\; |y\tfrac{\partial}{\partial y} G_0(x,y)| \;\le\; C\, y^{\frac{\lambda-1}{2}} \qquad (\, 0 < y \le 1\,)\,,$$

with $C = C(\tau, v')$ – recall that $G_0(x,y) = F_{\tau,v'}(x,y)$. This immediately implies the bound for $|y\frac{\partial}{\partial x} G_1(x,y)|$ since $\frac{\partial}{\partial x} G_1(x,y) = G_0(x,y)$. I now follow the pattern of inequalities (5.49), simultaneously with the parameter λ and its negative; now the two cases are similar since the exponent of y in the first term is strictly negative with either choice of sign. Skipping some of the intermediary steps and using

(5.50), I find

(5.51)

$$\begin{aligned}
&\left| y^{-\frac{1\pm\lambda}{2}} \left(\tfrac{1\mp\lambda}{2} - y\tfrac{\partial}{\partial y}\right) G_1(x,y) \right| - \left| 2^{\frac{1\pm\lambda}{2}} \left(\tfrac{1\mp\lambda}{2} - \tfrac{1}{2}\tfrac{\partial}{\partial y}\right) G_1(x,\tfrac{1}{2}) \right| \\
&\qquad \leq \left| \int_y^{\frac{1}{2}} s^{-\frac{1\pm\lambda}{2}} \left(s\tfrac{\partial}{\partial x} - ir\right) G_0(x,s)\, ds \right| \\
&\qquad \leq D \int_y^{\frac{1}{2}} s^{\frac{\lambda-1}{2}-\frac{1\pm\lambda}{2}}\, ds \leq \begin{cases} D\,|\log y| & \text{for the upper choice of sign} \\ D\,\lambda^{-1} y^{\lambda} & \text{for the lower choice of sign}, \end{cases}
\end{aligned}$$

for $0 < y \leq \frac{1}{2}$, and with $D = (1+|r|)C(\tau,\nu)$. I multiply through by $y^{\frac{1\pm\lambda}{2}}$, rearrange terms, and take appropriate linear combinations of the two resulting inequalities to get the desired conclusion. □

5.52 Lemma. *1) If* $\operatorname{Re}\lambda \leq 1$ *and if* τ *corresponds to a cuspidal embedding, the limit*

$$\lim_{y\to 0^+}\left(y^{\frac{\lambda-1}{2}} (\tfrac{1+\lambda}{2} - y\tfrac{\partial}{\partial y}) F_{\tau,\nu'}(x,y)\right)$$

exists in $C^{-\infty}(\mathbb{R})$*, equipped with the strong distribution topology, and the limit lies in* $C^{\frac{1}{2}(\operatorname{Re}\lambda-1)}(\mathbb{R})$ *if* $\frac{1}{2}(\operatorname{Re}\lambda - 1) \notin \mathbb{Z}$*, or in* $C^{<\frac{1}{2}(\operatorname{Re}\lambda-1)}(\mathbb{R})$ *if* $\frac{1}{2}(\operatorname{Re}\lambda-1) \in \mathbb{Z}$.
2) If $\lambda \in \mathbb{R}$*,* $0 < \lambda < 1$*, with* τ *corresponding to a non-cuspidal embedding, the limits*

$$\lim_{y\to 0^+}\left(y^{\frac{\lambda-1}{2}} (\tfrac{1+\lambda}{2} - y\tfrac{\partial}{\partial y}) F_{\tau,\nu'}(x,y)\right), \quad \lim_{y\to 0^+}\left(y^{\frac{-\lambda-1}{2}} (\tfrac{1-\lambda}{2} - y\tfrac{\partial}{\partial y}) F_{\tau,\nu'}(x,y)\right)$$

exist and lie in $C^{\lambda-1}(\mathbb{R})$ *and* $C^{<(-1)}(\mathbb{R})$*, respectively.*
3) If π *belongs to the limits of the discrete series – in which case* $\lambda = 0$ *– also*

$$\lim_{y\to 0^+}\left(y^{\frac{1}{2}} F_{\tau,\nu'}(x,y)\right)$$

exists in the strong distribution topology, and the limit lies in $C^{-\frac{1}{2}}(\mathbb{R})$.

The reason for the exceptional statement for the limits of the discrete series will become apparent in the course of the proof of lemma 5.65 below. Formally, the assertion of the lemma remains correct even without hypothesis that $\operatorname{Re}\lambda \leq 1$ – in the other cases, the limits can be shown to be zero.

Proof. Note that $1 - \operatorname{Re}\lambda \geq 0$. I set $k = [\frac{1-\operatorname{Re}\lambda}{2}]$ and let $G_k(x,y)$ denote the function whose existence is asserted in lemma 5.36 when $k > 0$; when $k = [\frac{1-\operatorname{Re}\lambda}{2}] = 0$, I set $G_0(x,y) = F_{\tau,\nu'}(x,y)$. In both cases I regard $y \mapsto G_k$ as a function of the variable y, with values in the space of distributions on the real line. In view of (5.18), for

$u > 0$,

(5.53)

$$\begin{aligned}
&\lim_{y\to 0^+}\Big(y^{\frac{\lambda-1}{2}}\big(\tfrac{1+\lambda}{2}-y\tfrac{\partial}{\partial y}\big)G_k(x,y)\Big) \;=\\
&= \;u^{\frac{\lambda-1}{2}}\big(\tfrac{1+\lambda}{2}-u\tfrac{\partial}{\partial y}\big)G_k(x,u)\;-\;\lim_{y\to 0^+}\int_y^u s^{\frac{\lambda-1}{2}}\Big(s\tfrac{\partial^2}{\partial x^2}-ir\tfrac{\partial}{\partial x}\Big)G_k(x,s)\,ds\\
&= \;u^{\frac{\lambda-1}{2}}\big(\tfrac{1+\lambda}{2}-u\tfrac{\partial}{\partial y}\big)G_k(x,u)\;-\;\tfrac{\partial}{\partial x}\Big(\int_0^u s^{\frac{\lambda-1}{2}}\Big(s\tfrac{\partial}{\partial x}-ir\Big)G_k(x,s)\,ds\Big)\,,
\end{aligned}$$

provided the integral on the right hand side converges, locally uniformly in the variable x. I suppose initially that the embedding is cuspidal. According to (5.36) and (5.23), for $0 < y \le \frac{1}{2}$, $|G_k(x,y)|$ and $|\,y\frac{\partial}{\partial x}G_k(x,y)\,|$ are bounded by Cy^k or by $Cy^k|\log y|$, depending on whether $\frac{1-\mathrm{Re}\,\lambda}{2}$ is non-integral or integral; the constant $C = C(x,k,\tau,v')$ can be chosen locally uniformly in x. Recall that $1-\mathrm{Re}\,\lambda \ge 0$ by assumption, and $k = [\frac{1-\mathrm{Re}\,\lambda}{2}]$. Now, for $0 < y_1 < y_2 \le \frac{1}{2}$,

