Deformations of algebras over operads and the Deligne conjecture

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Abstract. In the present paper we develop the deformation theory of operads and algebras over operads. Free resolutions (constructed via Boardman-Vogt approach) are used in order to describe formal moduli spaces of deformations. We apply the general theory to the proof of the Deligne conjecture. The latter says that the Hochschild complex of an associative algebra carries a canonical structure of a dg-algebra over the chain operad of the little discs operad. In the course of the proof we construct an operad of geometric nature which acts on the Hochschild complex. It seems to be different from the brace operad (the latter was used in the previous approaches to the Deligne conjecture). It follows from our results that the Grothendieck-Teichmüller group acts (homotopically) on the moduli space of structures of 2-algebras on the Hochschild complex. In the Appendix we develop a theory of piecewise algebraic chains and forms. It is suitable for real semialgebraic manifolds with corners (like Fulton-MacPherson compactifications of configuration spaces of points).

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Introduction

The deformation theory of associative algebras is a guide for developing the deformation theory of many algebraic structures. Conversely, all the concepts of what should be the "deformation theory of everything" must be tested in the case of associative algebras.

An associative algebra is an algebra over an operad. This fact, along with the observation that in many examples we are dealing with algebras over operads, "explains" the universality of associative algebras. It also suggests how to develop the deformation theory of algebras over operads. This theory is one of the topics of the present paper.

Another remarkable fact is the relationship of the deformation theory of associative algebras to the geometry of configuration spaces of points on surfaces. One

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of its incarnations is the Deligne conjecture (formulated in [4]). A brief history of the conjecture as well as its generalizations can be found in [14]. The Deligne conjecture is the second topic of our paper.

The theme which motivated the whole project is the Grothendieck-Teichmüller group (*GT* for short) and its role in deformation theory. The Grothendieck-Teichmüller group can be defined as the automorphism group of the tower of the pro-nilpotent completions of pure braid groups (see [5]). The pure braid group of n strings is the fundamental group of the configuration space of n points in the plane. The Deligne conjecture shows its relation to the Hochschild complex. More evidence for that relation has been found in deformation quantization. The coefficients in the explicit formulas for the deformed multiplication given in [12] are periods of algebraic varieties defined over the field of rational numbers. Through the hypothetical relation to mixed Tate motives, this fact leads to some conjectures about the action of the Grothendieck-Teichmüller group on the moduli space of quantized algebras of functions on a manifold (see [14]). The Deligne conjecture says that the Hochschild complex $C^\bullet(A,A)$ of an associative (or more generally, A_∞) algebra A carries a structure of 2-algebra (i.e., an algebra over the operad of chains of the little discs operad). It follows from our results that the Grothendieck-Teichmüller group acts (homotopically) on the moduli space of structures of 2-algebras on $C^\bullet(A,A)$. This action is closely related to the action of the motivic Galois group described in [14]. We hope to discuss this topic in detail elsewhere.

The paper is organized in the following way.

Section 1 is devoted to the review of operads. It includes a brief introduction to operads, language of trees, polynomial functors, etc.

The deformation theory of algebras over free operads is discussed in Section 2. The "general" deformation theory is based on the notion of a formal pointed dg-manifold (a formal $\mathbf{Z}$-graded manifold with a vector field d of degree $+1$ such that $[d,d]=0$). We do not use any deep results concerning dg-manifolds. All necessary facts and definitions can be found in [12] (note that in [12], [13] formal pointed dg-manifolds were called formal pointed Q-manifolds). We explain how to construct a formal pointed dg-manifold which controls the deformation theory of an algebra over an operad. In traditional language this corresponds to a construction of the deformation functor as a functor of the category of local Artinian rings. We explain how this approach leads to the (homotopy) action of the Grothendieck-Teichmüller group on the dg-manifold controlling the deformation theory of the Hochschild complex of an A_∞-algebra.

In Section 3 we outline the strategy and state the theorem (proved in Section 6) which explain why the *GT* group appears in the deformation theory of associative (and more generally A_∞) algebras.

Our approach is based on the notion of a free resolution of an operad. This is the subject of Section 4. We construct canonical free resolutions via an approach similar to the one developed by Boardman and Vogt in [3]. We use free resolutions of operads instead of the conventional approach which uses free resolutions of algebras. This approach to the deformation theory of algebras over operads is not used very often (see however [15]).

In Section 5 we construct an operad M which acts on the Hochschild complex of an A_∞-algebra. It turns out that M is closely related to the compactifications of configuration spaces of point introduced by Fulton and MacPherson in [6].

Section 6 is devoted to the proof of the theorem from Section 3.

Section 7 is devoted to the Deligne conjecture and its proof. The proof presented in this paper is based on the general deformation theory developed in the previous sections. The strategy is explained in detail in Section 7. We also suggest certain generalizations of the original Deligne conjecture as well as some conjectures about the cell structure of the spaces which appear in the course of its study. It seems that the proof admits a generalization to the higher-dimensional version of the Deligne conjecture proposed in [14].

Other approaches to the original Deligne conjecture were proposed in [18], [21], [22].

The Appendix is devoted to a theory of piecewise algebraic chains. It is suitable for real semialgebraic manifolds with corners. A typical example is the compactification of the configuration space of points in the plane (Fulton-MacPherson compactification). This theory is useful for the proof of formality of the operad of chains of the little discs operad (in the approach of [14]). It is also useful (but not necessary) in the proof of the Deligne conjecture. The usual techniques of simplicial homology will do the job. At the same time we feel that the theory of piecewise algebraic chains is appropriate to the nature of the topic. For this reason we have decided to include it in the paper.

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1. Generalities on operads

1.1. Polynomial functors, operads, algebras

The material of this subsection is very well-known (see e.g. [9], [7], [8], [17]). We present it here for completeness and in order to fix the notation.

Let k be a field of characteristic zero. All vector spaces below will be k-vector spaces unless we say otherwise.

We fix a category $\mathcal{C}$ which is assumed to be k-linear abelian symmetric monoidal and closed under infinite sums and products. We will also assume that it has internal $Hom's$. Our main examples will be the category of k-vector spaces, the category $Vect_{\mathbf{Z}}$ of $\mathbf{Z}$-graded vector spaces (with Koszul rule of signs), and the category of complexes of k-vector spaces.

Suppose we have a collection of representations $F = (F_n)_{n\geq 0}$ of the symmetric groups S_n, $n \geq 0$, in $\mathcal{C}$ (i.e., we have a sequence of objects F_n together with an action of the group S_n on F_n for each n).

Definition 1 *A polynomial functor $F:\mathscr{C}\to\mathscr{C}$ is defined on objects by the formula*

$$F(V)=\oplus_{n\geq 0}(F_n\otimes V^{\otimes n})_{S_n}$$

where for a group H and an H-module W we denote by W_H the space of coinvariants. The functor F is defined on morphisms in an obvious way.

Notice that having a sequence F_n as above we can define F_I for any finite set I using isomorphisms of I with the standard set $\{1,\dots,|I|\}$, where $|I|$ is the cardinality of I. Thus $F_{\{1,\dots,n\}}=F_n$. Technically speaking, we consider a functor Φ from the groupoid of finite sets (morphisms are bijections) to the symmetric monoidal category $\mathscr{C}$. Then we set $F_I=\Phi(I)$.

Polynomial functors on $\mathscr{C}$ form a category $\mathscr{PF}$ if we define morphisms between two such functors F and G as a vector space of S_n-intertwiners

$$Hom(F,G)=\prod_{n=0}^{\infty}Hom_{S_n}(F_n,G_n).$$

There is a composition operation $\circ$ on polynomial functors such that $(F\circ G)(V)$ is naturally isomorphic to $F(G(V))$ for any $V\in\mathscr{C}$. We also have a polynomial functor $\mathbf{1}$ such that $\mathbf{1}_1=1_{\mathscr{C}}$ and $\mathbf{1}_n=0$ for all $n\neq 1$. Here $1_{\mathscr{C}}$ is the unit object in the monoidal category $\mathscr{C}$. It is easy to see that in this way we get a monoidal structure on $\mathscr{PF}$.

