A generalization of the Bott-Borel-Weil theorem and Euler number multiplets of representations

Bertram Kostant*
Department of Mathematics, MIT, Cambridge, MA 02139, USA
e-mail: `kostant@math.mit.edu`

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Dedicated to the memory of a much missed friend, Moshé Flato

0. Introduction

0.1. In Remark 0.27 of [3] we have stated that results in [3] yield an extensive generalization of the Bott-Borel-Weil theorem (BBW). The purpose of the present paper is to explain and justify this statement in detail.

Let $G_{\mathbb{C}}$ be a simply-connected complex semisimple Lie group and let $\mathfrak{g} = Lie\, G_{\mathbb{C}}$. Let $P \subset G_{\mathbb{C}}$ be a parabolic subgroup and let $\mathfrak{p} = Lie\, P$. As one knows the complex $G_{\mathbb{C}}$-homogeneous space $X = G/P$ is a compact projective variety. Let E be a holomorphic homogeneous vector bundle over X defined by an irreducible representation of P. Then BBW, established as Theorem IV$'$ in [1], determines the multiplicity of any irreducible representation $\pi_\lambda : \mathfrak{g} \to End\, V_\lambda$ of $\mathfrak{g}$ in the cohomology $H^j(X, \mathscr{S}E)$ where $\mathscr{S}E$ is the sheaf of local holomorphic sections of E. The subscript λ is the highest weight of π_λ. Let

$$\mathfrak{p} = \mathfrak{g}_1 + \mathfrak{n}$$

be the Levi-decomposition of $\mathfrak{p}$ where $\mathfrak{g}_1$ is a Levi-factor of $\mathfrak{p}$ and $\mathfrak{n}$ is the nilradical of $\mathfrak{p}$. It is implicit in [1] and made more explicit in [2] that BBW is equivalent to knowing the $\mathfrak{g}_1$-module structure on the Lie algebra cohomology $H^j(\mathfrak{n}, V_\lambda)$. In [2] we obtained an algebraic proof of BBW by determining this $\mathfrak{g}_1$-structure. The technique was to introduce a certain Laplacian L_λ on the cochain complex $C(\mathfrak{n}, V_\lambda)$ which defines $H^*(\mathfrak{n}, V_\lambda)$ so that one has a graded $\mathfrak{g}_1$-module isomorphism

$$Ker\, L_\lambda \cong H^*(\mathfrak{n}, V_\lambda).$$

Now $C(\mathfrak{n}, V_\lambda) = V_\lambda \otimes \wedge \mathfrak{n}^*$. What was needed, and what was directly established in [2], was that the $\mathfrak{g}_1$-submodule $Ker\, L_\lambda$ of $V_\lambda \otimes \wedge \mathfrak{n}^*$ is multiplicity free and

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decomposes into a direct sum of $d = Eul(X)$ irreducibles

$$Ker L_\lambda = \sum_{\tau \in W^1} (Ker L_\lambda)^{\tau * \lambda}. \tag{0.1}$$

Here W^1 is a special set of coset representatives of the Weyl group of $\mathfrak{g}_1$ in the Weyl group of $\mathfrak{g}$. Also $\tau * \lambda = \tau(\lambda + \rho) - \rho$ where ρ has its usual meaning. The superscript denotes the primary $\sigma_{\tau*\lambda}$-component where, in general, σ_μ is an irreducible $\mathfrak{g}_1$-module with highest weight μ. In addition, what was needed and what was proved was the inclusion

$$(Ker L_\lambda)^{\tau * \lambda} \subset V_\lambda \otimes \wedge^{l(\tau)} \mathfrak{n}^*, \tag{0.2}$$

where $l(\tau)$ is the length of τ. Actually the result in [2] is stronger than (0.1) and (0.2). For one thing the result in [2] names the highest weight vector of $(Ker L_\lambda)^{\tau*\lambda}$. But more significantly, it established the equality $(Ker L_\lambda)^{\tau*\lambda} = C(\mathfrak{n}, V_\lambda)^{\tau*\lambda}$ so that (0.1) can be replaced by the stronger statement

$$Ker L_\lambda = \sum_{\tau \in W^1} C(\mathfrak{n}, V_\lambda)^{\tau * \lambda}. \tag{0.3}$$

Now $\mathfrak{g}_1$ is a reductive Lie subalgebra of $\mathfrak{g}$ with the same rank as $\mathfrak{g}$. Much more generally, let $\mathfrak{r}$ be an arbitrary reductive Lie subalgebra of $\mathfrak{g}$ with the same rank as $\mathfrak{g}$. We refer to the special case when $\mathfrak{r} = \mathfrak{g}_1$ as the Levi-factor case. Of course $\mathfrak{r}$ can be semisimple whereas $\mathfrak{g}_1$ is never semisimple — except in the trivial situation when $\mathfrak{g}_1 = \mathfrak{g}$. Let $B_\mathfrak{g}$ be the Killing form on $\mathfrak{g}$ and let $B_\mathfrak{r} = B|\mathfrak{r}$. Let $\mathfrak{q}$ be the $B_\mathfrak{g}$-orthocomplement of $\mathfrak{r}$ in $\mathfrak{g}$ so that

$$\mathfrak{g} = \mathfrak{r} + \mathfrak{q}$$

and let $B_\mathfrak{q} = B_\mathfrak{g}|\mathfrak{q}$. Let $Cliff(\mathfrak{q})$ be the Clifford algebra over $\mathfrak{q}$ with respect to $B_\mathfrak{q}$. The dimension of $\mathfrak{q}$ is even so that $Cliff(\mathfrak{q})$ has a unique irreducible module S — the spin module. The natural action ν of $\mathfrak{r}$ on $\mathfrak{q}$ (since $[\mathfrak{r}, \mathfrak{q}] \subset \mathfrak{q}$) defines a homomorphism $\nu_* : \mathfrak{r} \to Cliff(\mathfrak{q})$ and hence a representation $Spin\,\nu : \mathfrak{r} \to End\,S$. Taking the tensor product of $Spin\,\nu$ with $\pi_\lambda|\mathfrak{r}$ defines a representation

$$\zeta_\lambda : \mathfrak{r} \to End\,(V_\lambda \otimes S).$$

The point to be made here is that in the Levi-factor case $V_\lambda \otimes S$ differs from the cochain complex $C(\mathfrak{n}, V_\lambda)$, as $\mathfrak{g}_1$-modules, only by tensoring with a 1-dimensional representation, $\sigma_{\rho(\mathfrak{n})}$, of $\mathfrak{g}_1$. That is, in the Levi-factor case one can readily go back and forth between $C(\mathfrak{n}, V_\lambda)$ and $V_\lambda \otimes S$ whereas in the general case one has only $V_\lambda \otimes S$. (If $\mathfrak{r}$ is semisimple it has no nontrivial 1-dimensional representations.)

In the general case let G be a compact form of $G_\mathbb{C}$ chosen so that there exists a connected Lie subgroup $R \subset G$ which corresponds to a real form of $\mathfrak{r}$. The compact homogeneous space $X = G/R$ need no longer be complex but is, in fact, the most general simply-connected compact homogeneous space of positive Euler characteristic. To sense the generality consider the example of when X is a sphere. In the Levi-factor case one is looking only at the 2-sphere whereas in the general case all even dimensional spheres are examples.

In the general case one still has the set W^1 of Weyl group coset representatives and if $d = card\, W^1$ then $d = Eul(X)$. Also, as in the Levi-factor case, σ_μ will denote an irreducible representation of $\mathfrak{r}$ with highest weight μ. The paper [3] is much concerned with the d-multiplet of irreducible $\mathfrak{r}$-representations $\{\sigma_{\tau\bullet\lambda} \mid \tau \in W^1\}$ defined by any highest weight λ of $\mathfrak{g}$. Here $\tau\bullet\lambda = \tau(\lambda+\rho) - \rho_{\mathfrak{r}}$. Note that in the Levi-factor case

$$\tau\bullet\lambda = \rho(\mathfrak{n}) + \tau * \lambda.$$

Let $Cas_{\mathfrak{r}} \in Cent\, U(\mathfrak{r})$ be the Casimir element which corresponds to $B_{\mathfrak{r}}$. The generalization of (0.3) will be an easy consequence (Theorem B) of Theorem A below. Theorem A is part of the statement of Theorem 4.17 in [3].