(5.54)

$$\begin{aligned}
&\Big|\int_{y_1}^{y_2} s^{\frac{\lambda-1}{2}}\Big(s\tfrac{\partial}{\partial x}-ir\Big)G_k(x,s)\,ds\Big|\\
&\le\begin{cases}(1+|r|)\,C\displaystyle\int_{y_1}^{y_2} s^{k-\frac{1-\mathrm{Re}\lambda}{2}}\,ds \;\le\; D\,y_2^{[\frac{1-\mathrm{Re}\lambda}{2}]-\frac{1-\mathrm{Re}\lambda}{2}+1} & \text{if } \frac{1-\mathrm{Re}\lambda}{2}\notin\mathbb{Z}\\[2ex]
(1+|r|)\,C\displaystyle\int_{y_1}^{y_2} |\log s|\,s^{k-\frac{1-\mathrm{Re}\lambda}{2}}\,ds \;\le\; D\,y_2\,|\log y_2| & \text{if } \frac{1-\mathrm{Re}\lambda}{2}\in\mathbb{Z}\,.\end{cases}
\end{aligned}$$

Here D is a positive constant with the same type of dependence on parameters as C, in particular, locally uniform dependence on x. This does imply the locally uniform convergence of the last integral in (5.53). It also implies: For $0 < u \le \frac{1}{2}$,

$$\text{(5.55)}\qquad\begin{aligned}
&\text{a)}\quad \phi_u(x)\;=_{\mathrm{def}}\;-\int_0^u s^{\frac{\lambda-1}{2}}\Big(s\tfrac{\partial}{\partial x}-ir\Big)G_k(x,s)\,ds\;\in\;C^0(\mathbb{R})\,,\\
&\text{b)}\quad |\,\phi_u(x)\,|\;\le\;D\,u^{[\frac{1-\mathrm{Re}\lambda}{2}]-\frac{1-\mathrm{Re}\lambda}{2}+1}\qquad\text{or}\quad\le\;D\,u\,|\log u|\,,
\end{aligned}$$

depending on whether $\frac{1-\mathrm{Re}\,\lambda}{2}$ is non-integral or integral. It follows that the limit on the left hand side of (5.53) is the first derivative of a continuous function,

(5.56)

$$\begin{aligned}
&\lim_{y\to 0^+}\Big(y^{\frac{\lambda-1}{2}}\big(\tfrac{1+\lambda}{2}-y\tfrac{\partial}{\partial y}\big)G_k(x,y)\Big)\;=\;\tfrac{\partial}{\partial x}\,\psi_u(x)\,,\\
&\qquad\text{where}\quad \psi_u(x)\;=_{\mathrm{def}}\;u^{\frac{\lambda-1}{2}}\int_0^x\big(\tfrac{1+\lambda}{2}-u\tfrac{\partial}{\partial y}\big)\,G_k(s,u)\,ds\;+\;\phi_u(x)\,.
\end{aligned}$$

Since partial differentiation is continuous with respect to the strong distribution topology, I can take the k-th partial derivative with respect to x and conclude

$$\text{(5.57)}\qquad \lim_{y\to 0^+}\Big(y^{\frac{\lambda-1}{2}}\big(\tfrac{1+\lambda}{2}-y\tfrac{\partial}{\partial y}\big)F_{\tau,v'}(x,y)\Big)\;=\;\tfrac{\partial^{k+1}}{\partial x^{k+1}}\,\psi_u(x)\,.$$

As defined, $\psi_u(x)$ depends on u. However, the partial derivative of $\psi_u(x)$ with respect to x is a distribution in the variable x alone, so $a(u_1,u_2) = \psi_{u_1}(x) - \psi_{u_2}(x)$ depends on u_1 and u_2, but not on x. I conclude that the difference $\psi_u(x_1) - \psi_u(x_2)$ depends on x_1 and x_2 alone, not on u. Thus, for $x, x+\delta \in \mathbb{R}$, with $0 < \delta \leq \frac{1}{2}$,

(5.58)
$$\begin{aligned}|\,\psi_u(x+\delta) - \psi_u(x)\,| \ &= \ |\,\psi_\delta(x+\delta) - \psi_\delta(x)\,| \\ &\leq \ |\,\phi_\delta(x+\delta)\,| \ + \ |\,\phi_\delta(x)\,| \ + \ \delta^{-\frac{1-\mathrm{Re}\lambda}{2}} \left| \int_x^{x+\delta} (\tfrac{1+\lambda}{2} - \delta \tfrac{\partial}{\partial y})\, G_k(s,\delta)\, ds \right| .\end{aligned}$$

To simplify the notation, I set $\alpha = [\frac{1-\mathrm{Re}\lambda}{2}] - \frac{1-\mathrm{Re}\lambda}{2} + 1$ when $\frac{1-\mathrm{Re}\lambda}{2} \notin \mathbb{Z}$, and let α denote an arbitrary number strictly between 0 and 1 when $\frac{1-\mathrm{Re}\lambda}{2} \in \mathbb{Z}$; in either case, $0 < \alpha < 1$. All three terms on the right of (5.58) are bounded by multiples of δ^α – the first two because of (5.55b) and the third because of (5.36) in case $k > 0$, or because of (5.23) in case $k = 0$. Conclusion: For fixed $u > 0$, the function $\psi_u(x)$ is locally Hölder continuous of index α. In view of (5.57), this establishes the first assertion of the lemma.