Definition 2 *An operad in $\mathscr{C}$ is a monoid in the monoidal category $\mathscr{PF}$. In other words it is a polynomial functor $R\in\mathscr{PF}$ together with morphisms $m:R\circ R\to R$ and $u:\mathbf{1}\to R$ satisfying the associativity and the unit axioms.*

To shorten the notation we will denote the operad (R,m,u) simply by R. An operad R gives rise to a triple in the category $\mathscr{C}$. There is the notion of an algebra over a triple in a category. Hence we can use it in order to give a definition of an algebra in $\mathscr{C}$ over the operad R. It is given by an object $V\in\mathscr{C}$ and a morphism $R(V)\to V$ satisfying natural properties of compatibility with the structure of a triple. Equivalently, V is an R-algebra iff there is a morphism of operads $R\to\mathscr{E}nd(V)$, where $\mathscr{E}nd(V)$ is the endomorphism operad of V defined by $(\mathscr{E}nd(V))_n=\underline{Hom}(V^{\otimes n},V)$, and $\underline{Hom}$ denotes the internal Hom in $\mathscr{C}$. The category of R-algebras will be denoted by $R-alg$. There are two adjoint functors $Forget_R:R-alg\to\mathscr{C}$ and $Free_R:\mathscr{C}\to R-alg$ such that $Forget_R\circ Free_R=R$.

Definition 3 *For $X\in Ob(\mathscr{C})$, we call $Free_R(X)$ the free R-algebra generated by X.*

We remind the reader that there are operads As, Lie, $Comm$ such that the algebras over them in the category of vector spaces are associative, Lie and commutative algebras correspondingly.

1.2. Colored operads

There is a generalization of the notion of operad. It is useful in order to describe in operadic terms pairs *(associative algebra A, A-module)*, homomorphisms of algebras over operads, etc.

Let I be set. We consider the category $\mathscr{C}^I$ consisting of families $(V_i)_{i\in I}$ of objects of $\mathscr{C}$.

A polynomial functor $F:\mathscr{C}^I\to\mathscr{C}^I$ is defined by the following formula:

$$(F((V_i)_{i\in I}))_j=\oplus_{a:I\to\mathbf{Z}_{\geq 0}}F_{a,j}\otimes_{\prod_i S_{a(i)}}\otimes_{i\in I}(V_i^{\otimes a(i)}),$$

where $a:I\to\mathbf{Z}_{\geq 0}$ is a map with the finite support, and $F_{a,j}$ is a representation in $\mathscr{C}$ of the group $\prod_{i\in I}S_{a(i)}$.

Polynomial functors in $\mathscr{C}^I$ form a monoidal category with the tensor product given by the composition of functors.

Definition 4 *A colored operad is a monoid in this category.*

Similarly, as in the case of usual operads, it defines a triple in the category $\mathscr{C}^I$. Therefore we have the notion of an algebra over a colored operad.

There exists a colored operad $\mathscr{O}\mathscr{P}$ such that the category of $\mathscr{O}\mathscr{P}$-algebras is equivalent to the category of operads.

Namely, let us consider the forgetful functor $Operads\to\mathscr{P}\mathscr{F}$. It has a left adjoint functor. Thus we have a triple in $\mathscr{P}\mathscr{F}$. As we have noticed before, the category $\mathscr{P}\mathscr{F}$ can be described as a category of sequences $(P_n)_{n\geq 0}$ of S_n-modules. Then using the representation theory of symmetric groups, we conclude that the category $\mathscr{P}\mathscr{F}$ is equivalent to the category $\mathscr{C}^{I_0}$, where I_0 is the set of all Young diagrams (partitions). Hence a polynomial functor $F:\mathscr{P}\mathscr{F}\to\mathscr{P}\mathscr{F}$ can be described as a collection $F_{(m_i),n}$ of the representations of the groups $S_{n,(m_k)}:=S_n\times\prod_{k\geq 0}(S_{m_k}\triangleright S_k^{m_k})$, where $\triangleright$ denotes the semidirect product of groups.

Having these data we can express any polynomial functor F on $\mathscr{P}\mathscr{F}$ by the formula:

$$(F((U_k)_{k\geq 0}))_n=\bigoplus_{(m_k)}F_{(m_k),n}\otimes_{S_{1,(m_k)}}\bigotimes_{k\geq 0}(U_k^{\otimes m_k}).$$

In particular, one has a functor $\mathscr{O}\mathscr{P}:\mathscr{P}\mathscr{F}\to\mathscr{P}\mathscr{F}$, which is the composition of the forgetful functor $Operads\to\mathscr{P}\mathscr{F}$ with its adjoint. It gives rise to an I_0-colored operad $\mathscr{O}\mathscr{P}=(\mathscr{O}\mathscr{P}_{(m_i),n})$. We will describe it explicitly in the subsection devoted to trees.

1.3. Non-linear operads

We remark that operads and algebras over operads can be defined for any symmetric monoidal category $\mathscr{C}$, not necessarily k-linear. In particular, we are going to use operads in the categories of sets, topological spaces, etc.

Namely, an operad in $\mathscr{C}$ is a collection $(F_n)_{n\geq 0}$ of objects in $\mathscr{C}$, each equipped with an S_n-action, as well as composition maps:

$$F_n \otimes F_{k_1} \otimes \cdots \otimes F_{k_n} \to F_{k_1+\cdots+k_n},$$

for any $n \geq 0$ and $k_1, \ldots, k_n \geq 0$. Another datum is the unit, which is a morphism $\mathbf{1}_{\mathscr{C}} \to F_1$. All the data are required to satisfy certain axioms (see [17]). Analogously one describes colored operads and algebras over operads. Notice that in this framework one cannot speak about polynomial functors and free algebras.

This approach has some advantages and drawbacks (like the description of analytic functions in terms of Taylor series vs their description in terms of Taylor coefficients).

1.4. Pseudo-tensor categories

The notion of colored operad has been rediscovered many times. In [1] the notion of *pseudo-tensor category* was introduced as a generalization of the notion of symmetric monoidal (=tensor) category. The terminology stresses the similarity of operads with tensor categories. Similar notion was introduced in [2] under the name *multi-linear category*.

This notion is essentially equivalent to the notion of colored operad. We recall it below, using the name suggested in [1].

Definition 5 *A pseudo-tensor category is given by the following data:*

1) A class $\mathscr{A}$ called the class of objects, and a symmetric monoidal category $\mathscr{V}$ called the category of operations.

2) For every finite set I, a family $(X_i)_{i\in I}$ of objects, and an object Y, an object $P_I((X_i),Y) \in \mathscr{V}$ called the space of operations from $(X_i)_{i\in I}$ to Y.

3) For any map of finite sets $\pi : J \to I$, two families of objects $(Y_i)_{i\in I}, (X_j)_{j\in J}$ and an object Z, a morphism in $\mathscr{V}$

$$P_I((Y_i),Z) \otimes (\otimes_i P_{\pi^{-1}(i)}((X_{j_i}),Y_i)) \to P_J((X_j),Z),$$

called composition of operations. Here we denote by $\otimes$ the tensor product in $\mathscr{M}$.

4) For an 1*-element set $\{pt\}$ and an object X, a unit morphism $\mathbf{1}_{\mathscr{V}} \to P_{\{pt\}}((X),X)$.*

These data are required to satisfy natural conditions. In particular, compositions of operations are associative with respect to morphisms of finite sets, and the unit morphisms satisfy properties analogous to those of the identity morphisms (see [1] or [19] for details).

If $\mathscr{A}$ is a set, then a pseudo-tensor category is exactly the same as an $\mathscr{A}$-colored operad in the tensor category $\mathscr{V}$.