Theorem A *Let λ be any highest weight for $\mathfrak{g}$ and let $\mathfrak{r}$ be any reductive subalgebra of $\mathfrak{g}$ having the same rank as $\mathfrak{g}$. Then the highest eigenvalue of the operator $\zeta_\lambda(Cas_{\mathfrak{r}})$ on $V_\lambda \otimes S$ is $|\lambda+\rho|^2 - |\rho_{\mathfrak{r}}|^2$. Furthermore if M is the corresponding eigenspace then M is a multiplicity-free $\mathfrak{r}$-module whose irreducible components are the elements of the d-multiplet $\{\sigma_{\tau\bullet\lambda} \mid \tau \in W^1\}$. In fact*

$$M = \sum_{\tau\in W^1} (V_\lambda \otimes S)^{\tau\bullet\lambda} \tag{0.4}$$

and the primary component $(V_\lambda \otimes S)^{\tau\bullet\lambda}$ is irreducible for any $\tau \in W^1$.

Assume the Levi-factor case and let $Z_{\rho(\mathfrak{n})}$ be the 1-dimensional $\mathfrak{g}_1$-module which affords the representation $\sigma_{\rho(\mathfrak{n})}$ so that as $\mathfrak{g}_1$-modules

$$V_\lambda \otimes S = C(\mathfrak{n}, V_\lambda) \otimes Z_{\rho(\mathfrak{n})}. \tag{0.5}$$

Recalling the Laplacian L_λ, let L_λ^e be the operator on $V_\lambda \otimes S$ defined by putting $L_\lambda^e = L_\lambda \otimes 1$ with respect to the right side of (0.5).

Theorem B *Let the notation be as in Theorem A except that now we assume the Levi-factor case ($\mathfrak{r} = \mathfrak{g}_1$). Then*

$$2L_\lambda = (|\lambda+\rho|^2 - |\rho_{\mathfrak{r}}|^2) - \zeta_{\mathfrak{r}}(Cas_{\mathfrak{r}}) \tag{0.6}$$

so that the $\mathfrak{g}_1$-module structure on $Ker\, L_\lambda$ (and hence on $H^*(\mathfrak{n}, V_\lambda)$) is given by the $\mathfrak{g}_1$-module structure on M. More precisely, as $\mathfrak{g}_1$-modules,

$$(Ker\, L_\lambda) \otimes Z_{\sigma(\mathfrak{n})} \cong M \tag{0.7}$$

and hence (0.3) follows from (0.4).

0.2. In the general case one does not have the Laplacian L_λ. Instead one has the cubic Dirac operator $\square_\lambda$, defined in [3]. The relation of $\square_\lambda$ to L_λ in the Levi-factor case is given in

Theorem C *With the assumptions of Theorem B one has*

$$2L^e_\lambda = \square_\lambda. \tag{0.8}$$

The equation (0.6) is Theorem 2.9 in this paper and Theorem C is Theorem 3.10.

1. The Bott-Borel-Weil theorem and nilpotent Lie algebra cohomology

1.1. Let $G_\mathbb{C}$ be a complex semisimple Lie group and let $\mathfrak{g} = Lie\, G_\mathbb{C}$. Let $l = rank\,\mathfrak{g}$. Let $\mathfrak{r}$ be a reductive Lie subalgebra of $\mathfrak{g}$ and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{r}$. Assume $rank\,\mathfrak{r} = rank\,\mathfrak{g}$ so that $\mathfrak{h}$ is also a Cartan subalgebra of $\mathfrak{g}$ and $dim\,\mathfrak{h} = l$. Let $\mathfrak{h}^*$ be the dual space to $\mathfrak{h}$ and let $\Delta \subset \mathfrak{h}^*$ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let $\Delta_\mathfrak{r} \subset \Delta$ be the set of roots of $\mathfrak{r}$ with respect to $\mathfrak{h}$. Let (z,z'), for $z,z' \in \mathfrak{g}$, be the Killing form, $B_\mathfrak{g}$, of $\mathfrak{g}$ and let $B_\mathfrak{r} = B_\mathfrak{g}|\mathfrak{r}$. For any $\varphi \in \Delta$ let $e_\varphi \in \mathfrak{g}$ be a corresponding root vector where the choice is normalized so that $(e_\varphi, e_{-\varphi}) = 1$ for any $\varphi \in \Delta$. Let $\mathfrak{h}_\mathbb{R}$ be the space of hyperbolic elements in $\mathfrak{h}$ and let $\mathfrak{h}^*_\mathbb{R}$ be the real span of Δ so that $\mathfrak{h}_\mathbb{R}$ (resp. $\mathfrak{h}^*_\mathbb{R}$) is a real form of $\mathfrak{h}$ (resp. $\mathfrak{h}^*$) and $\mathfrak{h}^*_\mathbb{R}$ can be taken as the real dual space to $\mathfrak{h}_\mathbb{R}$. The restriction $B_\mathfrak{g}|\mathfrak{h}$ is nonsingular and induces a nonsingular symmetric bilinear form (δ,γ) on $\mathfrak{h}^*$ that is positive definite on $\mathfrak{h}^*_\mathbb{R}$. We write $|\mu| = \sqrt{(\mu,\mu)}$ for $\mu \in \mathfrak{h}^*_\mathbb{R}$. Let $\Gamma \subset \mathfrak{h}^*_\mathbb{R}$ be the lattice of $\mathfrak{g}$-integral linear forms on $\mathfrak{h}$. That is, $\Gamma = \{\mu \in \mathfrak{h}^* \mid 2(\mu,\varphi)/(\varphi,\varphi) \in \mathbb{Z}, \forall \varphi \in \Delta\}$.

For any subspace $\mathfrak{m} \subset \mathfrak{g}$ which is stable under $ad\,\mathfrak{h}$ let $\Delta(\mathfrak{m}) = \{\varphi \in \Delta \mid e_\varphi \in \mathfrak{m}\}$. Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ which contains $\mathfrak{h}$. Then $\mathfrak{b}_\mathfrak{r} = \mathfrak{b} \cap \mathfrak{r}$ is a Borel subalgebra of $\mathfrak{r}$. Positive root systems for $\mathfrak{g}$ and $\mathfrak{r}$ can then be given respectively by letting $\Delta^+ = \Delta(\mathfrak{b})$ and $\Delta^+_\mathfrak{r} = \Delta(\mathfrak{b}_\mathfrak{r})$. Let $\rho = \frac{1}{2}\sum_{\varphi\in\Delta^+}\varphi$ and $\rho_\mathfrak{r} = \frac{1}{2}\sum_{\varphi\in\Delta^+_\mathfrak{r}}\varphi$. One has $\rho \in \Gamma$ but $\rho_\mathfrak{r}$ is not necessarily in Γ. Of course $2\rho_\mathfrak{r} \in \Gamma$. Let $\Gamma_\mathfrak{r} \subset \mathfrak{h}^*_\mathbb{R}$ be the lattice generated by Γ and $\rho_\mathfrak{r}$. One of course has

$$2(\mu,\varphi)/(\varphi,\varphi) \in \mathbb{Z}, \quad \forall \varphi \in \Delta_\mathfrak{r} \text{ and } \forall \mu \in \Gamma_\mathfrak{r}.$$

Let D be the $\mathfrak{g}$-Weyl chamber in $\mathfrak{h}^*_\mathbb{R}$ defined by $D = \{\gamma \in \mathfrak{h}^*_\mathbb{R} \mid (\gamma,\varphi) \geq 0, \forall \varphi \in \Delta^+\}$ and let $D_\mathfrak{r}$ be the $\mathfrak{r}$-Weyl chamber in $\mathfrak{h}^*$ defined by $D_\mathfrak{r} = \{\gamma \in \mathfrak{h}^*_\mathbb{R} \mid (\gamma,\varphi) \geq 0, \forall \varphi \in \Delta^+_\mathfrak{r}\}$. Now let $\Lambda = D \cap \Gamma$ and let $\Lambda_\mathfrak{r} = D_\mathfrak{r} \cap \Gamma_\mathfrak{r}$. For each $\lambda \in \Lambda$ let

$$\pi_\lambda : \mathfrak{g} \to End\, V_\lambda$$

be an irreducible representation with highest weight λ and, for any $\mu \in \Lambda_\mathfrak{r}$, let

$$\sigma_\mu : \mathfrak{r} \to End\, Z_\mu$$

be an irreducible representation with highest weight μ. If Y is any completely reducible $\mathfrak{r}$-module we will let Y^μ be the σ_μ-primary component of Y.