When the embedding corresponding to τ is non-cuspidal and $0 < \lambda < 1$, most of the preceeding argument still applies, with $k = 0$ and accordingly, with $G_k = F_{\tau,\nu'}$. However, I have to make the following modifications: In the present situation, according to (5.23),

$$(5.59) \quad |F_{\tau,\nu'}(x,y)|\,, \ |y\tfrac{\partial}{\partial x}F_{\tau,\nu'}(x,y)\,|\,, \ |y\tfrac{\partial}{\partial y}F_{\tau,\nu'}(x,y)\,| \ \leq \ C y^{\frac{\lambda-1}{2}} \quad \text{if } \ 0 < y \leq \ 1\,,$$

for some $C = C(\tau,\nu')$. This results in weaker bounds, namely y_2^λ instead of $y_2^{\frac{1+\mathrm{Re}\lambda}{2}}$ in (5.54) (recall $k = [\frac{1-\mathrm{Re}\lambda}{2}] = 0$), with u^λ instead of $u^{\frac{1+\mathrm{Re}\lambda}{2}}$ in (5.55b), and with multiples of δ^λ bounding the three terms in (5.58). This makes $\psi_u(x)$ Hölder continuous of index λ, and implies the first non-cuspidal case in the statement of the lemma. For the remaining non-cuspidal case, I use the equations (5.53–59) with $-\lambda$ in place of λ, with $k = 1$, and with $G_1(x,y)$ chosen as in the second part of lemma 5.36. The analogue of (5.53) in the present situation is

(5.60)
$$\begin{aligned}&\lim_{y\to 0^+}\Big(y^{-\frac{\lambda+1}{2}}(\tfrac{1-\lambda}{2} - y\tfrac{\partial}{\partial y})G_1(x,y)\Big) \ = \\ &= \ u^{-\frac{\lambda+1}{2}}(\tfrac{1-\lambda}{2} - u\tfrac{\partial}{\partial y})G_1(x,u) \ - \ \tfrac{\partial}{\partial x}\left(\int_0^u s^{-\frac{\lambda+1}{2}}\left(s\tfrac{\partial}{\partial x} - ir\right)G_1(x,s)\,ds\right),\end{aligned}$$

The weaker bounds on $|G_1(x,y)|$ and $|y\frac{\partial}{\partial x}G_1(x,y)|$ result in the inequality

(5.61)
$$\begin{aligned}&\left|\int_{y_1}^{y_2} s^{-\frac{\lambda+1}{2}}\left(s\tfrac{\partial}{\partial x} - ir\right)G_k(x,s)\,ds\right| \\ &\qquad \leq \ (1+|r|)\,C\int_{y_1}^{y_2}|\,\log s\,|\,ds \ \leq \ D y_2\,|\,\log y_2\,|\,,\end{aligned}$$

which now replaces (5.54). In the definitions of $\phi_u(x)$ and $\psi_u(x)$, the quantity λ must be replaced by $-\lambda$. Otherwise the earlier argument applies unchanged, and shows that $\psi_u(x)$ is locally Hölder continuous of index α, for any α strictly between 0 and 1, as asserted by the lemma.

Only the limits of the discrete series remain to be dealt with. I first suppose that v' is an extreme weight vector, so that $r=-1$ or $r=1$, depending on whether π' has a highest or lowest weight; in either case, (5.9) forces $\lambda=0$. I argue again as in (5.53–59), but now using lemma 5.19 instead of corollary 5.18, and with $k=0$, $G_0(x,y)=F_{\tau,v'}(x,y)$, $\lambda=0$. Now (5.53) gets replaced by

(5.62)

$$\lim_{y\to 0^+}\left(y^{-\frac{1}{2}}F_{\tau,v'}(x,y)\right) \;=\; u^{-\frac{1}{2}}F_{\tau,v'}(x,u) \,-\, \tfrac{\partial}{\partial x}\left(i\eta\int_0^u s^{-\frac{1}{2}}F_{\tau,v'}(x,s)\,ds\right),$$

and (5.54–59) get modified accordingly. The limit is now the first derivative of $\psi_u(x)$, and $\psi_u(x)$ is locally Hölder continuous of index $\frac{1}{2}$. This establishes the third part of the lemma, in the special case of an extreme weight vector v'. The general case then follows formally; the reduction of the general to the special case involves ideas used in the proof of lemma 5.36. I shall skip the details, since the special case suffices for the proof of theorem 2.18. □

To make the connection with the statement of theorem 2.18, I suppose that (π,V) has been identified with a subrepresentation of a principal series representation. Since the Casimir operator acts as a multiple of the identity in any principal series representation, it must act by the same constant on V and on the ambient principal series representation. From (5.8) and the definition (2.5), one sees that $\pi_{\lambda,\varepsilon}(\Omega)=(\lambda^2-1)1_{V_{\lambda,\varepsilon}}$. Comparing this with (5.9) and noting that the Casimir operator is formally self-adjoint, I conclude that the constant λ in the statement of theorem 2.18 coincides with $\pm\lambda$ in this section. Since $\pm\lambda$ play symmetric roles, I may as well suppose that the constants λ in theorem 2.18 and in lemma 5.52 agree. Thus

$$V^{-\infty} \hookrightarrow V_{\lambda,\varepsilon}^{-\infty}, \qquad V_{-\lambda,\varepsilon}^{\infty} \twoheadrightarrow V'^{\infty}, \tag{5.63}$$

with $\varepsilon=+1$ or $\varepsilon=-1$, depending on whether V has only even or only odd $K_{\mathbb{R}}$-weights. When π is a representation of the principal or complementary series, the maps (5.63) are isomorphisms, and $\pm\lambda$ occur on an equal footing. On the other hand, (5.63) forces

$$\begin{aligned}&\lambda\in\mathbb{Z}_{\leq 0},\quad \lambda\equiv\tfrac{\varepsilon+1}{2}\mod(2),\quad \text{if } \pi \text{ belongs to}\\ &\text{the discrete series or limits of the discrete series,}\end{aligned} \tag{5.64}$$

and $(\lambda,\varepsilon)=(1,+1)$ when π is the trivial representation. In the latter two cases, the symmetry between λ and $-\lambda$ is broken. However, the hypothesis $\operatorname{Re}\lambda\leq 1$ of lemma 5.52 is satisfied whenever embeddings and quotient maps (5.63) exist. Recall that τ_λ denotes the image in $(V_{\lambda,\varepsilon}^{-\infty})^\Gamma$ of the distribution vector (1.2–3).