If we take $\mathscr{V}$ to be the category of sets, and take I above to be 1-element sets only, we obtain a category with the class of objects equal to $\mathscr{A}$.

Colored operad with one color gives rise to an ordinary operad. A symmetric monoidal category $\mathscr{A}$ produces the colored operad with

$$P_I((X_i),Y) = Hom_{\mathscr{A}}(\otimes_i X_i, Y).$$

The notion of pseudo-tensor category admits a generalization to the case when no action of the symmetric group is assumed. This means that we consider *sequences* of objects instead of *families* (see [19]). The new notion generalizes monoidal categories. In the terms of the next subsection, this would mean that one uses planar trees instead of all trees. One can go one step further by generalizing braided categories. This leads to colored braided operads (or pseudo-braided categories). In this case, trees in $\mathbf{R}^3$ should be used. The deformation theory of such structures can be developed along the lines of the present paper. It will be explained in detail elsewhere.

1.5. Trees

Definition 6 *A tree T is defined by the following data:*

1) A finite set $V(T)$ whose elements are called vertices;

2) A distinguished element $root_T \in V(T)$ called root vertex;

3) Subsets $V_i(T)$ and $V_t(T)$ of $V(T) \setminus \{root_T\}$ called the set of internal vertices and the set of tails respectively. Their elements are called internal and tail vertices respectively;

4) A map $N = N_T : V(T) \to V(T)$.

These data are required to satisfy the following properties:

a) $V(T) = \{root_T\} \sqcup V_i(T) \sqcup V_t(T)$;

b) $N_T(root_T) = root_T$, and $N_T^k(v) = root_T$ for all $v \in V(T)$ and $k \gg 1$;

c) $N_T(V(T)) \cap V_t(T) = \emptyset$;

d) There exists a unique vertex $v \in V(T), v \neq root_T$ such that $N_T(v) = root_T$.

We denote by $|v|$ the *valency* of a vertex v, which we understand as the cardinality of the set $N_T^{-1}(v)$.

We call the pairs $(v,N(v))$ *edges* in the case where $v \neq root_T$. If both elements of the pair belong to $V_i(T)$ we call the corresponding edge *internal*. The only edge e_r defined by the condition d) above is called the *root edge*. All edges of the type $(v,N(v)), v \in V_t(T)$ are called *tail edges*. We use the notation $E_i(T)$ and $E_t(T)$ for the sets of internal and tail edges respectively. We have a decomposition of the set of all edges $E(T) = E_i(T) \sqcup (E_t(T) \cup \{e_r\})$. There is a unique tree T_e such that $|V_t(T_e)| = 1$ and $|V_i(T_e)| = 0$. It has the only tail edge which is also the root edge.

A *numbered* tree with n tails is by definition a tree T together with a bijection of sets $\{1,\dots,n\} \to V_t(T)$. We can picture trees as in Figure 1.

Let R be an operad. Any tree T gives a natural way to compose elements of R, $comp_T : \otimes_{i \in V_i(T)} R_{N^{-1}(v)} \to R_{V_t(T)}$.

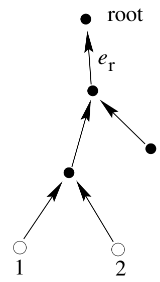

Figure 1. Numbered tree, non-numbered vertices are black.

Let us return to the colored operad $\mathscr{O}\mathscr{P}$ and give its description using the language of trees.

Namely, $\mathscr{O}\mathscr{P}_{(m_i),n}$ is a k-vector space generated by the isomorphism classes of trees T such that:

a) T has n tails numbered from 1 to n;

b) T has $\sum_i m_i$ internal vertices all numbered in such a way that first m_0 vertices have valency 0, and they are numbered from 1 to m_0, next m_1 internal vertices have valency 1, and they are numbered from 1 to m_1, and so on;

c) for every internal vertex $v \in V_i(T)$ the set of incoming edges $N_T^{-1}(v)$ is also numbered.

An action of the group $S_{(m_k),n}$ is defined naturally: the factor S_n permutes numbered tails, the factor S_{m_k} permutes numbered internal vertices and the factor $S_k^{m_k}$ permutes their incoming edges numbered from 1 to k.

The composition is given by the procedure of inserting of a tree into an internal vertex of another one. The new numeration is clear. We leave these details as well as the proof of the following proposition to the reader.

Proposition 1 *The category of $\mathscr{O}\mathscr{P}$-algebras is equivalent to the category of k-linear operads.*

Let F be a polynomial functor on $\mathscr{C}$ (see Section 1.1). Let us consider a category $\mathscr{C}_F$, objects of which are pairs $(V, \phi : F(V) \to V)$ where V is an object of $\mathscr{C}$ and ϕ is a morphism in $\mathscr{C}$. Morphisms of pairs are defined in the natural way. The following lemma is easy to prove.

Lemma 1 *The category $\mathscr{C}_F$ is equivalent to the category of $Free_{\mathscr{O}\mathscr{P}}(F)$-algebras.*

We call $P = Free_{\mathscr{O}\mathscr{P}}(F)$ the *free operad* generated by F.

Components P_n of the functor P can be defined explicitly as follows.

Let $Tree(n)$ denotes the groupoid of numbered trees with n tails, $|Tree(n)|$ denotes the set of classes of isomorphisms of these trees. We denote the class of isomorphism of T by $[T]$. Then we have

$$P_n = Free_{\mathscr{O}\mathscr{P}}(F)_n = \oplus_{[T]\in|Tree(n)|}(\otimes_{v\in V_i(T)}F_{N^{-1}(v)})_{AutT}.$$

2. Deformations and differentials in free operads

Let F be a polynomial functor, $P = Free_{\mathscr{O}\mathscr{P}}(F)$ be the corresponding free operad. Let g_P be the Lie algebra (in the symmetric monoidal category $\mathscr{C}$) of derivations of the operad P. Then, as an object of $\mathscr{C}$:

$$g_P = \prod_{n\geq 0} \underline{Hom}(F_n, P_n)^{S_n}$$

where W^H denotes the space of H-invariants of an H-module W and $\underline{Hom}$ denotes the internal Hom in $\mathscr{C}$. This observation follows from the fact that

$$Hom_{\mathscr{P}\mathscr{F}}(F, Forget_{\mathscr{O}\mathscr{P}}(G)) = Hom_{\mathscr{O}\mathscr{P}-alg}(Free_{\mathscr{O}\mathscr{P}}(F), G).$$

From now on we suppose that $\mathscr{C}$ is the category $Vect_{\mathbf{Z}}$ of $\mathbf{Z}$-graded vector spaces. Then g_P is a graded Lie algebra with the graded components g_P^n.

Definition 7 *A differential on P is an element $d_P \in g_P^1$ such that $[d_P, d_P] = 0$.*

If d_P is a differential then $\hat{P} := (P, d_P)$ is an operad in the symmetric monoidal category of complexes. We should distinguish it from P, which is a free operad in the category $Vect_{\mathbf{Z}}$.

For the purposes of the deformation theory it is very convenient to work with operads of the type $\hat{P}$. A general dg-operad could be replaced by its free resolution.

Definition 8 *A dg-algebra over (P, d_P) is an algebra over $\hat{P}$ in the category of complexes.*

Later we will sometimes abuse notations speaking about dg-algebras over P meaning that the differential d_P is obvious from the context.

We recall that usually a deformation theory of a mathematical structure is controlled by a formal pointed dg-manifold (see [12]). In what follows we shall define deformation theories of operads and algebras by declaring the dg-manifolds which control them. Thus:

Definition 9 *The deformation theory of $\hat{P}$ is controlled by the formal dg-manifold associated in the usual way with the differential graded Lie algebra $(g_P, [d_P, \cdot])$.*

The motivation for this definition is that in order to deform the pair (P, d_P) it is enough to deform d_P (since P itself is free and therefore rigid).