Let $\mathfrak{q}$ be the $B_{\mathfrak{g}}$-orthocomplement of $\mathfrak{r}$ in $\mathfrak{g}$ and let $B_{\mathfrak{q}} = B_{\mathfrak{g}}|\mathfrak{q}$. Much of the notation of [2] will be adopted here. However we are writing $\mathfrak{q}$ here for $\mathfrak{p}$ in [2] to avoid confusion with the Lie algebra of a parabolic subalgebra to be considered in §1.2. Let $n = dim\,\mathfrak{q}$ and let $\nu : \mathfrak{r} \to Lie\,SO(\mathfrak{q})$ be the representation defined so that

$$\nu(x)y = [x,y] \tag{1.1}$$

for $x \in \mathfrak{r}$ and $y \in \mathfrak{q}$. Of course $\Delta(\mathfrak{q})$ is the complement of $\Delta(\mathfrak{r})$ in Δ and one of course has $card\,\Delta(\mathfrak{q}) = n$. Let $\Delta^+(\mathfrak{q}) = \Delta^+ \cap \Delta(\mathfrak{q})$ so that $\Delta^+(\mathfrak{q})$ is the complement of $\Delta_{\mathfrak{r}}^+$ in Δ^+. Put $m = card\,\Delta^+(\mathfrak{q})$ and since one readily has the partition

$$\Delta(\mathfrak{q}) = \Delta^+(\mathfrak{q}) \cup -\Delta^+(\mathfrak{q})$$

it follows that

$$n = 2m. \tag{1.2}$$

Let $\mathfrak{q}_+$ (resp. $\mathfrak{q}_-$) be the span of e_φ (resp. $e_{-\varphi}$) for $\varphi \in \Delta^+(\mathfrak{q})$ so that as linear spaces

$$\mathfrak{q} = \mathfrak{q}_+ \oplus \mathfrak{q}_-. \tag{1.3}$$

Let W (resp. $W_{\mathfrak{r}}$) be the Weyl group of $\mathfrak{g}$ (resp. $\mathfrak{r}$) operating as usual in both $\mathfrak{h}$ and $\mathfrak{h}^*$. Of course $W_{\mathfrak{r}}$ is a subgroup of W. Let d be the index of $W_{\mathfrak{r}}$ in W. A cross-section, W^1, of the set of right $W_{\mathfrak{r}}$ cosets is defined (using the Weyl chambers $D \subset D_{\mathfrak{r}}$, see §3.1) by putting

$$W^1 = \{\tau \in W \mid \tau(D) \subset D_{\mathfrak{r}}\}. \tag{1.4}$$

That is, the map

$$W_{\mathfrak{r}} \times W^1 \to W, \qquad (w,\tau) \mapsto w\tau \tag{1.5}$$

is clearly a bijection. In particular

$$card\,W^1 = d. \tag{1.6}$$

Let Γ^{reg} be the set of W-regular elements in the lattice $\Gamma \subset \mathfrak{h}_{\mathbb{R}}^*$. One has the partition

$$\Gamma^{reg} = \bigcup_{w \in W} w(\Lambda + \rho). \tag{1.7}$$

The partition (1.7) clearly defines a partition of the subset, $\Gamma^{reg} \cap D_{\mathfrak{r}}$, of elements in Γ^{reg} which are $\Delta_{\mathfrak{r}}^+$-dominant

$$\Gamma^{reg} \cap D_{\mathfrak{r}} = \bigcup_{\tau \in W^1} \tau(\Lambda + \rho). \tag{1.8}$$

Obviously $\Gamma^{reg} \cap D_{\mathfrak{r}} \subset \Lambda_{\mathfrak{r}}$. Furthermore it is also obvious that the elements in $\Gamma^{reg} \cap D_{\mathfrak{r}}$ are $W_{\mathfrak{r}}$-regular. For any $\tau \in W^1$ and $\lambda \in \Lambda$ let

$$\tau \bullet \lambda = \tau(\lambda + \rho) - \rho_{\mathfrak{r}}. \tag{1.9}$$

It then follows that $\tau \bullet \lambda \in \Lambda_{\mathfrak{r}}$. In fact this isolates a distinguished subset, $\Lambda_{\mathfrak{r}}^{*}$, of $\Lambda_{\mathfrak{r}}$. Let

$$\Lambda_{\mathfrak{r}}^{*} = \{\mu \in \Lambda_{\mathfrak{r}} \mid \mu + \rho_{\mathfrak{r}} \in \Gamma^{reg}\}. \tag{1.10}$$

The following proposition (stated as Proposition 3.31 in [2]) is an immediate consequence of the partition (1.8).

Proposition 1.1 *For any $\tau \in W^1$ and $\lambda \in \Lambda$ one has $\tau \bullet \lambda \in \Lambda_{\mathfrak{r}}^{*}$. Moreover the map*

$$W^1 \times \Lambda \to \Lambda_{\mathfrak{r}}^{*}, \qquad (\tau,\lambda) \mapsto \tau \bullet \lambda \tag{1.11}$$

is a bijection.

Now observe that the elements of Λ parameterize subsets of $\Lambda_{\mathfrak{r}}^{*}$. For each $\lambda \in \Lambda$ let

$$\Lambda_{\mathfrak{r}}^{\lambda} = \{\tau \bullet \lambda \mid \tau \in W^1\}. \tag{1.12}$$

From Proposition 3.3 one has

$$card\, \Lambda_{\mathfrak{r}}^{\lambda} = d \tag{1.13}$$

so that all the subsets have the same cardinality and

$$\Lambda_{\mathfrak{r}}^{*} = \bigcup_{\lambda \in \Lambda} \Lambda_{\mathfrak{r}}^{\lambda}. \tag{1.14}$$

is a partition. By abuse of notation we will use (as in [3]) the word multiplet (or d-multiplet) to refer, either to any subset of $\Lambda_{\mathfrak{r}}^{*}$ of the form $\Lambda_{\mathfrak{r}}^{\lambda}$, or to the corresponding d-set $\{\sigma_\mu\}$, $\mu \in \Lambda_{\mathfrak{r}}^{\lambda}$, of irreducible $\mathfrak{r}$-representations, or again, to the corresponding d-set $\{Z_\mu\}, \mu \in \Lambda_{\mathfrak{r}}^{\lambda}$, of irreducible $\mathfrak{r}$-modules.

1.2. The Bott-Borel-Weil theorem (BBW) has to do with a special choice of $\mathfrak{r}$ (and Δ_+). By abuse of notation we will say that $\mathfrak{r}$ is a Levi-factor if there exists $x \in \mathfrak{h}_{\mathbb{R}}$ such that $\mathfrak{r}$ is the centralizer of x in $\mathfrak{g}$. In this case, for $\varphi \in \Delta$, $\langle \varphi, x\rangle$ is zero or not zero according as $\varphi \in \Delta(\mathfrak{r})$ or $\varphi \in \Delta(\mathfrak{q})$. If $\mathfrak{r}$ is a Levi-factor we may (and shall) choose $\mathfrak{b}$ such that $\langle \varphi, x\rangle > 0$ for $\varphi \in \Delta^+(\mathfrak{q})$. When $\mathfrak{r}$ is a Levi-factor, to partially conform to the notation of [3], we will write $\mathfrak{g}_1$ for $\mathfrak{r}$, W_1 for $W_{\mathfrak{r}}$, and $\mathfrak{n}$ for $\mathfrak{q}_+$. In that case

$$\mathfrak{p} = \mathfrak{g}_1 \oplus \mathfrak{n} \tag{1.15}$$

is a parabolic subalgebra of $\mathfrak{g}$, $\mathfrak{g}_1$ is a Levi-factor of $\mathfrak{p}$ and $\mathfrak{n}$ is the nilradical of $\mathfrak{p}$. The most general parabolic Lie subalgebra of $\mathfrak{g}$ is of this form.