5.65 Lemma. *When* $(\lambda,\varepsilon) \neq (0,-1)$, $\lim_{y\to 0^+}\Big(y^{\frac{\lambda-1}{2}}(\frac{1+\lambda}{2} - y\frac{\partial}{\partial y})F_{\tau,v'}(x,y)\Big)$ *is a non-zero multiple of the image of* τ_λ *in* $C^{-\infty}(\mathbb{R})$*, via the distribution analogue of the isomorphism (2.8). In the remaining case* $(\lambda,\varepsilon) = (0,-1)$*, the limit* $\lim_{y\to 0^+} y^{-\frac{1}{2}}F_{\tau,v'}(x,y)$ *is a non-zero multiple of the image of* τ_λ *in* $C^{-\infty}(\mathbb{R})$*, again via the distribution analogue of the isomorphism (2.8).*

Note that the existence of the limits referred to in the preceeding statement has already been shown in the proof of lemma 5.52, where τ was assumed to arise from an embedding $V' \hookrightarrow L^2(\Gamma\backslash G_{\mathbb{R}})$. The proof of the current lemma also establishes the existence of the limits, with respect to the strong distribution topology, without any particular hypotheses on the distribution vector $\tau \in V^{-\infty}$; this will be used in the proof of theorem 2.20.

Proof. It is possible to argue by direct computation. However, I shall give an argument that uses representation-theoretic facts valid in any semisimple group $G_{\mathbb{R}}$; these facts can, of course, be verified directly in the special case $G_{\mathbb{R}} = SL(2,\mathbb{R})$. I let V_0 and V_0' denote the subspaces of $K_{\mathbb{R}}$-finite vectors in V and V', respectively – in other words, the linear span of all $K_{\mathbb{R}}$-isotypic vectors in the two spaces. The complexified Lie algebra $\mathfrak{g}$ and its universal enveloping algebra $U(\mathfrak{g})$ act on V_0 and V_0'. For

$$v' \in V_0', \qquad v \in V_0, \tag{5.66}$$

the function $t \mapsto \langle v, \pi'(a_t)v'\rangle$ has a convergent series expansion near $t = -\infty$, in powers of e^{2t},

$$\begin{aligned}\langle v, \pi'(a_t)v'\rangle \ &= \ e^{(1-\lambda)t}\left(a_0(v,v') + a_1(v,v')\,e^{2t} + a_2(v,v')\,e^{4t} + \dots\right)\\ &\quad + \ e^{(1+\lambda)t}\left(b_0(v,v') + b_1(v,v')\,e^{2t} + b_2(v,v')\,e^{4t} + \dots\right)\end{aligned} \tag{5.67a}$$

when π is a principal series or complementary series representation and $(\lambda,\varepsilon) \neq (0,+1)$,

$$\begin{aligned}\langle v, \pi'(a_t)v'\rangle \ &= \ t\,e^{t}\left(a_0(v,v') + a_1(v,v')\,e^{2t} + a_2(v,v')\,e^{4t} + \dots\right)\\ &\quad + \ e^{t}\left(b_0(v,v') + b_1(v,v')\,e^{2t} + b_2(v,v')\,e^{4t} + \dots\right)\end{aligned} \tag{5.67b}$$

when π is the principal series representation with $(\lambda,\varepsilon) = (0,+1)$, and

$$\langle v, \pi'(a_t)v'\rangle \ = \ e^{(1-\lambda)t}\left(a_0(v,v') + a_1(v,v')\,e^{2t} + a_2(v,v')\,e^{4t} + \dots\right) \tag{5.67c}$$

when π is a representation of the discrete series or limits of the discrete series and $\lambda \leq 0$, as in (5.64) [7]. The function $t \mapsto \langle v, \pi'(a_t)v' \rangle$, with v and v' $K_{\mathbb{R}}$-isotypic, satisfies an ordinary differential equation in the variable $y = e^{2t}$, with a regular singular point at $y = 0$. The indicial equation has distinct roots precisely when $\lambda \neq 0$, and this explains the appearance of the linear term in (5.67b); the fact that no linear term appears in (5.67c) with $\lambda = 0$ – in the other case when the two roots coincide – reflects the reducibility of the principal series representation $V_{0,-1}$. The coefficients are uniquely determined, except in the situation (5.67a) with $\lambda \in \mathbb{Z}$. This exceptional case never occurs for an irreducible unitary representation, and so the coefficients *are* uniquely determined in the cases of interest.

The series (5.67) can be differentiated term-by-term, since the change of variables $y = e^{2t}$ transforms them into slight variants of power series. Also, certain properties of the coefficients hold for formal reasons. Specifically, we shall use:

$$(5.68) \qquad \begin{aligned} a_0(\pi(h)v, v') &= (\lambda - 1)\, a_0(v, v'), & a_0(\pi(f)v, v') &= 0, \\ a_0(v, \pi'(h)v') &= (-\lambda + 1)\, a_0(v, v'), & a_0(v, \pi'(e)v') &= 0. \end{aligned}$$

Indeed, the two identities on the left reflect the fact that h is the infinitesimal generator of the one-parameter group $\{a_t\}$. As for the other two, $\mathrm{Ad}\, a_t(f) = e^{-2t} f$, so $a_0(\pi(f)v, v')$ is the coefficient of $e^{(-\lambda-1)t}$, or of te^{-t} if $(\lambda, \varepsilon) = (0, +1)$, in the series expansion of $t \mapsto \langle \pi(f)v, \pi'(a_t)v' \rangle$, but this term does not occur at all; $a_0(v, \pi'(e)v')$ vanishes for the same reason. These assertions hold uniformly for the series (5.67a-c), with $\lambda = 0$ in the case of (5.67b).

The series may not converge when v is only a C^∞ vector, and not a $K_{\mathbb{R}}$-finite vector, as has been assumed so far. However,

(5.69)

a) for $v' \in V_0'$ fixed, the linear functions $v \mapsto a_j(v, v')$ and $v \mapsto b_j(v, v')$ are continuous in the topology of V^∞;

b) for $v \in V^\infty$ and $v' \in V_0'$, the series (5.67) remain valid in the weaker sense of asymptotic expansions for $t \to -\infty$,

with the k-th remainder bounded by a multiple of the same power of e^t as that of the next term, and a coefficient that is one of the semi-norms defining the topology of V^∞ [11]. Because of a), the asymptotic expansions can still be differentiated term-by-term.