Now we are going to describe the deformation theory of dg-algebras over P ($\widehat{P}$-algebras). In what follows we will often speak about points of $\mathbf{Z}$-graded manifolds.

It will always mean Λ-points, where Λ is an auxiliary $\mathbf{Z}$-graded commutative associative algebra (in general without the unit). In the case of formal manifolds we take Λ to be nilpotent.

Let us describe the formal pointed dg-manifold controlling deformations of a P-algebra V. We have the following graded vector space

$$\mathscr{M} = \mathscr{M}(P,V) = (\underline{Hom}(V,V))[1] \oplus \underline{Hom}(F(V),V).$$

We denote by $\mathscr{M}^n, n \in \mathbf{Z}$, the graded components of $\mathscr{M}$.

The structures of a complex on a graded vector space V and an action of P on V define a point $(d_V,\rho) \in \mathscr{M}^0 = Hom_{Vect_{\mathbf{Z}}}(k,\mathscr{M})$. We consider here d_V and ρ as morphisms of graded vector spaces. The equation $d_V^2 = 0$ and the condition of compatibility of d_V and ρ can be written in the form $d_{\mathscr{M}}(d_V,\rho) = 0$, where $d_{\mathscr{M}}(d_V,\rho) = (d_V^2, \xi(d_V,\rho)) \in \mathscr{M}^1$, for some $\xi(d_V,\rho) \in \underline{Hom}(F(V),V)$. It is easy to see that the assignment $(d_V,\rho) \mapsto d_{\mathscr{M}}(d_V,\rho)$ defines an odd vector field $d_{\mathscr{M}}$ on the "infinite-dimensional graded manifold" $\mathscr{M}$. A zero of this vector field corresponds to a structure of a complex on V together with a compatible structure of a dg-algebra over P. This gives a bijection between the set of zeros and the set of such structures.

It is easy to check that $[d_{\mathscr{M}}, d_{\mathscr{M}}] = 0$. Therefore a formal neighborhood of a fixed point (d_V,ρ) of $d_{\mathscr{M}}$ becomes a formal pointed dg-manifold.

Definition 10 *The deformation theory of a dg-algebra V is controlled by the formal pointed dg-manifold $M(P,V)$.*

Remark 1 Operads are algebras over the colored operad $\mathscr{O}\mathscr{P}$. One can show that deformation theories for an $\mathscr{O}\mathscr{P}$-algebra $\widehat{P}$ described in the last two definitions are in fact equivalent.

Let $P^{(0)}$ be an operad in $Vect_k$. In order to define the deformation theory of $P^{(0)}$-algebras, one needs to choose a resolution $\phi : P \to P^{(0)}$, where $P = Free_{\mathscr{O}\mathscr{P}}(F)$ as a $\mathbf{Z}$-graded operad, and ϕ is a quasi-isomorphism. One wants to be sure that the deformation theory does not depend on the choice of the resolution. This can be achieved by assuming that (see e.g. [16]):

a) F admits a filtration (as a polynomial functor) $F = \cup_{j\geq 1} F^{(j)}, F^{(j)} \subset F^{(j+1)}$ such that $d_P(F^{(0)}) = 0$ and $d_P(F^{(j)}) \subset Free_{\mathscr{O}\mathscr{P}}(F^{(j-1)}), j \geq 1$;

b) $\phi : P \to P^{(0)}$ is a an epimorphism.

Property a) means that F is a cofibrant object in the category of operads (see [10]). All resolutions used in the paper will satisfy these properties.

2.1. Example: A_∞-operad and A_∞-algebras

Let $V \in Vect_{\mathbf{Z}}$ and $m_n : V^{\otimes n} \to V[n-2]$, $n \geq 2$, be a sequence of morphisms. It gives rise to an action on V of the free operad $P = Free_{\mathscr{O}\mathscr{P}}(F)$ where

$$F(V) = \oplus_{n\geq 2} V^{\otimes n}[n-2].$$

Then $F_n = k[S_n]m_n \otimes k[1]^{\otimes(n-2)}$. This notation means that we consider F_n as a space (with the grading shifted by $n-2$) of the regular representation of the group algebra of the symmetric group S_n. This space is generated by an element which we denote by m_n.

The differential $d_P \in g_P$ (equivalently, a structure of a dg-operad on P) is defined by the standard formulas:

$$\begin{aligned}
&d_P(m_2) = 0,\\
&d_P(m_n)(v_1 \otimes \cdots \otimes v_n) =\\
&\quad \sum_{k+l=n} \pm m_k(v_1 \otimes \cdots \otimes v_i \otimes m_l(v_{i+1} \otimes \cdots \otimes v_{i+l}) \otimes \cdots \otimes v_n), \quad n > 2.
\end{aligned}$$

We do not specify signs in these well-known formulas. In Section 5 we propose a general framework allowing to fix signs in the formulas like the one above.

Definition 11 *The dg-operad $\mathscr{A}_\infty = (P, d_P)$ is called the A_∞-operad. Algebras over this dg-operad are called A_∞-algebras.*

Deformations of an A_∞-algebra A are controlled by the truncated Hochschild complex

$$C^\bullet_+(A,A) = \prod_{n\geq 1} Hom_{Vect_{\mathbf{Z}}}(A^{\otimes n}, A)[-n].$$

More precisely, let A be a graded vector space. We define a graded vector space of all Hochschild cochains of A as

$$C^\bullet(A,A) = \prod_{n\geq 0} Hom_{Vect_{\mathbf{Z}}}(A^{\otimes n}, A)[-n].$$

Then $C^\bullet(A,A)[1]$ can be equipped with the structure of a graded Lie algebra with the Gerstenhaber bracket (the latter appears naturally if we interpret Hochschild cochains as coderivations of the free coalgebra $\oplus_{n\geq 0}(A[1])^{\otimes n}$).

Let us consider an element $m = (m_1, m_2 ...) \in C^\bullet_+(A,A)[1]$ of degree $+1$ such that $[m,m] = 0$. Such an element defines a differential $d = m_1$ on A, and the sequence $(m_2, m_3, ...)$ gives rise to a structure of an A_∞-algebra on (A, m_1).

Then we can make $C^\bullet(A,A)$ into a complex (Hochschild complex) with the differential $d_m = [m, \cdot]$. It is easy to see that in this way we get a differential graded Lie algebra (DGLA for short) $(C^\bullet(A,A)[1], d_m)$. The truncated Hochschild complex $C^\bullet_+(A,A)[1]$ is a DGLA subalgebra. According to the general theory (see [12]) both DGLAs define formal pointed dg-manifolds, and therefore give rise to the deformation functors. This is a straightforward generalization of the well-known deformation theory of associative algebras.

In some sense, the full Hochschild complex controls deformations of the A_∞-category with one object, such that its endomorphism space is equal to A.

The deformation theory of A_∞-categories is not in the scope of present paper. It will be explained elsewhere. Nevertheless we will refer to the formal dg-manifold associated with $C^\bullet(A,A)[1]$ as to the *moduli space of A_∞-categories*. Similarly, the formal dg-manifold associated with $C^\bullet_+(A,A)[1]$ will be called the *moduli space of A_∞-algebras*. (All the terminology assumes that we deform a given A_∞-algebra A).

The moduli space of A_∞-algebras is the same as $\mathscr{M}(\mathscr{A}_\infty,A)$ in the previous notation. Similarly we will denote the moduli space of A_∞-categories by $\mathscr{M}_{cat}(\mathscr{A}_\infty,A)$. The natural inclusion of DGLAs $C^\bullet_+(A,A)[1] \to C^\bullet(A,A)[1]$ induces a dg-map (dg-map is a morphism of dg-manifolds):

$$\mathscr{M}(\mathscr{A}_\infty,A) \to \mathscr{M}_{cat}(\mathscr{A}_\infty,A).$$

Let us remark that the operad $\mathscr{A}_\infty$ gives rise to a free resolution of the operad As. Algebras over the latter are associative algebras without the unit.