Assume we are in the Levi-factor case and $\mathfrak{p}$ is given by (1.15). Let P be the (parabolic) Lie subgroup of $G_{\mathbb{C}}$ corresponding to $\mathfrak{p}$. Thus $P = (G_1)_{\mathbb{C}} N$ where $(G_1)_{\mathbb{C}}$ (resp. N) is the subgroup of $G_{\mathbb{C}}$ corresponding to $\mathfrak{g}_1$ (resp. $\mathfrak{n}$). Let $X = G_{\mathbb{C}}/P$ so that X is a complex (in fact projective) compact simply-connected $G_{\mathbb{C}}$ homogeneous space, and for the Euler characteristic one has

$$Eul(X) = d.$$

For $\mathfrak{r} = \mathfrak{g}_1$ put $\Lambda_1 = \Lambda_{\mathfrak{r}}$. Also let $\Lambda_{(1)} = D_{\mathfrak{r}} \cap \Gamma$ so that

$$\Lambda_{(1)} \subset \Lambda_1.$$

Let $\xi \in \Lambda_{(1)}$. Note that the representation σ_ξ of $\mathfrak{g}_1$ integrates to a representation $\sigma_\xi : (G_1)_{\mathbb{C}} \to Aut\, Z_\xi$ and extends, by trivializing the action on N, to a representation $\sigma_\xi : P \to Aut\, Z_\xi$. Let $\sigma_\xi^* : P \to Aut\, Z_\xi^*$ be the corresponding contragredient representation of P on the dual space Z_ξ^* to Z_ξ. Let $E^{-\xi}$ be the homogeneous holomorphic vector bundle on X corresponding to σ_ξ^*. Let $\mathscr{S}E^{-\xi}$ be the sheaf, on X, of local holomorphic sections of $E^{-\xi}$ and let $H^*(X, \mathscr{S}E^{-\xi})$ be the cohomology of $\mathscr{S}E^{-\xi}$. See §6, p. 364, in [2]. Then $H^*(X, \mathscr{S}E^{-\xi})$ has the natural structure of a graded $G_{\mathbb{C}}$ (or equivalently $\mathfrak{g}$)-module. The Bott-Borel-Weil theorem explicitly describes this structure in representation theoretic terms.

In order to state the BBW Theorem (Theorem 1.3. below) we need to introduce a certain subset $\Lambda_{(1)}^*$ (written D_1^0 in [2]) of $\Lambda_{(1)}$. Let $\Lambda_{(1)}^* = \{\xi \in \Lambda_{(1)} \mid \xi + \rho \text{ is } W\text{-regular}\}$. (Note that ρ is written as g in [2] and in the definition of D_1^0 on p. 370 in [2], D_1 was misprinted as D). For any $\tau \in W^1$ and $\lambda \in \Lambda$, let $\tau * \lambda = \tau(\lambda + \rho) - \rho$. Let $\rho(\mathfrak{n}) = \frac{1}{2}\sum_{\varphi \in \Delta(\mathfrak{n})} \varphi$. Note that

$$\tau \bullet \lambda = \rho(\mathfrak{n}) + \tau * \lambda \tag{1.16}$$

for any $(\tau, \lambda) \in W^1 \times \Lambda$. The following result has been established as Lemma 6.4 in [2].

Proposition 1.2 *One has $\tau * \lambda \in \Lambda_{(1)}^*$ for any $(\tau, \lambda) \in W^1 \times \Lambda$ and the map*

$$W^1 \times \Lambda \to \Lambda_{(1)}^*, \qquad (\tau, \lambda) \mapsto \tau * \lambda \tag{1.17}$$

is a bijection.

For any $\sigma \in W$ let $l(\sigma)$ be the length of σ with respect to reflections defined by the simple roots in Δ^+. The Bott-Borel-Weil theorem is due to Bott and has been proved as Theorem IV$'$ in [1]. It was later reproved as Theorem 6.4, p. 371 in [2]. The notation in Theorem 1.3 below is essentially the same as in Theorem 6.4 in [2].

Theorem 1.3 *Let $\xi \in \Lambda_{(1)}$. Then $H^*(X, \mathscr{S}E^{-\xi}) = 0$ in case $\xi \notin \Lambda_{(1)}^*$. Assume $\xi \in \Lambda_{(1)}^*$ so that $\xi = \tau * \lambda$ uniquely for $(\tau, \lambda) \in W^1 \times \Lambda$. Then $H^j(X, \mathscr{S}E^{-\xi}) = 0$ if $j \neq l(\tau)$. On the other hand, as a $\mathfrak{g}$-module, $H^{l(\tau)}(X, \mathscr{S}E^{-\xi})$ is irreducible and affords the representation π_λ^* where π_λ^* is the representation contragredient to π_λ.*

The determination of the $\mathfrak{g}$-module structure of $H^*(X, \mathscr{S}E^{-\xi})$ may be given in terms of nilpotent Lie algebra cohomology. Let $\lambda \in \Lambda$. The restriction, $\pi_\lambda|\mathfrak{n}$ defines the structure of an $\mathfrak{n}$-module on V_λ and consequently one defines the cohomology $H^*(\mathfrak{n},V_\lambda)$. Since $\mathfrak{n}$ is normalized by $\mathfrak{g}_1$ the restriction $\pi_\lambda|\mathfrak{g}_1$ defines the structure of a graded $\mathfrak{g}_1$-module on $H^*(\mathfrak{n},V_\lambda)$. The following Frobenius-like multiplicity result, due to Bott, is implicit in (1.6) of [1]. Explicitly it is given as Proposition 6.3, p. 367 in [2].

Proposition 1.4 *Let $\xi \in \Lambda_{(1)}, \lambda \in \Lambda$ and $j \in \mathbb{Z}_+$. Then*

$$\textit{mult. of } \pi_\lambda^* \textit{ in } H^j(X,\mathscr{S}E^{-\xi}) = \textit{mult. of } \sigma_\xi \textit{ in } H^j(\mathfrak{n},V_\lambda). \tag{1.18}$$

As an immediate consequence of Proposition 1.4 one has

Proposition 1.5 *The Bott-Borel-Weil theorem is equivalent to the statement that, for any $\lambda \in \Lambda$,*
(1) $H(\mathfrak{n},V_\lambda)$ is a multiplicity free $\mathfrak{g}_1$-module such that, using the primary component notation of §1.1,
(2)

$$H(\mathfrak{n},V_\lambda) = \sum_{\tau\in W^1} H(\mathfrak{n},V_\lambda)^{\tau*\lambda}. \tag{1.19}$$

and
(3)

$$0 \neq H(\mathfrak{n},V_\lambda)^{\tau*\lambda} \subset H^{l(\tau)}(\mathfrak{n},V_\lambda). \tag{1.20}$$

2. The determination of $H^*(\mathfrak{n},V_\lambda)$ and an algebraic proof of BBW

2.1. Retain the notation of §1.2. Let $\lambda \in \Lambda$. We recall results in [2] and for suitability here we will make some changes in the notation of [2]. Let $C(\mathfrak{n},V_\lambda) = V_\lambda \otimes \wedge\mathfrak{n}^*$, where $\mathfrak{n}^*$ is the dual space to $\mathfrak{n}$, so that $C(\mathfrak{n},V_\lambda)$ has the structure of a cochain complex whose cohomology is $H^*(\mathfrak{n},V_\lambda)$. The coboundary operator is denoted here by d_λ. See §3.1 in [2] where d_λ is denoted by d_π and we have reversed the order of the factors V_λ and $\wedge\mathfrak{n}^*$ in $C(\mathfrak{n},V_\lambda)$. Let $d_\lambda^* \in End\, C(\mathfrak{n},V_\lambda)$ be the Hermitian adjoint of d_λ with respect to the positive definite inner product on $C(\mathfrak{n},V_\lambda)$ defined in §3.1 of [2]. Then the Laplacian $L_\lambda = d_\lambda d_\lambda^* + d_\lambda^* d_\lambda$ is a positive semidefinite operator of degree 0 on $C(\mathfrak{n},V_\lambda)$. Furthermore L_λ commutes with the natural $\mathfrak{g}_1$-module structure on $C(\mathfrak{n},V_\lambda)$. In particular one then has a graded $\mathfrak{g}_1$-module isomorphism

$$Ker L_\lambda \cong H^*(\mathfrak{n},V_\lambda). \tag{2.1}$$

Let $\Lambda_{(1)}(\lambda) = \{\xi \in \Lambda_{(1)} \mid C(\mathfrak{n},V_\lambda)^\xi \neq 0\}$. The following result is a restatement of Theorem 5.7, p. 253, in [2].