I set $\mathfrak{n} = \mathbb{C}e$, $\mathfrak{n}^- = \mathbb{C}f$; this notation is consistent with the conventions of §2. Since a_0 is a bilinear function of its two arguments, (5.68) implies that it induces a bilinear pairing between $V_0/\pi(\mathfrak{n}^-)V_0$ and $V_0'/\pi'(\mathfrak{n})V_0'$. Since $\{a_t\}$ normalizes $\mathfrak{n}$ and $\mathfrak{n}^-$, h operates on both quotients. For $\mu \in \mathbb{C}$, I define

$$(5.70) \qquad \begin{aligned} \left(V_0'/\pi'(\mathfrak{n})V_0'\right)_\mu &=_{\text{def}} (\mu + 1) - \text{eigenspace of } \pi'(h), \\ \left(V_0/\pi(\mathfrak{n}^-)V_0\right)_\mu &=_{\text{def}} (\mu - 1) - \text{eigenspace of } \pi'(h); \end{aligned}$$

the reason for the shifts by ± 1 will become apparent soon. Now, invoking (5.68) once more, I find

(5.71)
$$a_0(\ ,\ )\ \text{ induces a bilinear pairing }\ \big(V_0/\pi(\mathfrak{n}^-)\big)_\lambda \times \big(V_0'/\pi'(\mathfrak{n})V_0'\big)_{-\lambda} \longrightarrow \mathbb{C}.$$

For the next statement, I set $\varepsilon = +1$ or $\varepsilon = -1$, depending on whether π has even or odd $K_\mathbb{R}$-weights. Then, according to Casselman's reciprocity theorem,

$$\begin{gathered}\big(V_0/\pi(\mathfrak{n}^-)V_0\big)_\lambda \cong \operatorname{Hom}_{G_\mathbb{R}}(V^\infty, V^\infty_{\lambda,\varepsilon}),\\ \big(V_0'/\pi'(\mathfrak{n})V_0'\big)_{-\lambda} \cong \operatorname{Hom}_{G_\mathbb{R}}(V'^\infty, {V'}^{\infty}_{\lambda,\varepsilon}),\\ \text{and the pairing (5.71) is non-degenerate.}\end{gathered} \tag{5.72}$$

In the case of the group $G_\mathbb{R} = SL(2,\mathbb{R})$, embeddings into (possibly non-unitary) principal series representations are unique up to scaling. Thus, in the situation (5.63),

$$\dim\big(V_0/\pi(\mathfrak{n}^-)V_0\big)_\lambda = \dim\big(V_0'/\pi'(\mathfrak{n})V_0'\big)_{-\lambda} = 1. \tag{5.73}$$

The isomorphism (5.72) can be described explicitely, as follows. I fix a vector $v' \in V_0'$ whose image generates $(V_0'/\pi'(\mathfrak{n})V_0')_{-\lambda}$. Then $v \mapsto a_0(v,v')$ spans the dual space of $(V_0/\pi(\mathfrak{n}^-)V_0)_\lambda$, and

$$\begin{gathered}V^\infty \hookrightarrow V^\infty_{\lambda,\varepsilon} \cong \{\, f \in C^\infty(\mathbb{R}) \mid (\operatorname{sgn} x)^{\frac{1+\varepsilon}{2}}|x|^{\lambda-1} f(\tfrac{1}{x}) \in C^\infty(\mathbb{R})\,\}\\ V^\infty \ni v \longmapsto f_v \in C^\infty(\mathbb{R}),\ \ f_v(x) = a_0(\pi(n_{-x})v, v');\end{gathered} \tag{5.74a}$$

this involves the isomorphism (2.8), the notation (2.3), and (5.69a), which ensures that $a_0(\pi(n_{-x})v,v')$ is well-defined for $v \in V^\infty$. The embedding is usually described in terms of the "compact realization" (2.7) of $V^\infty_{\lambda,\varepsilon}$, but the two statements are equivalent because of the formal properties (5.68) of the bilinear function a_0. Embeddings into principal series representations extend continuously to the spaces of distribution vectors. In particular,

$$\text{the inclusion (5.74a) extends continuously to } V^{-\infty} \hookrightarrow V^{-\infty}_{\lambda,\varepsilon}. \tag{5.74b}$$

The general facts I have used here are proved in [5, 11, 6]. The inclusion (5.74b) is unique up to scaling, and thus coincides with (5.63), again up to scaling. The function corresponding to a non-zero $K_\mathbb{R}$-isotypic vector $v \in V^\infty_{\lambda,\varepsilon}$ via the "compact realization" (2.7), suitably renormalized, is a character of $K_\mathbb{R}$, and therefore takes the value 1 at the identity. In terms of the "unbounded realization" (2.8), this means that $f_v(0) = 1$. Thus, by (5.68) and (5.73–74), any $K_{\mathbb{R}}$-isotypic vector in V maps to a generator of $\left(V_0/\pi(\mathfrak{n}^-)V_0\right)_\lambda$. Since π and π' play symmetric roles in (5.68),

(5.75)

$$\text{the image of any non-zero } K_{\mathbb{R}}\text{-isotypic vector in } V' \text{ spans } \left(V_0'/\pi'(\mathfrak{n})V_0'\right)_{-\lambda}.$$

In particular, the vector (5.5) which was used in the definition of $F_{\tau,v'}$ can and shall play the role of v' in (5.74). Multiplying v' by a suitable multiple, I can arrange that (5.74b) coincides exactly with (5.63), not just up to a scale factor.

Distributions do not have values at points. For that reason, one cannot describe (5.74b) quite as explicitly as the embedding on the level of C^∞-vectors. However, one can get a similar expression after smoothing distribution vectors by integration against a test function. For $\psi \in C_c^\infty(\mathbb{R})$, I define

(5.76a)
$$\pi_{N_{\mathbb{R}}}(\psi) \;=\; \int_{\mathbb{R}} \psi(x)\,\pi(n_x)\,dx \;\in\; \operatorname{End}(V^\infty)\,.$$

The integral is well defined because the $\operatorname{End}(V^\infty)$-valued function $x \mapsto \psi(x)\pi(n_x)$ is continuous in the strong operator topology and has compact support. I claim:

(5.76b)
$$\pi_{N_{\mathbb{R}}}(\psi) \text{ extends continuously to } V^{-\infty}, \text{ and is a smoothing operator,}$$

in the sense that the extension is a linear map $V^{-\infty} \to V^\infty$, continuous with respect to the strong distribution topology on $V^{-\infty}$ and the C^∞ topology on V^∞. To see this, I pick $v \in V^\infty$ and let $f_v \in C^\infty(\mathbb{R})$ denote the function which corresponds to v via the embedding $V^\infty \hookrightarrow V^\infty_{\lambda,\varepsilon}$ and the isomorphism (2.8). Recall that $N_{\mathbb{R}} \cong \mathbb{R}$ acts on f_v via translation – cf. (2.9). Thus, with $w = \pi_{N_{\mathbb{R}}}(\psi)v$,