Remark 2 It is interesting to describe deformation theories of free resolutions of the classical operads As, Lie, $Comm$. It seems that for an arbitrary free resolution P of either of these operads the following is true: $H^i(g_P) = 0$ for $i \neq 0$, $H^0(g_P) = k$. This one-dimensional vector space gives rise to the rescaling of operations, like $m_n \mapsto \lambda^n m_n$ in the case of A_∞-algebras.

2.2. Homotopical actions of Lie algebras

Let g be a Lie algebra acting on a formal dg-manifold (Y,d_Y). This means that we have a homomorphism of Lie algebras $g \to Der(Y), \gamma \mapsto \hat{\gamma}$ where $Der(Y)$ is the Lie algebra of vector fields on Y preserving $\mathbf{Z}$-grading an d_Y.

We can make $Z = Y \times g[1]$ into a formal dg-manifold by introducing an odd vector field by the following formula

$$d_Z(y,\gamma) = (d_Y(y) + \hat{\gamma}, [\gamma,\gamma]/2).$$

Then $[d_Z,d_Z] = 0$. We can make $g[1]$ into a formal dg-manifold by using the odd vector field $d_{g[1]}$ arising from the Lie bracket. Then the natural projection $(Z,d_Z) \to (g[1],d_{g[1]})$ becomes a dg-bundle (cf. [13]).

This construction of a dg-bundle out of a group or Lie algebra acting on a dg-manifold motivates the following definition.

Definition 12 *Let g be a Lie algebra. Homotopical g-action on a formal dg-manifold (Y,d_Y) is a dg-bundle $\pi : (Z,d_Z) \to (g[1],d_{g[1]})$ together with an isomorphism of dg-manifolds $(\pi^{-1}(0),d_Z) \simeq (Y,d_Y)$.*

Remark 3 It was pointed out in [13] that in this case g acts on the cohomology of all complexes naturally associated with (Y,d_Y) (like the tangent space at a zero point of d_Y, the space of formal functions on Y, etc.).

We remark also that if g is a DGLA then the same definition can be given. It can be also generalized to the case when the base of a equivariant dg-bundle is an arbitrary dg-manifold with a marked d-stable point.

Suppose that F is a polynomial functor in the category of $\mathbf{Z}$-graded vector spaces, $P = Free(F), V \in Vect_{\mathbf{Z}}$. We apply the general scheme outlined above to the case $Y = \mathscr{M}(P,V)$, $g = g_P$. Obviously g acts on the dg-manifold $\underline{Hom}(F(V),V)$ equipped with the trivial odd vector field.

Let us consider the graded vector space

$$\mathscr{N} = \underline{Hom}(V,V)[1] \oplus \underline{Hom}(F(V),V) \oplus g_P[1].$$

Let $d_V \in \underline{Hom}(V,V)[1]$, it makes V into a complex, $\gamma = d_P \in g_P[1]$ satisfies the equation $[d_P,d_P] = 0$ and $\rho \in \underline{Hom}(F(V),V)$ makes V into a dg-algebra over (P,d_P).

We consider the formal neighborhood of the point (d_V,ρ,d_P) in $\mathscr{N}$, and define an odd vector field by the formula

$$d_{\mathscr{N}}(d_V,\rho,d_P) = (d_V^2, \xi(d_V,\rho) + \hat{d}_P, [d_P,d_P]/2).$$

The notation here is compatible with the one for $\mathscr{M}$.

One can check that $[d_{\mathscr{N}},d_{\mathscr{N}}] = 0$. Thus the formal neighborhood becomes a formal dg-manifold. It controls deformations of pairs *(an operad, an algebra over this operad)*.

The natural projection $\pi : \mathscr{N} \to g_P[1]$ is a morphism of formal dg-manifolds. Here on $g_P[1]$ we use the odd vector field $d_{g_P[1]}$ defined by the Lie bracket. Then the formal scheme of zeros of $d_{g_P[1]}$ corresponds to the structures of a dg-operad on P. The fiber over a fixed point $x \in g_P[1]$ is a dg-manifold with the differential induced from $\mathscr{N}$. Then the formal neighborhood of a fixed point in $\pi^{-1}(x)$ controls deformations of $\widehat{P}$-algebras.

We conclude that the Lie algebra of derivations of an operad acts homotopically on the moduli space of algebras over this operad.

3. Hochschild complex and operads

This section serves as a sort of a "second introduction", outlining objectives and the strategy of the rest of the paper.

One of our aims will be to construct a dg-operad of the type $\widehat{P}$ (i.e. free as a graded operad) acting naturally on the Hochschild complex of an arbitrary A_∞-algebra.

In Section 5 we are going to construct an operad M which acts naturally on the full Hochschild complex $C^\bullet(A,A)$ of an A_∞-algebra A as well as on $C_+^\bullet(A,A)$.

There is a natural free resolution P of the operad M, so that $C := C^\bullet(A,A)$ becomes a $\widehat{P}$-algebra. Then we can say that there is a dg-map of the moduli space of A_∞-categories to the moduli space $\mathscr{M}(P,C)$ of structures of $\widehat{P}$-algebras on the graded vector space C.

From the point of view of deformation theory it is not very natural to make constructions of the type *algebraic structure* $\to$ *another algebraic structure* (like our construction $A_\infty - algebras \to M - algebras$). It is more natural to extend them to morphisms between the formal pointed dg-manifolds controlling the deformation theories of structures.

We will construct an explicit dg-map $\mathscr{M}(\mathscr{A}_\infty,A) \to \mathscr{M}(P,C)$ as well as a dg-map $\mathscr{M}_{cat}(\mathscr{A}_\infty,A) \to \mathscr{M}(P,C)$, such that the one is obtained from another by the restriction from the moduli space of algebras to the moduli space of categories.

The operad $\mathscr{A}_\infty$ is augmented, i.e., equipped with a morphism of dg-operads $\eta : \mathscr{A}_\infty \to Free(0)$. Here $Free(0)$ is the trivial operad:

$$Free(0)_1 = k \quad \text{and} \quad Free(0)_{(n\neq 1)} = 0.$$

Since A (as any graded vector space) is an algebra over $Free(0)$, it becomes also an algebra over $\mathscr{A}_\infty$. Any structure of an $\mathscr{A}_\infty$-algebra on A can be considered as a deformation of this trivial structure. Notice also that in the previous notation the augmentation morphism defines a point in the dg-manifold $\mathscr{M}(\mathscr{A}_\infty,C)$, where $C = C^\bullet(A,A)$. Therefore it is sufficient to work in the formal neighborhood of this point.

Notice that we can also consider the moduli space of structures of a complex on the graded vector space $C^\bullet(A,A)$ where A is an arbitrary graded vector space. It gives rise to a formal dg-manifold. There are natural morphisms to it from the formal dg-manifold of the moduli space of A_∞-categories and from the formal dg-manifold of the moduli space of structures of $\widehat{P}$-algebras on $C^\bullet(A,A)$. Theorem 1 below combines all three morphisms discussed above into a commutative diagram. Let us make it more precise.

First we formulate a simple general lemma, which will be applied in the case $V = C[2]$.

Let V be an arbitrary graded vector space, d_V an odd vector field on V (considered as a graded manifold) such that $[d_V,d_V] = 0$. Thus we get a dg-manifold. The graded vector space $H = H(V) = \underline{Hom}(V,V)[1]$ is a dg-manifold with $d_H(\gamma) = \gamma^2$. To every point $v \in V$ we assign a point in H by taking the first Taylor coefficient $d_V^{(1)}(v)$ of d_V at v. In this way we obtain a map $\nu : V \to H$.