Theorem 2.1 *Let $\lambda \in \Lambda$. Then L_λ reduces to scalars on the $\mathfrak{g}_1$-primary components of $C(\mathfrak{n},V_\lambda)$. In fact for any $\xi \in \Lambda_{(1)}(\lambda)$,*

$$L_\lambda = \frac{1}{2}(|\lambda+\rho|^2 - |\xi+\rho|^2) \textit{ on } C(\mathfrak{n},V_\lambda)^\xi. \tag{2.2}$$

From the positive semidefiniteness of L_λ, it follows from (2.2) that

$$|\lambda+\rho| \geq |\xi+\rho|, \qquad \forall\, \xi \in \Lambda_{(1)}(\lambda). \tag{2.3}$$

Let $\Lambda_{(1)}[\lambda] = \{\xi \in \Lambda_{(1)}(\lambda) \mid |\lambda+\rho| = |\xi+\rho|\}$. As a consequence of Theorem 2.1 one has

Theorem 2.2 *For any $\lambda \in \Lambda$,*

$$Ker L_\lambda = \sum_{\xi\in\Lambda_{(1)}[\lambda]} (Ker L_\lambda)^\xi \tag{2.4}$$

so that by (2.1)

$$H^*(\mathfrak{n}, V_\lambda) = \sum_{\xi\in\Lambda_{(1)}[\lambda]} H^*(\mathfrak{n}, V_\lambda)^\xi. \tag{2.5}$$

2.2. To introduce notation which will serve us now and also in Section 3 we return "momentarily" to the general case of §1.1. For any subset $\Phi \subset \Delta_+$ let $\langle\Phi\rangle \in \Gamma$ be defined by putting $\langle\Phi\rangle = \sum_{\varphi\in\Phi}\varphi$. Also for any $\sigma \in W$ let $\Phi_\sigma = \Delta_+ \cap \sigma(\Delta_-)$. One has

$$-\langle\Phi_\sigma\rangle = \sigma(\rho) - \rho \tag{2.6}$$

by (5.10.1) in [2]. Now order the roots in $\Delta(\mathfrak{q}_+)$ so that $\Delta(\mathfrak{q}_+) = \{\beta_1, \dots, \beta_m\}$. Let $\Phi \subset \Delta(\mathfrak{q}_+)$ so that for a sequence $i_j \in \mathbb{N}$, $j = 1, \dots, k$, in increasing order, where $k = card\,\Phi$, we can write $\Phi = \{\beta_{i_1}, \dots, \beta_{i_k}\}$. Let $e_{-\Phi} \in \wedge^k \mathfrak{q}_-$ be defined by putting $e_{-\Phi} = e_{-\beta_{i_1}} \wedge \cdots \wedge e_{-\beta_{i_k}}$. It is clear that the set $\{e_{-\Phi}\}$, $\Phi \subset \Delta(\mathfrak{q}_+)$, is a basis of $\wedge\,\mathfrak{q}_-$.

Remark 2.3 If $\sigma \in W$ then one notes that $\Phi_\sigma \subset \Delta(\mathfrak{q}_+)$ if and only if $\sigma \in W^1$. This easy fact is established in the second from the bottom paragraph on p. 485 in [3]. In particular, for $\tau \in W^1$ one has

$$e_{-\Phi_\tau} \in \wedge^{l(\tau)} \mathfrak{q}_-. \tag{2.7}$$

Now return to the notation of §1.2 where $\mathfrak{r} = \mathfrak{g}_1$, $\mathfrak{q}_+ = \mathfrak{n}$. Put $\mathfrak{n}_- = \mathfrak{q}_-$. The Killing form defines a nonsingular pairing of $\mathfrak{n}$ and $\mathfrak{n}_-$, and using this pairing we will identify $\mathfrak{n}^*$ with $\mathfrak{n}_-$. Let $\lambda \in \Lambda$. Then $C(\mathfrak{n}, V_\lambda) = V_\lambda \otimes \wedge\,\mathfrak{n}_-$. If $\gamma \in \Gamma$ is an extremal weight of V_λ, let $v_\gamma \in V_\lambda$ be a nonzero γ-weight vector. For $\tau \in W^1$ let $c(\tau,\lambda) = v_{\tau(\lambda)} \otimes e_{-\Phi_\tau} \in C(\mathfrak{n}, V_\lambda)$.

Remark 2.4 Note that, by (2.6) and (2.7), $c(\tau,\lambda) \in C^{l(\tau)}(\mathfrak{n}, V_\lambda)$ is a weight vector of weight $\tau * \lambda$ with respect to the $\mathfrak{g}_1$-module structure on $C(\mathfrak{n}, V_\lambda)$.

The following result is a reformulation of Theorem 5.14 in [2]. Using Proposition 1.5 it establishes the Bott-Borel-Weil theorem.

Theorem 2.5 *Let the notation be as in Theorem 2.2. Then* $\Lambda_{(1)}[\lambda] = \{\tau * \lambda \mid \tau \in W^1\}$ *so that (2.4) becomes*

$$Ker L_\lambda = \sum_{\tau \in W^1} C(\mathfrak{n}, V_\lambda)^{\tau*\lambda}. \tag{2.8}$$

Furthermore $Ker L_\lambda$ *is a multiplicity-free* $\mathfrak{g}_1$*-module so that* $C(\mathfrak{n}, V_\lambda)^{\tau*\lambda}$ *is* $\mathfrak{g}_1$*-irreducible for any* $\tau \in W^1$*. In addition* $c(\tau,\lambda)$ *is a highest weight vector in* $C(\mathfrak{n}, V_\lambda)^{\tau*\lambda}$ *so that* $C(\mathfrak{n}, V_\lambda)^{\tau*\lambda} \subset C^{l(\tau)}(\mathfrak{n}, V_\lambda)$*. Finally, using (2.1), the conditions (1), (2), and (3) of Proposition 1.5 are consequently satisfied, thereby establishing BBW.*

Remark 2.6 Note that (2.8) trivially implies $(Ker L_\lambda)^{\tau*\lambda} = C(\mathfrak{n}, V_\lambda)^{\tau*\lambda}$ so that (2.8) implies the weaker equation

$$Ker L_\lambda = \sum_{\tau \in W^1} (Ker L_\lambda)^{\tau*\lambda}. \tag{2.9}$$

If one replaces (2.8) by (2.9) then note, using Proposition 1.5, that this weakened version of Theorem 2.5 is equivalent to BBW (except for the more refined statement in Theorem 2.5 explicitly naming the highest weight vector in $(Ker L_\lambda)^{\tau*\lambda}$, and not just the degree in which $(Ker L_\lambda)^{\tau*\lambda}$ occurs). For a further comment on the equivalence, in the case where X is the flag manifold (i.e. $\mathfrak{p} = \mathfrak{b}$), see the discussion of Schmid on p. 42 in [4]. We will reformulate Theorem 2.5 by introducing some slight changes. The altered form, Theorem 2.10, of Theorem 2.5 will be seen in Section 3 to be a special case of a more general theorem involving $\mathfrak{r}$ of §1.1 instead of the Levi-factor $\mathfrak{g}_1$.

2.3. In this paragraph adopt the notation of §1.1. Let $U(\mathfrak{r})$ be the universal enveloping algebra of $\mathfrak{r}$. Given a representation

$$\sigma : \mathfrak{r} \to End Y \tag{2.10}$$

we will also let $\sigma : U(\mathfrak{g}) \to End Y$ denote the homomorphism extending (2.10) to $U(\mathfrak{g})$. Let $Cas_\mathfrak{r} \in Cent\, U(\mathfrak{g})$ be the Casimir element with respect to $B_\mathfrak{r}$. Thus if $\{x_i\}$ is a basis of $\mathfrak{r}$ and $\{y_i\}$ is the dual basis with respect to $B_\mathfrak{r}$ then $Cas_\mathfrak{r} = \sum_i x_i y_i$. For any $\mu \in \Lambda_\mathfrak{r}$ let $\chi_\mu : Cent\, U(\mathfrak{r}) \to \mathbb{C}$ be the infinitesimal character of σ_μ. One knows that

$$\chi_\mu(Cas_\mathfrak{r}) = |\mu + \rho_\mathfrak{r}|^2 - |\rho_\mathfrak{r}|^2. \tag{2.11}$$

Return to the case of §1.2 so that $\mathfrak{r} = \mathfrak{g}_1$, $q_+ = \mathfrak{n}$, and $\mathfrak{q}_- = \mathfrak{n}_-$. We write Cas_1 for $Cas_\mathfrak{r}$ and ρ_1 for $\rho_\mathfrak{r}$. Recall the notation of (1.16). One clearly has

$$\rho = \rho_1 + \rho(\mathfrak{n}). \tag{2.12}$$

It is clear from (2.12) that $\rho(\mathfrak{n}) \in \Gamma_\mathfrak{r}$ for $\mathfrak{r} = \mathfrak{g}_1$ (recalling the definition of $\Gamma_\mathfrak{r}$ in §1.1). But more is true.