(5.77)
$$f_w(x) \;=\; \int_{\mathbb{R}} f_v(x-u)\,\psi(u)\,du\,.$$

Since $\psi \in C_c^\infty(\mathbb{R})$, the smoothing property is apparent except for the behavior of f_w at ∞. To deal with the missing point, I let $g_v, g_w \in C^\infty(K_{\mathbb{R}})$ denote the functions which correspond to v and $w = \pi_{N_{\mathbb{R}}}(\psi)v$ via the "compact realization" (2.7) of $V^\infty_{\lambda,\varepsilon}$. They are related to f_v and f_w by the formulas

(5.78)
$$f_v(\tan\theta) \,=\, (\cos\theta)^{1-\lambda} g_v(k_\theta)\,,\;\; f_w(\tan\theta) \,=\, (\cos\theta)^{1-\lambda} g_w(k_\theta)\;\; \left(-\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2}\right),$$

which is a consequence of the definition (2.5) and the identity

(5.79)
$$k_\theta \,=\, n_x a_t n^- \quad \text{when} \quad -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2}\,,\; x = \tan\theta\,,\; e^{-t} = \cos\theta\,,\; n^- \in N^-_{\mathbb{R}}\,.$$

With this change of variables, the equation (5.77) takes the form

$$g_w(k_\theta) \;=\; \int_{\mathbb{R}} \left(\frac{\cos\theta}{\cos\alpha(\theta,u)}\right)^{\lambda-1} g_v(k_{\alpha(\theta,u)})\,\psi(u)\,du\,, \qquad \text{where} \quad \alpha(\theta,u) \;=\; \tan^{-1}(\tan\theta-u)\,. \tag{5.80}$$

Near $\theta=\frac{\pi}{2}$, the function $\cot\theta$ is a coordinate, and

$$\cot\alpha(\theta,u) \;=\; (1-u\cot\theta)^{-1}\cot\theta\,, \quad \alpha(\tfrac{\pi}{2},u) \;=\; \tfrac{\pi}{2}\,. \tag{5.81}$$

It follows that $\alpha(\theta,u)$ is a real analytic function, and $\frac{\cos\theta}{\cos\alpha(\theta,u)}$ takes the value 1 at $\theta=\frac{\pi}{2}$, independently of the value of u. These facts, coupled with the hypothesis that ψ have compact support, imply the smoothing property of $\pi_{N_{\mathbb{R}}}$ also at the point at infinity.

I continue with the notation f_v, f_w, for $v\in V^\infty$ and $w=\pi_{N_{\mathbb{R}}}(\psi)v$. From the formulas (5.74a), (5.77), one sees that

$$a_0(\pi_{N_{\mathbb{R}}}(\psi)v,v') \;=\; f_w(0) \;=\; \int_{\mathbb{R}} f_v(u)\,\psi(-u)\,du\,. \tag{5.82}$$

In this identity, I let v run through a sequence $\{v_n\}$ in V^∞ which converges to the distribution τ in the topology of $V^{-\infty}$. The corresponding sequence of functions f_{v_n} then converges in $C^{-\infty}(\mathbb{R})$ to f_τ – i.e., the image of τ via (5.74b); note that f_τ is the image of τ_λ under the distribution analogue of (2.8). In the statement of corollary 2.19, this distribution is denoted by $\tilde{\tau}$. As $v_n\to\tau$, the sequence of C^∞ vectors $\pi_{N_{\mathbb{R}}}(\psi)v_n$ converges in V^∞ to $\pi_{N_{\mathbb{R}}}(\psi)\tau$; here I am using the smoothing property of $\pi_{N_{\mathbb{R}}}$. Taking limits, I find

$$a_0(\pi_{N_{\mathbb{R}}}(\psi)\tau,v') \;=\; \int_{\mathbb{R}} f_\tau(u)\,\psi(-u)\,du\,. \tag{5.83}$$

The integral on the right is to be taken in the symbolic sense: I formally write the natural pairing between distributions and smooth measures as an integral.

I now substitute $\pi_{N_{\mathbb{R}}}(\psi)\tau\in V^\infty$ for the C^∞ vector v in the asymptotic expansion (5.70b). I shall show:

$$\begin{aligned} &\lim\nolimits_{t\to-\infty}\left(e^{(\lambda-1)t}(1+\lambda-\tfrac{\partial}{\partial t})\,\langle\,\pi_{N_{\mathbb{R}}}(\psi)\tau,\pi'(a_t)v'\,\rangle\right) \;= \\ &\qquad\qquad = \begin{cases} 2\lambda\,a_0(\pi_{N_{\mathbb{R}}}(\psi)\tau,v') & \text{if } \lambda\neq 0\,,\\ -\,a_0(\pi_{N_{\mathbb{R}}}(\psi)\tau,v') & \text{if } (\lambda,\varepsilon)=(0,+1)\,,\\ 0 & \text{if } (\lambda,\varepsilon)=(0,-1)\,. \end{cases} \end{aligned} \tag{5.84}$$

This is evidently correct on the level of leading terms; the only potential obstacle would be a non-zero coefficient $b_j(\pi_{N_{\mathbb{R}}}(\psi)\tau,v')$ with $0<j<-\mathrm{Re}\,\lambda$. However,

$\mathrm{Re}\,\lambda > -1$ unless π is a discrete series representation, in which case all the b_j vanish identically. In view of (5.6),

$$\begin{aligned} \langle\, \pi_{N_{\mathbb{R}}}(\psi)\tau, \pi'(a_t)v'\,\rangle \;&=\; \int_{\mathbb{R}} \psi(x)\langle\, \pi(n_x)\tau, \pi'(a_t)v'\,\rangle\, dx \\ &=\; \int_{\mathbb{R}} \psi(x)\langle\, \tau, \pi'(n_{-x}a_t)v'\,\rangle\, dx \\ &=\; \int_{\mathbb{R}} \psi(-x)\langle\, \tau, \pi'(n_x a_t)v'\,\rangle\, dx \;=\; \int_{\mathbb{R}} \psi(-x)\, F_{\tau,v'}(x,y)\, dx\,, \end{aligned} \tag{5.85}$$

with $y = e^{2t}$, as usual. At this point, (5.83–84) imply

$$\begin{aligned} \lim\nolimits_{y\to 0^+} \left(y^{\frac{\lambda-1}{2}} \left(\tfrac{1+\lambda}{2} - y\,\tfrac{\partial}{\partial y}\right) \int_{\mathbb{R}} \psi(-x)\, F_{\tau,v'}(x,y)\, dx \right) \;&= \\ =\; c(\lambda,\varepsilon) \int_R \psi(-x)\, f_\tau(x)\, dx\,; \end{aligned} \tag{5.86}$$

the constant $c(\lambda,\varepsilon)$ equals λ, $-\frac{1}{2}$, or 0, depending on whether $\lambda \neq 0$, $(\lambda,\varepsilon) = (0,+1)$, or $(\lambda,\varepsilon) = (0,-1)$.