Lemma 2 *The map ν is a morphism of dg-manifolds.*

Proof. Let us write in local coordinates $x = (x_1,\ldots,x_n)$ the vector field $d_V = \sum_i \phi_i \partial_i$ where ∂_i denotes the partial derivative with respect to x_i, and ϕ_i are functions on V. Then the map ν assigns to a point x the matrix $M = (M_{ij}(x))$ with $M_{ij} = \partial_j \phi_i$. Then direct computation shows that the condition $[d_V,d_V] = 0$ implies that the vector field $\dot{x} = d_V(x)$ is mapped to the vector field $\dot{M} = d_H(M) = M^2$. ■

Let A be a graded vector space endowed with the trivial A_∞-structure, and $C = C^\bullet(A,A) = \prod_{n\geq 0} Hom_{Vect_{\mathbf{Z}}}(A^{\otimes n}, A)$ be the graded space of Hochschild cochains. Since $C[1]$ carries a structure of a graded Lie algebra when endowed with the Gerstenhaber bracket, it gives rise to the structure of a dg-manifold on $C[2]$, which is the same as $\mathcal{M}_{cat}(\mathcal{A}_\infty, A)$. We will denote it by (X, d_X) (or simply by X for short).

Even for the trivial A_∞-algebra structure on A, we get a non-trivial P-algebra structure on C. The corresponding moduli space $\mathcal{M}(P,C)$ is denoted by (Y, d_Y) (or Y for short).

There is a natural morphism of dg-manifolds $p: Y \to \underline{Hom}(C,C)[1] = H$ (projection of $Y = \mathcal{M}(P,C)$ to the first summand).

The following theorem will be proved later in the paper.

Theorem 1 *There exists a $GL(A)$-equivariant morphism of dg-manifolds $f: X \to Y$ such that $pf = \nu$.*

Moreover we will present an explicit construction of the morphism.

Suppose that A is an A_∞-algebra. Geometrically the structure of an A_∞-algebra on the graded vector space A gives rise to a point $\gamma \in X = C[2], C = C^\bullet(A,A)$ such that $d_X(\gamma) = 0$. Indeed, the definition can be written as $[\gamma,\gamma] = 0$. Thus we get a differential in C (commutator with γ) making it into a complex. The structure of a complex on the graded vector space C gives rise to a zero of the field d_H in the dg-manifold $H = \underline{Hom}(C,C)[1]$. Theorem 1 implies that $f(\gamma)$ is a zero of the vector field $d_{\mathcal{M}(P,C)}$. Therefore the Hochschild complex $(C, [\gamma,\cdot])$ carries a structure of a dg-algebra over P.

Our next aim is to uncover the geometric origin of the operad P. It will be related to the configuration space of discs inside of the unit discs in the plane. More precisely, the operad $Chains(E_2)$ of chains on the little discs operad (see [14] and Section 7 below) is quasi-isomorphic to $\widehat{P}$. Here we use either usual singular chains or piecewise algebraic chains (see the Appendix). In fact we are going to construct explicitly a morphism $\widehat{P} \to Chains(E_2)$ which gives the homotopy equivalence (to be more precise we will do that for the operad $Chains(FM_2)$ which is quasi-isomorphic to E_2). Then using the fact that both dg-operads are free as graded operads, we invert this quasi-isomorphism. This gives a structure of an $Chains(E_2)$-algebra on the Hochschild complex of an A_∞-algebra. This result is known as the Deligne conjecture.

Let us recall that there is a notion of d-algebra, $d \in \mathbf{Z}_+$ (see e.g. [14]). Namely, a graded vector space V is called a d-algebra if it is an algebra over the operad $Chains(E_d)$ (chains of the topological operad of little d-dimensional discs). Thus the moduli space $\mathcal{M}(P,C)$ can be thought as a moduli space of structures of a 2-algebra on a graded vector space C. Then our Theorem 1 says that there is a $GL(A)$-equivariant morphism of the moduli space of A_∞-categories with one object to the moduli space of 2-algebras.

Remark 4 In the unpublished paper [7] the name d-algebras was reserved for algebras over the operad $H_\bullet(Chains(E_d))$. It was proved by Tamarkin in [21] and by the first author in [14] that there exists a (non-canonical) quasi-isomorphism between the operad $Chains(E_d)$ and its homology operad $H_\bullet(Chains(E_d))$ (in other words the operad $Chains(E_d)$ is formal).

Let $g_P = \underline{Der}P$ denote as before the DGLA of derivations of $\widehat{P}$. Then g_P acts on the moduli space of $\widehat{P}$-algebras. On the other hand, from the point of view of deformation theory, we can replace the operad of little discs E_2 by the operad of configurations of points in the plane (properly compactified). This is the operad FM_2 mentioned above. There is a natural action of the Grothendieck-Teichmüller groups on the rational homotopy type of the latter. It gives rise to a morphism of L_∞-algebras $Lie(GT)[1] \to (g_P, [d_P, \cdot])$ where GT is the Grothendieck-Teichmüller group.

Therefore one has a homotopical action of the Lie algebra $Lie(GT)$ on the moduli space of $\widehat{P}$-algebras.

4. Free resolution of a dg-operad

In this section we recall well-known constructions of free resolutions of operads (see e.g. [7]).

Let k be a field as before, R be a dg-operad over k. The aim of this subsection is to construct canonically a dg-operad P_R over k, which is free as a graded operad, and a quasi-isomorphism $P_R \to R$. Then P_R will be a free resolution of R. In this subsection we will assume that R is non-trivial, which means that the unit operation from R_1 is not equal to zero.

4.1. Topological construction

We will mainly follow [3].

Let $O = (O_n)_{n\geq 0}$ be a topological operad (i.e., all O_n are S_n-topological spaces and all operadic morphisms are continuous). We describe (following [3]) a construction of a topological operad $B(O) = (B(O)_n)_{n\geq 0}$ together with a morphism of topological operads $B(O) \to O$ which is homotopy equivalence.

To simplify the exposition we assume that S_n acts freely on O_n for all n.

Each space $B(O)_n$ will be the quotient of

$$\overline{B(O)_n} = \bigsqcup_{[T], T\in Tree(n)} ([0,+\infty]^{E_i(T)} \times \prod_{v\in V_i(T)} O_{N^{-1}(v)})/AutT$$

under the relations described in the following way.

Let us consider the elements of $\overline{B(O)_n}$ as numbered trees with elements of O attached to the internal vertices, and the *length* $l(e) \in [0,+\infty]$ attached to every edge e. We require that all external edges (i.e., root edge and the tail edges) have lengths $+\infty$.

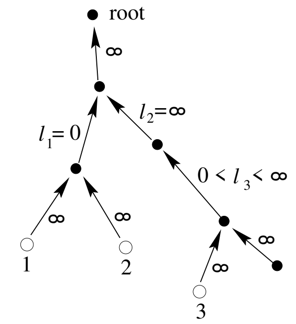

Figure 2. Tree with lengths.

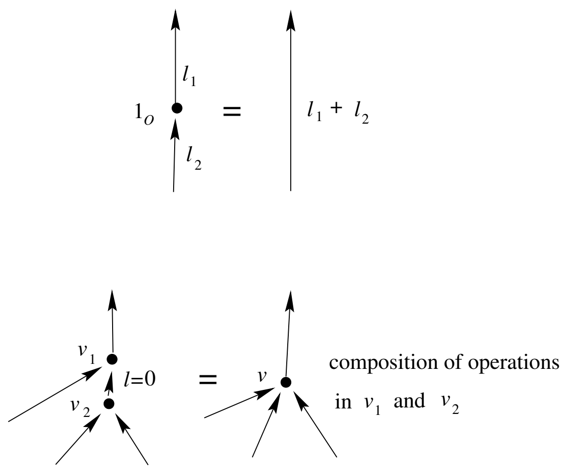

Figure 3. Two basic identifications between trees with lengths.