Proposition 2.7 *One has $\rho(\mathfrak{n}) \in \Lambda_1$ and the irreducible $\mathfrak{g}_1$-module $Z_{\rho(\mathfrak{n})}$ (defined by the representation $\sigma_{\rho(\mathfrak{n})}$) is 1-dimensional.*

Proof. Using notation in §1.1 clearly $\mathfrak{n}$ is stable under $\nu(\mathfrak{g}_1)$. Let $f : \mathfrak{g}_1 \to \mathbb{C}$ be defined by putting $f(x) = \frac{1}{2} tr(\nu(x)|\mathfrak{n})$. Since $f([\mathfrak{g}_1, \mathfrak{g}_1]) = 0$ it follows that f defines a representation of $\mathfrak{g}_1$ on $\mathbb{C}$. On the other hand note that $f(x) = \langle \rho(\mathfrak{n}), x\rangle$ for $x \in \mathfrak{h}$. Thus $\rho(\mathfrak{n})$ is the unique weight of this 1-dimensional representation. This clearly proves the proposition. ■

If Y is a completely reducible $\mathfrak{g}_1$-module with respect to a representation σ we will let $Y^e = Y \otimes Z_{\rho(\mathfrak{n})}$ be the completely reducible $\mathfrak{g}_1$-module defined by the tensor product representation $\sigma \otimes \sigma_{\rho(\mathfrak{n})}$. We will write σ^e for $\sigma \otimes \sigma_{\rho(\mathfrak{n})}$. Also for $\xi \in \Lambda_1$ let $\xi^e = \xi + \rho(\mathfrak{n})$. Clearly $\xi^e \in \Lambda_1$.

Remark 2.8 Note that if $\Gamma_1(Y) \subset \mathfrak{h}^*$ is the set of set weights of Y then $\Gamma_1(Y) + \rho(n)$ is the set of weights of Y^e. Also if $\xi \in \Lambda_1$ then one has the equivalence

$$Z_\xi^e \cong Z_{\xi^e}$$

of irreducible $\mathfrak{g}_1$-modules.

Let $\lambda \in \Lambda$ and let $\eta_\lambda : \mathfrak{g}_1 \to End\, C(\mathfrak{n}, V_\lambda)$ be the representation which defines the given $\mathfrak{g}_1$-module structure on the cochain complex $C(\mathfrak{n}, V_\lambda)$. We can regard $\eta_0 : \mathfrak{g}_1 \to End \wedge \mathfrak{n}_-$ as the representation of $\mathfrak{g}_1$ on $\wedge \mathfrak{n}_-$ which defines the given $\mathfrak{g}_1$-module structure on $\wedge\mathfrak{n}_-$. To align notation here with that in [3], put $\zeta_{(\lambda)} = \eta_\lambda^e$ and $\zeta_{(0)} = \eta_0^e$ so that

$$\zeta_{(\lambda)} : \mathfrak{g}_1 \to End\,(C(\mathfrak{n}, V_\lambda) \otimes Z_{\rho(\mathfrak{n})}) \tag{2.13}$$

and

$$\zeta_{(0)} : \mathfrak{g}_1 \to End\,(\wedge\mathfrak{n}_- \otimes Z_{\rho(\mathfrak{n})}). \tag{2.14}$$

Recall the Laplacian L_λ defined in §2.1. Let L_λ^e be the operator on $C(\mathfrak{n}, V_\lambda)^e = C(\mathfrak{n}, V_\lambda) \otimes Z_{\rho(\mathfrak{n})}$ defined by putting $L_\lambda^e = L_\lambda \otimes 1$.

Theorem 2.9 *Let $\lambda \in \Lambda$. Then*

$$2L_\lambda^e = (|\lambda + \rho|^2 - |\rho_1|^2) - \zeta_{(\lambda)}(Cas_1). \tag{2.15}$$

Proof. Recalling the definition of $\Lambda_{(1)}(\lambda)$ in §2.1 one has the primary decomposition

$$C(\mathfrak{n}, V_\lambda)^e = \sum_{\xi \in \Lambda_{(1)}(\lambda)} (C(\mathfrak{n}, V_\lambda)^\xi)^e. \tag{2.16}$$

But $2L_\lambda^e$ reduces to the scalar $|\lambda + \rho|^2 - |\xi + \rho|^2$ on $C(\mathfrak{n}, V_\lambda)^e)^\xi$ by Theorem 2.2. But

$$\xi + \rho = \xi^e + \rho_1 \tag{2.17}$$

by (2.11). One the other hand

$$(C(\mathfrak{n},V_\lambda)^\xi)^e = (C(\mathfrak{n},V_\lambda)^e)^{\xi^e} \tag{2.18}$$

by (2.12) so that $\zeta_{(\lambda)}(Cas_1)$ reduces to the scalar $|\xi^e+\rho_1|^2-|\rho_1|^2$ on $(C(\mathfrak{n},V_\lambda)^\xi)^e$ by (2.11). But, by (2.17),

$$|\lambda+\rho|^2-|\xi+\rho|^2 = |\lambda+\rho|^2-|\rho_1|^2-(|\xi^e+\rho_1|^2-|\rho_1|^2)$$

so that the left and right sides of (2.15) take the same value on $(C(\mathfrak{n},V_\lambda)^\xi)^e$ for any $\xi \in \Lambda_{(1)}(\lambda)$. But this establishes (2.15). ■

Let $\lambda \in \Lambda$. By (2.18) $\{\chi_{\xi^e}(Cas_1)\}$, $\xi \in \Lambda_{(1)}(\lambda)$, is the set of eigenvalues of $\zeta_{(\lambda)}(Cas_1)$. Since L_λ is a positive semidefinite operator the equality (2.15) implies that for any $\xi \in \Lambda_{(1)}(\lambda)$,

$$|\lambda+\rho|^2-|\rho_1|^2 \geq \chi_{\xi^e}(Cas_1) \tag{2.19}$$

where, recalling Theorem 2.2, equality occurs if and only if $\xi \in \Lambda_{[1]}(\lambda)$.

Now recall that Theorem 2.5 yields BBW. In addition Theorem 2.5 determines $\Lambda_{[1]}(\lambda)$ for any $\lambda \in \Lambda$. In fact the theorem asserts $\Lambda_{[1]}(\lambda) = \{\tau * \lambda \mid \tau \in W^1\}$. On the other hand $(\tau*\lambda)^e = \tau \bullet \lambda$ by (1.16). In the notation of [3] the set $\{\tau\bullet\lambda \mid \tau \in W^1\}$ is called a multiplet. This multiplet plays a key role in Theorem 2.10 below. Using (2.15), Theorem 2.10 is just a reformulation of Theorem 2.5 and hence Theorem 2.10 also implies BBW. Let $0 \neq z_{\rho(\mathfrak{n})} \in Z_{\rho(\mathfrak{n})}$.

Theorem 2.10 *Let $\lambda \in \Lambda$. Then $|\lambda+\rho|^2-|\rho_1|^2$ is the maximal eigenvalue of $\zeta_{(\lambda)}(Cas_1)$ on $C(\mathfrak{n},V_\lambda)^e$. Let $M_1 \subset C(\mathfrak{n},V_\lambda)^e$ be the corresponding eigenspace. Then*

$$M_1 = \sum_{\tau\in W^1} (C(\mathfrak{n},V_\lambda)^e)^{\tau\bullet\lambda}. \tag{2.20}$$

Furthermore $C(\mathfrak{n},V_\lambda)^e)^{\tau\bullet\lambda}$ is irreducible for any $\tau \in W^1$ so that M_1 is a multiplicity free $\mathfrak{g}_1$-module whose irreducible components are a multiplet in the sense of [3]. Finally, in the notation of Theorem 2.5, $c(\tau,\lambda)\otimes z_{\tau,\lambda}$ is a highest weight vector in $C(\mathfrak{n},V_\lambda)^e)^{\tau\bullet\lambda}$.

It will be seen in Section 3 that Theorem 2.10 generalizes to the case where the Levi-factor $\mathfrak{g}_1$ is replaced by the much more general subalgebra $\mathfrak{r}$ of §1.1.

3. The spin module S and Euler number multiplets

3.1. We return to the notation and definitions of §1.1. We will also use results in [3]. We will use the notation of [3] with some exceptions. For example $\mathfrak{q}$ here replaces $\mathfrak{p}$ in [3] and $Cliff(\mathfrak{q})$ replaces $C(\mathfrak{p})$.