The limit mentioned in the statement of the lemma exists, in the strong distribution topology, because $\pi_{N_{\mathbb{R}}}(\psi)$ is continuous with respect to that topology. Since ψ is an arbitrary test function, (5.86) implies the first part of the lemma – recall the definition of f_τ as the image of τ_λ under the distribution analogue of (2.8). The second assertion of the lemma follows from

$$\lim\nolimits_{y\to 0^+} \left(y^{-\frac{1}{2}}\, \psi(-x)\, F_{\tau,v'}(x,y)\, dx \right) \;=\; \int_R \psi(-x)\, f_\tau(x)\, dx\,, \tag{5.87}$$

which is entirely analogous to (5.86). □

Taken together, lemmas 5.52 and 5.65 show that τ_λ has the degree of regularity asserted by theorem 2.18, except possibly near the point at infinity. But the $G_{\mathbb{R}}$ acts transitively on $\mathbb{R}\cup\{\infty\}$, so no point plays an exceptional role.

Proof of theorem 2.20. I must show that when τ satisfies the hypotheses of the theorem, the functions

$$F_{\tau,v'}(x,y) \;=\; \langle\, \tau\,, \pi'(n_x a_t)v'\,\rangle\,, \qquad \text{with } e^{2t} \;=\; y\,, \tag{5.88}$$

for some non-zero, $K_{\mathbb{R}}$-isotypic vector $v' \in V'$, is square-integrable over $\mathscr{H} \cong \Gamma\backslash G_{\mathbb{R}}$. Note that $F_{\tau,v'}$ is a well-defined, C^∞ function, because v' is a C^∞ vector. Much of the discussion at the beginning of this section applies to this function – anything that does not depend on knowing the square-integrability of $F_{\tau,v'}$ in the first place. The fact that τ is a distribution vector makes $F_{\tau,v'}$ have at most polynomial growth,

$$|F_{\tau,v'}(x,y)| \;\leq\; C\max\{\, y^a\,, y^{-a}\,\}\,, \qquad \text{for some } C>0\,,\, a>0\,. \tag{5.89}$$

To see this, one can express τ as $\pi(\Omega_K)^m(v)$, an integral power of the Casimir operator of $K_{\mathbb{R}}$ applied to a vector v in the unitary representation space V. This makes $F_{\tau,v'}$ a – possibly high – derivative of a bounded function satisfying an elliptic differential equation. From here on, one can argue as in the proof of Lemma 5.23.

The square-integrability of $F_{\tau,v'}$ is entirely a matter of the behavior at the cusps: Because of the conditions on Γ, there are finitely many cusps, and away from the cusps, $\Gamma\backslash G_{\mathbb{R}}$ is compact. The group $G_{\mathbb{R}}$ acts transitively on $\mathbb{R}\cup\{\infty\}$, so nothing is lost by looking only at a potential cusp at the point at infinity. Further, I may assume that $z\mapsto z+1$ generates the action of $\Gamma\cap N_{\mathbb{R}}$. I can then express $F_{\tau,v'}$ as a Fourier series

$$F_{\tau,v'}(x,y) \;=\; \sum\nolimits_{-\infty<k<\infty} a_k(y)\, e^{2\pi i k x}\,, \tag{5.90}$$

as in (5.29). The Fourier coefficients $a_k(y)$ satisfy the differential equation (5.31), for the same reason as in the earlier case. The boundedness condition (5.89) implies that the $a_k(y)$, for $k\neq 0$, decay exponentially – again, as explained in the proof of lemma 5.23. Thus square-integrabilty near the cusp at infinity hinges on $a_0(y)$: In the notation of (5.35), the square-integrability around the cusp at infinity is equivalent to

(5.91)

a) $c_+ \;=\; c_- \;=\; 0$ if τ corresponds to a cuspidal embedding of a representation π' of the unitary principal series or complementary series;

b) $c_+ \;=\; 0$ if τ corresponds to a possibly non-cuspidal embedding of a complementary series representation and $\lambda>0$;

c) $c_- \;=\; 0$ if τ corresponds to a possibly non-cuspidal embedding of a complementary series representation and $\lambda<0$;

d) $c_- \;=\; 0$ if τ corresponds to an embedding of a representation of the discrete series or limits of the discrete series;

in the latter case λ is a non-negative integer and c_+ vanishes automatically, as a consequence of the first order equation (5.22).

As before, I write $\tilde\tau_\lambda$ for the image of τ via $V^{-\infty}\hookrightarrow V^{-\infty}_{\lambda,\varepsilon}$ and the distribution analogue of (2.8). This is a distribution, periodic of period 1,

$$\tilde\tau_\lambda(x) \;=\; \sum\nolimits_{-\infty<k<\infty} a_k\, e^{2\pi i k x}\,. \tag{5.92}$$

According to the hypotheses, τ_λ – and therefore also $\tilde\tau_\lambda$ – is the n-th derivative of a continuous function, with

$$n \;=\; \begin{cases} 1 & \text{if } \pi \text{ belongs to the principal or complementary series} \\ [\frac{3}{2}-\frac{\lambda}{2}] & \text{if } \pi \text{ belongs to the discrete series or its limits;} \end{cases} \tag{5.93}$$

in the first case $-1 < \operatorname{Re}\lambda < 1$, and in the latter λ is a non-positive integer. Thus I can write

$$\tilde{\tau}_\lambda(x) \;=\; a_0 \;+\; \psi_\lambda^{(n)}(x)\,, \tag{5.94}$$

where ψ_λ is continuous, periodic, with zero constant term.