We impose two type of relations:

1) We can delete every vertex v of valency 1 if it contains the unit of the operad, replacing it and the attached two edges of lengths $l_i, i = 1,2$ by the edge with the length $l_1 + l_2$. We use here the usual assumption: $l + \infty = \infty + l = \infty$.

2) We can contract every internal edge $e = (v_1, v_2), v_2 = N(v_1)$ of the length 0 and compose in O the operations attached to $v_i, i = 1,2$.

We depict the trees and relations in Figures 2 and 3.

Let us describe how $B(O)$ can act naturally on a topological space.

Let X be a topological space, and $g^t : X \to X, t \in [0, +\infty)$ a 1-parameter semigroup of continuous maps acting on X.

We assume that the map $[0, +\infty) \times X \to X, (t, x) \mapsto g^t x$ extends continuously to $[0, +\infty] \times X$. We denote by Y the image of $g^\infty := lim_{t \to \infty} g^t$. Clearly the subspace Y is a homotopy retract of X.

Suppose that a topological operad O acts on X, i.e., that we are given continuous maps $O_n \times X^n \to X, n \geq 0$, satisfying the usual properties. We can construct an action of $B(O)$ on Y as follows. Let $\gamma \in O_n$, $t, t_i \in \mathbf{R}_+ \cup \{+\infty\}$, $x_i \in X$, $1 \leq i \leq n$. Then we assign to these data the point $x = g^t \gamma(g^{t_1}x_1, ..., g^{t_n}x_n)$ of X. We define the composition of such operations in the natural way.

We can interpret the parameters t, t_i above as lengths of edges of trees. Putting $t = +\infty$ we obtain an action of $B(O)$ on the homotopy retract Y.

4.2. Free resolutions of linear operads

Let R be a dg-operad over a field k. To describe its free resolution P_R we need a special class of trees described below.

For every $n \geq 0$ we introduce a groupoid $\mathscr{T}(n)$ of *marked trees* with n tails. An object of $\mathscr{T}(n)$ is a numbered tree $T \in Tree(n)$ and a map to a 3-element set $l = l_T : E(T) \to \{0, finite, +\infty\}$ such that $l_T(\{e_r\} \cup E_t(T)) = \{+\infty\}$. Notice that in the case of topological operads the component $\overline{B(O)_n}$ is stratified naturally with the strata labeled by equivalence classes $|\mathscr{T}(n)|$ of marked trees. The label of an edge e of the corresponding marked tree is 0 if $l(e) = 0$, is *finite* if $l(e) \in (0, +\infty)$ and is $+\infty$ if $l(e) = +\infty$. According to this description, we call them *zero, finite, infinite* edges respectively. We denote these sets of edges by E_{zero}, E_{finite} and $E_{infinite}$ correspondingly.

We will give three different but equivalent descriptions of the operad $P = P_R$ as a graded operad. Then we define a differential on P.

Description 1.

Let $\bar{P}_n = \bigoplus_{[T], T \in \mathscr{T}(n)} (\otimes_{v \in V(T)} R_{N^{-1}(v)}[J_T])_{AutT}$ where $AutT$ is the group of automorphisms of the tree T, $J_T = l_T^{-1}(finite)$, and for any graded vector space W and a finite set J we use the notation $W[J] = W \otimes k[1]^{\otimes J}$ (shift of the grading by J).

Notice that the dimension of the corresponding stratum of $\mathscr{T}(n)$ is the cardinality of the set J_T, i.e., the number of finite edges.

Then $(\bar{P}_n)_{n \geq 0}$ evidently form a graded operad $\bar{P}$. It is a k-linear analog of the operad $\overline{BO}$.

The operad $\bar{P}$ contains a subspace I_P generated by the following relations:

1) If the length of an edge (w, v) is 0 we contract it and make the composition in R of the operations attached to w and v (cf. description for $B(O)$);

2) If v is a vertex of $T \in \overline{P}$ such that $|v| = 1$ and $1_R \in R_1$ is attached to v, then T belongs to I_P if the following holds: Lengths of both edges attached to v are non-zero edges, and at least one of them is finite. If both edges are infinite, we remove the vertex and two attached edges, replacing them by an infinite edge.

One can check easily that I_P is a graded ideal in $\bar{P}$. We denote by P the quotient operad $\bar{P}/I_P$.

Description 2.

We define P_n in the same way, but making the summation over all trees without edges of zero length. We also drop the relation 1) from the list of imposed relations (there are no edges with $l = 0$).

Description 3.
We define an operad R' such as follows: $R'_n = R_n$ for $n \neq 1$, R'_1 is a complement to the subspace $k \cdot 1_R$ in R_1. Then we define P_n as in Description 2, but using R' instead of R and dropping both relations 1) and 2).

It is clear that this description defines a free graded operad. Equivalently, it can be described as a free graded operad P such that

$$P = Free(Cofree'(R'[1]))[-1].$$

Here $Cofree(L)$ means a dg-cooperad generated by L which is cofree as a graded co-operad, and $'$ denotes the procedure of taking a (non-canonical) complement to the subspace generated by the unit (or counit in the case of a co-operad) as described above in the case of R.

In this description the generators of P correspond to such trees T in $\mathscr{T} = (\mathscr{T}(n))_{n\geq 0}$ that every T has at least one internal vertex, all internal edges are finite and there are no zero-edges in T.

Proposition 2 *All three descriptions give rise to isomorphic graded free operads over k.*

Proof. Straightforward. ∎

We can make $\bar{P}$ into a dg-operad introducing a differential $d_{\bar{P}}$. We use the Description 1 for this purpose.

The differential $d_{\overline{P}}$ is naturally decomposed into the sum of two differentials: $d_{\overline{P}} = \tilde{d}_R + d_{\mathscr{T}}$ where:

a) The differential $\tilde{d}_R$ arises from the differential d_R in R;

b) The differential $d_{\mathscr{T}}$ arising from the stratification of $\mathscr{T}(n)$: It either contracts a finite edge or makes it into an infinite edge.

To be more precise, let us consider the following object Δ in $\mathscr{C} = Vect_{\mathbf{Z}}$: $\Delta^{-1} = 1_{\mathscr{C}}, \Delta^0 = 1_{\mathscr{C}} \oplus 1_{\mathscr{C}}$ where $1_{\mathscr{C}}$ is the unit object in the monoidal category $\mathscr{C}$. Then Δ can be made into a chain complex of the CW complex $[0,+\infty] = \{0\} \cup (0,+\infty) \cup \{+\infty\}$.

We see that as a graded space $\bar{P}_n$ is given by the formula

$$\bar{P}_n = \oplus_{[T],T\in Tree(n)} (\otimes_{v\in V_i(T)} R_{N^{-1}(v)} \otimes \Delta^{\otimes E_i(T)})_{AutT}.$$

Since we have here a tensor product of complexes, we get the corresponding differential $d_{\bar{P}}$ in $\bar{P}$.

Proposition 3 *The ideal I_P is preserved by $d_{\bar{P}}$.*

Proof. Straightforward computation. ■

Therefore $P = P_R$ is a dg-operad which is free as a graded operad.

There is a natural morphism of dg-operads $\phi : P \to R$. In terms of the Description 2 it can be defined as follows: ϕ sends to zero all generators of P corresponding to trees with at least one finite edge. Let $T \in P$ be a tree with all infinite edges. Then T gives rise to a natural rule of composing in R elements of $R_{N^{-1}(v)}$ assigned to the vertices of T. We define $\phi(T) \in R$ as the result of this composition.

It is easy to check that ϕ is a well-defined morphism of dg-operad.