Let $Cliff(\mathfrak{q})$ be the Clifford algebra over $\mathfrak{q}$ with respect to $B_{\mathfrak{q}}$. We recall some of the relations arising from Chevalley's identification of the underlying vector spaces of the exterior algebra $\wedge\mathfrak{q}$ and the Clifford algebra $Cliff(\mathfrak{q})$. We think of $\wedge\mathfrak{q}$ as having two multiplicative structures. If $u,w \in \wedge\mathfrak{q}$ then $uw \in \wedge\mathfrak{q}$ denotes the Clifford product of u and w and $u \wedge w \in \wedge\mathfrak{q}$ is the exterior product of u and w. For more details see §2 in [3]. If $u,w \in \wedge\mathfrak{q}$ the natural extension, (see e.g. §2.1 in [3]) (u,w), of $B_{\mathfrak{q}}$ to $\wedge\mathfrak{q}$ is again denoted by $B_{\mathfrak{q}}$. If $A \in End \wedge \mathfrak{q}$ then $A^t \in End \wedge \mathfrak{q}$ denotes the transpose of A with respect to $B_{\mathfrak{q}}$. If $w \in \wedge\mathfrak{q}$ then $\varepsilon(w) \in End \wedge \mathfrak{q}$ denotes the operator of left exterior multiplication by w and $\iota(w) = \varepsilon(w)^t$. In particular $\iota(y)$, for $y \in \mathfrak{q}$, is the anti-derivation of degree -1 of the exterior algebra $\wedge\mathfrak{q}$ such that $\iota(y)y' = (y,y')$ if $y' \in \mathfrak{q}$. Clifford multiplication in $\wedge\mathfrak{q}$ is readily determined from the equality

$$yw = (\varepsilon(y) + \iota(y))w \tag{3.1}$$

for any $y \in \mathfrak{q}$ and $w \in \wedge\mathfrak{q}$.

For any $u \in \wedge^2\mathfrak{q}$ let $ad\,u \in End \wedge \mathfrak{q}$ be the operator defined so that $ad\,u(w) = uw - wu$ for any $w \in \wedge\mathfrak{q}$. Then $ad\,u$, clearly a derivation of the Clifford algebra structure of $\wedge^2\mathfrak{q}$, is also a derivation of degree 0 of the exterior algebra structure on $\wedge\mathfrak{g}$. In particular $\wedge^2\mathfrak{q}$ is a Lie algebra under Clifford product and

$$\wedge^2\mathfrak{q} \to End \wedge \mathfrak{q}, \quad u \mapsto ad\,u$$

is a Lie algebra representation. Next, one has that

$$t : \wedge^2\mathfrak{q} \to Lie\,SO(\mathfrak{q}) \tag{3.2}$$

is a Lie algebra isomorphism, where for $u \in \wedge^2\mathfrak{q}$ and $y \in \mathfrak{q}$, one has

$$\begin{aligned} t(u)(y) &= ad\,u(y) \\ &= -2\iota(y)u. \end{aligned} \tag{3.3}$$

For a proof of the above statements in this paragraph see Proposition 7 and Theorem 8 in [3].

Remark 3.1 The second equality in (3.3) is a special case of the general fact that if $w \in \wedge^k\mathfrak{q}$ and $y \in \mathfrak{q}$ then

$$yw - (-1)^k wy = 2\iota(y)w.$$

In particular $\iota(y)$ is an anti-derivation of the Clifford algebra structure of $\wedge\mathfrak{q}$ as well as the exterior algebra structure of $\wedge\mathfrak{q}$. See Lemma 5, p. 284 in [3].

It clearly follows from (3.3) that there exists a unique Lie algebra homomorphism
$$\nu_* : \mathfrak{r} \to \wedge^2 \mathfrak{q} \tag{3.4}$$
such that $t \circ \nu_* = \nu$. See Theorem 8 in [3]. Putting $u = \nu_*(x)$ in (3.3) one has
$$\nu(x)(y) = -2\iota(y)\nu_*(x).$$
For any $x \in \mathfrak{r}$ let $\theta_\nu(x)$ be the unique derivation of the exterior algebra $\wedge \mathfrak{q}$ which extends $\nu(x)$. Regard $\wedge \mathfrak{q}$ as an $\mathfrak{r}$-module where $x \in \mathfrak{r}$ maps to $\theta_\nu(x)$. One notes that the extended bilinear form $B_\mathfrak{q}$ is invariant under $\theta_\nu(x)$ and recalls (see (8) in [3] where $u = \nu_*(x)$) that
$$\theta_\nu(x) = ad\, \nu_*(x). \tag{3.5}$$

Remark 3.2 In the special case of §1.2 where $\mathfrak{g}_1 = \mathfrak{r}$ one obviously has $\mathfrak{n}_- \subset \mathfrak{q}$ and hence $\wedge \mathfrak{n}_- \subset \wedge \mathfrak{q}$. Note that the representation $\eta_0 : \mathfrak{g}_1 \to End \wedge \mathfrak{n}_-$ in §2.3 is just the restriction of θ_ν to the submodule $\wedge \mathfrak{n}_-$.

Since $n = 2m = dim\, \mathfrak{q}$ is even the Clifford algebra $Cliff(\mathfrak{q})$ is simple and hence has a unique (up to equivalence) faithful irreducible module S. One has $dim\, S = 2^m$. Let
$$\varepsilon : Cliff(\mathfrak{q}) \to End\, S \tag{3.6}$$
be the corresponding isomorphism. With this structure the space S is referred to as the spin module for $Cliff(\mathfrak{q})$. The Lie algebra homomorphism $\nu_* : \mathfrak{r} \to \wedge^2 \mathfrak{q}$, composed with ε, defines a representation
$$Spin\, \nu : \mathfrak{r} \to End\, S \tag{3.7}$$
which is referred to as the spin of ν.

For computational purposes it is convenient to take S to be a minimal left ideal in $Cliff(\mathfrak{q})$ and the map ε defining the action of $Cliff(\mathfrak{q})$ on S is left multiplication. Recall (1.3). Clearly $B_\mathfrak{q}|\mathfrak{q}_+ = 0$ and $B_\mathfrak{q}|\mathfrak{q}_- = 0$ so that Clifford multiplication and exterior multiplication are the same on both $\wedge \mathfrak{q}_+$ and $\wedge \mathfrak{q}_-$. On the other hand, if $w \in \wedge^k \mathfrak{q}$ and $w' \in \wedge^{k'} \mathfrak{q}$, it follows from (3.1) that
$$ww' - w \wedge w' \in \sum_{j=1}^{[\frac{k+k'}{2}]} \wedge^{k+k'-2j} \mathfrak{q}. \tag{3.8}$$
This readily implies that the map
$$\wedge \mathfrak{q}_- \otimes \wedge \mathfrak{q}_+ \to Cliff(\mathfrak{q}), \quad w \otimes w' \mapsto ww' \tag{3.9}$$
is a linear bijection. Recall (see §2.2) $\Delta(\mathfrak{q}_+) = \{\beta_1, \ldots, \beta_m\}$. Now put $e_{\mathfrak{q}_+} = e_{\beta_1} \cdots e_{\beta_m}$. Clearly since (Clifford multiplication) $\wedge \mathfrak{q}_+ e_{\mathfrak{q}_+} = \mathbb{C} e_{\mathfrak{q}_+}$, one has that
$$C(\mathfrak{q})\, e_{\mathfrak{q}_+} = \wedge \mathfrak{q}_-\, e_{\mathfrak{q}_+} \tag{3.10}$$

so that the left ideal $C(\mathfrak{q})\,e_{\mathfrak{q}_+}$ has dimension 2^m and consequently is a minimal ideal. We can therefore take

$$\begin{aligned} S &= Cliff\,(\mathfrak{q})\,e_{\mathfrak{q}_+} \\ &= \wedge\mathfrak{q}_-\; e_{\mathfrak{q}_+}. \end{aligned} \tag{3.11}$$

Let $\rho_{\mathfrak{q}} = \frac{1}{2}\sum_{\varphi\in\Delta(\mathfrak{q}_+)}\varphi$ so that

$$\rho = \rho_{\mathfrak{r}} + \rho_{\mathfrak{q}}. \tag{3.12}$$

Remark 3.3 In the special case of §1.2 where $\mathfrak{r} = \mathfrak{g}_1$ note that $\rho_{\mathfrak{q}} = \rho(\mathfrak{n})$ recalling the notation of (1.16).