The expression (5.94) determines $\tilde{\tau}_\lambda$ only away from the point at infinity. Let me show how to give meaning to $\psi_\lambda^{(n)}$ not only as a distribution on the real line, but as vector in $V_{\lambda,\varepsilon}^{-\infty}$. That means extending the distribution $x \mapsto (\operatorname{sgn} x)^{\frac{1-\varepsilon}{2}}\,|x|^{\lambda-1}\,\psi_\lambda^{(n)}(\frac{1}{x})$ across $x = 0$ – see (2.8). To simplify the discussion, I look at positive x separately; the case of negative x can be handled the same way. For $j > 0$, I let $\psi_\lambda^{(-j)}$ denote the j-th anti-derivative of the continuous function ψ_λ; it becomes well-defined when one requires it to be periodic with zero constant term. Using the formal identity

$$x^\alpha h'(\tfrac{1}{x}) \;=\; \tfrac{d}{dx}\left(-x^{\alpha+2}h(\tfrac{1}{x}) - (\alpha{+}2)x^{\alpha+3}h^{(-1)}(\tfrac{1}{x})\right) + (\alpha{+}2)(\alpha{+}3)x^{\alpha+2}h^{(-1)}(\tfrac{1}{x}) \tag{5.95}$$

repeatedly, I find

$$x^{\lambda-1}\,\psi_\lambda^{(n)}(\tfrac{1}{x}) \;=\; \textstyle\sum_{0\le i,j,\ell\le n} a_{i,j,\ell}\,\frac{d^\ell}{dx^\ell}\left(x^{\lambda-1+2n+i}\,\psi_\lambda^{(-j)}(\tfrac{1}{x})\right)\,, \tag{5.96}$$

with suitable coefficients $a_{i,j,\ell}$. Since $\operatorname{Re}\lambda - 1 + 2n + i > 0$ in the range of summation, (5.96) exhibits $(\operatorname{sgn} x)^{\frac{1-\varepsilon}{2}}\,|x|^{\lambda-1}\,\psi_\lambda^{(n)}(\frac{1}{x})$ as a distribution even across $x = 0$, a distribution which is also the n-th derivative of a continuous function. The contribution of the term a_0 in (5.94), in terms of the coordinate system at infinity, is

$$(\operatorname{sgn} x)^{\frac{1-\varepsilon}{2}}\,|x|^{\lambda-1}a_0 \;=\; \begin{cases} a_0(\operatorname{sgn} x)^{\frac{1-\varepsilon}{2}+n}\,\frac{d^n}{dx^n}\left(\frac{1}{\lambda(\lambda-1)\dots(\lambda+1-n)}|x|^{\lambda-1+n}\right) & \text{if } \lambda \neq 0 \\ a_0(\operatorname{sgn} x)^{\frac{1+\varepsilon}{2}}\,\frac{d}{dx}\log|x| & \text{if } \lambda = 0\,; \end{cases} \tag{5.97}$$

in the cases covered by theorem 2.20, this term does not have the required degree of regularity except in the situation b), with $0 < \lambda < 1$. The expression (5.94) cannot detect a delta function or derivative of a delta function at the point at infinity. However, a delta function has degree of regularity exactly $C^{<(-1)}$, and is not allowed by the hypotheses of theorem 2.20, except in the situation b), with $-1 < \lambda < 0$, and conceivably also when λ is a strictly negative integer.

To complete the argument, I must show that the hypotheses of theorem 2.20 translate into the conditions (5.91). Recall the definition (5.35). Because of the existence of the limit in 5.65 in the strong distribution topology, the Fourier coefficients in (5.90) and (5.92) are matched in the limiting process. Specifically

$$a_0 \;=\; 0 \;\iff\; \begin{cases} c_- \;=\; 0 & \text{if } (\lambda,\varepsilon) \neq (0,+1) \\ c_+ \;=\; 0 & \text{if } (\lambda,\varepsilon) = (0,+1)\,; \end{cases} \tag{5.98}$$

the factor $\log y$ in (5.35) when $(\lambda,\varepsilon)=(0,+1)$ accounts for the reversal of c_+ and c_- in the second case. The coefficient $c_\pm$ not identified by (5.98) disappears in the limiting process when one looks only at finite values of x. On general grounds, it must correspond to a distribution supported at the point at infinity, and invariant under the action of $N_{\mathbb{R}}$. The only such distribution is the delta function at infinity, hence

(5.99)

$$\tau_\lambda \text{ involves the delta function at } \infty \iff \begin{cases} c_+ = 0 & \text{if } (\lambda,\varepsilon)\neq(0,+1) \\ c_- = 0 & \text{if } (\lambda,\varepsilon)=(0,+1)\,. \end{cases}$$

One can see (5.98–99) also directly by computing $F_{\tau,v'}$ in terms of τ and v' from the defining formula (5.6). In the case of a representation of the discrete series or limits of the discrete series, (5.99) holds vacuously since c_+ is known to vanish; on the other hand, the delta function cannot arise as the boundary value of a holomorphic or anti-holomorphic function. The translation of the hypotheses of theorem 2.20 in the previous paragraph now connects directly with (5.91) and (5.98–99). □

References

1. Beardon, A. F.: *The Geometry of Discrete Groups*, Graduate Texts in Mathematics 91, Springer, New York-Berlin, 1983.
2. Bernstein, S.: Sur la convergence absolue des séries trigonometriques, *C. R. Acad. Sci. Paris* **158** (1914), 1661–1664.
3. Bernstein, J. N. and Reznikov, A.: Sobolev norms of automorphic functionals and Fourier coefficients of cusp forms, *C. R. Acad. Sci. Paris Sér. I Math.* **327** (1998), 111–116.
4. Bernstein, J. N. and Reznikov, A.: Analytic continuation of representation and estimates of automorphic forms, *Ann. of Math.* **150** (1999), 329–352.
5. Casselman, W.: Jacquet modules for real reductive groups, in: *Proceedings of the International Congress of Mathematicians, Helsinki, 1978*, Acad. Sci. Fennica, Helsinki, 1980, pp. 557–563.
6. Casselman, W.: Canonical extensions of Harish-Chandra modules to representations of *G*, *Canad. J. Math.* **41** (1989), 385–438.
7. Casselman, W. and Miličić, D.: Asymptotic behavior of matrix coefficients, *Duke Math. J.* **49** (1982), 869–930.
8. Coddington, E. A. and Levinson, N.: *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.
9. Folland, G. B.: *Real Analysis*, Wiley-Interscience, New York, 1984.
10. Maass, H.: Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen, *Math. Ann.* **121** (1949), 141–183.
11. Wallach, N. R.: Asymptotic expansions of generalized matrix entries of representations of real reductive groups, in: R. Herb, R. Lipsman, and J. Rosenbeg, (eds.), *Lie Group Representations I*, Lecture Notes in Math. 1024, Springer, Berlin-New York, 1983, pp. 287–369.