Recall the notation of §2.2. For any $\Phi \subset \Delta(\mathfrak{q}_+)$ let $s_\Phi = e_{-\Phi}e_{\mathfrak{q}_+}$. The following result is part of Proposition 3.17, p. 477, in [3].

Proposition 3.4 *The set* $\{s_\Phi\}$, $\Phi \subset \Delta(\mathfrak{q}_+)$ *is a weight basis of S where the weight of* s_Φ *is* $\rho_{\mathfrak{q}} - \langle\Phi\rangle$.

Note that Proposition 3.4 implies that $e_{\mathfrak{q}_+}$ is a highest weight vector in S. This has special significance in the case of §1.2 where $\mathfrak{r} = \mathfrak{g}_1$.

Proposition 3.5 *Assume the case of §1.2 where* $\mathfrak{r} = \mathfrak{g}_1$. *Then* $\mathbb{C}e_{\mathfrak{q}_+}$ *is stable under (left Clifford multiplication) the action of* $Spin\,\nu(\mathfrak{g}_1)$ *and affords the 1-dimensional representation* $\sigma_{\rho(n)}$.

Proof. Since $e_{\mathfrak{q}_+}$ is a highest weight vector and the weight is $\rho(\mathfrak{n})$, the $\mathfrak{g}_1$-module generated, under $Spin\,\nu$, by $e_{\mathfrak{q}_+}$ is irreducible and equivalent to $Z_{\rho(\mathfrak{n})}$. But $Z_{\rho(\mathfrak{n})}$ is 1-dimensional by Proposition 2.6. ■

As a corollary one has

Proposition 3.6 *Retain the notation and assumptions of Proposition 3.5. Recall also the notation of §2.3 (especially Theorem 2.9). Let*

$$\gamma : \wedge\mathfrak{n}_-^e \to S \tag{3.13}$$

be the linear isomorphism defined so that for $u \in \wedge\mathfrak{n}_-$,

$$\gamma(u\otimes z_{\rho(\mathfrak{n})}) = u\,e_{\mathfrak{q}_+}. \tag{3.14}$$

Then γ *intertwines the representations (see (2.14))* $\zeta_{(0)}$ *and* $Spin\,\nu$ *of* $\mathfrak{g}_1$.

Proof. Let $x \in \mathfrak{g}_1$ and let $u \in \wedge \mathfrak{q}_-$. Let $f : \mathfrak{g}_1 \to \mathbb{C}$ be defined as in the proof of Proposition 2.6. Then by (3.5), Remark 3.2 and Proposition 3.5 one has

$$\begin{aligned} Spin\,\nu(\gamma(u \otimes z_{\rho(\mathfrak{n})})) &= Spin\,\nu(x)(u\,e_{\mathfrak{q}_+}) \\ &= \nu_*(x) u\,e_{\mathfrak{q}_+} \\ &= [\nu_*(x), u] e_{\mathfrak{q}_+} + u\,\nu_*(x) e_{\mathfrak{q}_+} \\ &= \eta_0(x)(u)\,e_{\mathfrak{q}_+} + f(x) u\,e_{\mathfrak{q}_+} \\ &= \gamma(\zeta_{(0)}(u \otimes z_{\rho(\mathfrak{n})})) \end{aligned}$$

■

Let $\lambda \in \Lambda$. Assume the general case of §1.1. Then in the notation of (2.20) in [3],

$$\zeta_\lambda : \mathfrak{r} \to End\,V_\lambda \otimes S$$

is the tensor product of the representations $\pi_\lambda|\mathfrak{r}$ and $Spin\,\nu$. The result in [3] which generalizes Theorem 2.10 (and hence generalizes BBW) will be a statement, Theorem 3.8 below, about ζ_λ. Proposition 3.7 below insures that Theorem 3.8 generalizes Theorem 2.10.

Retain the notation and assumptions of Proposition 3.6. Clearly we can write

$$C(\mathfrak{n}, V_\lambda)^e = V_\lambda \otimes \wedge \mathfrak{n}_-^e. \tag{3.15}$$

Let

$$\gamma_\lambda : C(\mathfrak{n}, V_\lambda)^e \to V_\lambda \otimes S \tag{3.16}$$

be the linear isomorphism defined so that if $v \in V_\lambda$ and $w \in \wedge \mathfrak{n}_-^e$ then $\gamma_\lambda(v \otimes w) = v \otimes \gamma(w)$. The following corollary of Proposition 3.6 is immediate.

Proposition 3.7 *Let $\lambda \in \Lambda$. Retain the notation and assumptions of Proposition 3.6. Recall (2.13). Then γ_λ intertwines the representations $\zeta_{(\lambda)}$ and ζ_λ of $\mathfrak{g}_1$.*

3.2. Return to the general case of §1.1. Let $\lambda \in \Lambda$. Then $V_\lambda \otimes S$ is an $\mathfrak{r}$-module with respect to ζ_λ. If $\delta \in \Gamma$ is an extremal weight of V_λ then, as in §2.2, let $v_\delta \in V_\lambda$ be a nonzero δ-weight vector. For any $\tau \in W^1$ recall that $\tau \bullet \lambda = \tau(\lambda + \rho) - \rho_\mathfrak{r}$. Recall (2.7). Let $w_{\tau \bullet \rho} = v_{\tau(\lambda)} \otimes s_{\Phi_\tau}$ so that, by (2.6) and Proposition 3.4, $w_{\tau\bullet\rho} \in V_\lambda \otimes S$ is a weight vector of weight $\tau \bullet \lambda$. The following result is part of Theorem 4.17 in [3]

Theorem 3.8 *Let $\lambda \in \Lambda$. Then for any $\tau \in W^1$ the multiplicity of $\sigma_{\tau\bullet\lambda}$ in $V_\lambda \otimes S$ is* 1 *so that there is an equivalence*

$$(V_\lambda \otimes S)^{\tau\bullet\lambda} \cong Z_{\tau\bullet\lambda} \tag{3.17}$$

of $\mathfrak{r}$*-modules. In particular*

$$\{(V_\lambda \otimes S)^{\tau \bullet \lambda} \mid \tau \in W^1\} \tag{3.18}$$

is a $\Lambda_{\mathfrak{r}}^{\lambda}$*-multiplet (see (3.3) in [3]). Moreover* $w_{\tau\bullet\lambda}$ *is a highest weight vector of* $(V_\lambda \otimes S)^{\tau\bullet\lambda}$.

Furthermore the highest eigenvalue of $\zeta_\lambda(Cas_r)$ *is* $|\lambda+\rho|^2 - |\rho_{\mathfrak{r}}|^2$ *and the sum*

$$\sum_{\tau \in W^1} (V_\lambda \otimes S)^{\tau \bullet \lambda} \tag{3.19}$$

of the components of the $\Lambda_{\mathfrak{r}}^{\lambda}$*-multiplet (3.18) is the corresponding eigenspace.*

Remark 3.9 Recall that Theorem 2.10 implies BBW. It is obvious from Proposition 3.7 that Theorem 3.8 reduces to Theorem 2.10 under the special Levi-factor assumptions of §1.2.

Again under the special Levi-factor assumptions of §1.2 one also recovers a familiar relation between the square of a Dirac operator and Laplacians. Let $\lambda \in \Lambda$. Identify $C(\mathfrak{n}, V_\lambda)^e$ with $V_\lambda \otimes S$ using the isomorphism (3.16). But then the Laplacian L_λ^e (see (2.15)) can be regarded as an operator on $V_\lambda \otimes S$. Let $\square_\lambda$ be the cubic Dirac operator on $V_\lambda \otimes S$ defined by (2.19) in [3]. (The following result uses Theorem 2.21 in [3]. The statement of the latter result misprinted the word Theorem 2.16 and wrote Theorem 2.2 instead.)

Theorem 3.10 *Let* $\lambda \in \Lambda$*. Assume the special Levi-factor case of* §*1.2 so that* $\mathfrak{r} = \mathfrak{g}_1$*. Then*

$$2L_\lambda^e = (\square_\lambda)^2. \tag{3.20}$$

Proof. The result follows immediately from (2.15), Proposition 3.7 and Theorem 2.21 in [3]. ■

Remark 3.11 In the general case the maximal eigenspace, (3.19), of $\zeta_\lambda(Cas_r)$ is the kernel of the Dirac operator $\square_\lambda$. See Theorem 4.24 in [3].

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