Proposition 4 *The morphism ϕ is a quasi-isomorphism of dg-operads.*

Proof. Follows from the spectral sequence arising from the natural stratification of $\mathscr{T}$. To say it differently, let us consider the tautological embedding ψ of R into P. Then ψ is a right inverse to ϕ. It gives a splitting of P into the sum $P = \psi(R) \oplus P^{(0)}$. Here $P^{(0)}$ is spanned by the operations corresponding to trees with finite edges only. Such a tree can be contracted to a point which means that $P^{(0)}$ is contractible as a complex. Hence ϕ defines a quasi-isomorphism of complexes and dg-operads. ■

4.3. Example

Let us discuss an example when R is the operad of associative algebras without the unit. We denote it by As. Then for any $n \geq 1$ we have: As_n is isomorphic to the regular representation of the symmetric group S_n.

In this case the complex P_n from the previous subsection can be identified with the chain complex of the CW-complex K_n, $n \geq 2$, described below.

The cells of K_n are parametrized by *planar trees* with an additional structure on edges. By a planar tree here we understand a numbered tree T such that for any $v \in V_i(T)$ the cardinality of $N^{-1}(v)$ is at least 2 and this set is completely ordered. The additional structure is a map $E_i(T) \to \{\textit{finite,infinite}\}$. We call an edge finite or infinite according to its image under this map. Dimension of the cell is equal to the number of finite edges of the corresponding planar tree.

We can either contract a finite edge or make it infinite. This defines an incidence relation on the set of cells.

An example of planar tree is given in Figure 4. Here the dashed lines show the complete orders on the set of incoming edges. We will not show them on other figures in the text. Instead, we will tacitly assume that for a given vertex the incoming edges are completely ordered from the left to the right.

In this way we obtain cubical subdivisions of the Stasheff polyhedra. The case $n = 4$ is represented in Figure 5.

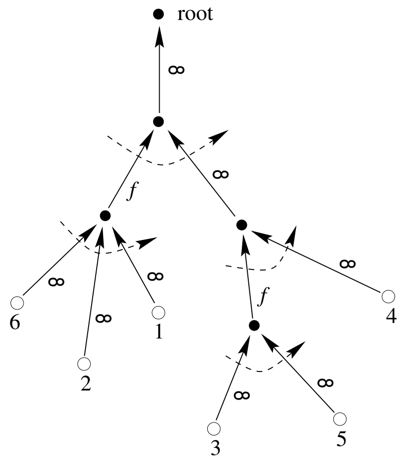

Figure 4. A planar tree which gives a 2-dimensional cell in K_6

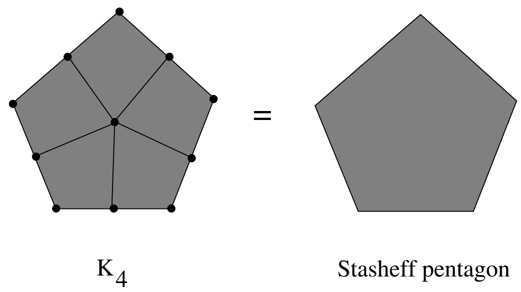

Figure 5. Cell complex K_4 has 11 vertices, 15 edges and 5 two-dimensional faces. It is homeomorphic to the Stasheff pentagon.

5. Minimal operad

In this section we describe a dg-operad $M = (M_n)_{n\geq 1}$ which acts naturally on the Hochschild complex of an A_∞-algebra. We call it *minimal* operad. Let us describe this operad informally. We treat elements $\gamma \in C^\bullet(A,A)$ as polylinear operations on A. For given operations $\gamma_1,\ldots,\gamma_n \in C^\bullet(A,A)$, and an A_∞-structure $m \in C^\bullet(A,A)$ we can make compositions in all possible ways, reading the arguments from left to right. For example we can make an expression like this:

$$\gamma(a_1 \otimes \cdots \otimes a_5) = \gamma_2(a_1 \otimes a_2 \otimes m_2(a_3 \otimes \gamma_1(a_4)) \otimes \gamma_3(a_5)), \quad \text{etc.}$$

Such compositions can be depicted by planar trees. The operad M is spanned by operations corresponding to certain trees, which we call admissible (see below).

It seems that the operad M is close to what is described in [18] as “natural transformations $(C^\bullet(A,A))^{\otimes n} \to C^\bullet(A,A)$” (this terminology is confusing because the assignment $A \mapsto C^\bullet(A,A)$ is not a functor).

In all previous works dealing with the Deligne conjecture the authors used operads which act on $C^\bullet(A,A)$ and generated by operations called braces. It seems that the braces generate the operad M, but it is not clear what is the complete list of relations. The advantage of our operad M is that it is described directly, not as a quotient of a free operad.

The defining properties of a dg-operad (i.e., associativity of the composition, Leibniz rule for the differential) will become clear later. We will give two definitions of the operad M. The first one is suitable for the pure algebraic descriptions of the operadic composition and the differential. But the signs in the formulas are not very transparent. Second description takes care about "parity of edges", so the correct signs come out automatically. In some formulas related to the first description we will write $\pm$ having in mind that the correct sign follows from the second description.

5.1. Basis of M

Definition 13 *For a finite set I we define an I-labeled planar tree as a triple (T, lab, ord) where T is a tree in the sense of Section 1.2, $lab : I \hookrightarrow V_i(T)$ is an embedding, and ord is a complete order on the sets $N^{-1}(v), v \in V_i(T)$.*

We call *labeled* a vertex from the image of the map lab. All other internal vertices are called *non-labeled*.

Definition 14 *We will call an I-labeled tree admissible if it has no tail vertices, and for every non-labeled internal vertex v we have $|v| \geq 2$.*

We denote the set of isomorphism classes of I-labeled planar trees by $Tree^{(p)}(I)$ where the upper script p stands for "planar". Notice that the automorphism group of an I-labeled planar tree is trivial. For $I = \{1, ..., n\}$ we will use the notation $Tree^{(p)}(n)$.

We are going to use admissible trees unless we say otherwise. If it will not lead to a confusion, we will simply call them trees. Slightly abusing the notation we will denote an I-labeled tree by T, skipping lab and ord. Notice that the terminology here is different from the one in Section 1.5. In particular, we label here internal vertices, not tails. Since we do not consider here numbered trees (in the terminology of Section 1.5), this change of terminology should not lead any confusion.

We can depict trees from $Tree^{(p)} = (Tree^{(p)}(n))_{n\geq 1}$ as in Figure 6.

Labeled vertices are depicted as circles with numbers inscribed, non-labeled vertices are depicted as black vertices.

We define M_I to be a k-vector space spanned by all elements of $Tree^{(p)}(I)$. For $I = \{1, ..., n\}$ we will use the notation M_n. The symmetric group S_n acts on M_n by permuting labeled vertices.

Abusing the notation further, we will denote the element of M_n corresponding to a tree T simply by T. Thus we have: $M_0 = 0$ and M_1 is a 1-dimensional vector

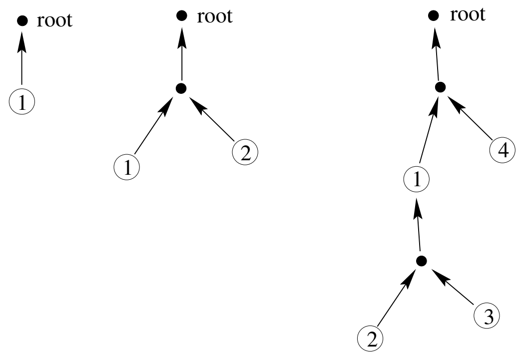

Figure 6. Three examples of admissible labeled planar trees.

space generated by T_e, the first tree on the figure above. In fact T_e corresponds to the unit $1_M \in M_1$ of the operad M.

The operadic composition in M and the differential will be described below.

The degree of the basis element corresponding to a tree T is equal to

$$deg(T) = \sum_{v \in V_{lab}(T)} (-|v|) + \sum_{v \in V_{nonl}(T)} (2 - |v|),$$

where $V_{lab}(T)$ and $V_{nonl}(T)$ denote the sets of labeled and non-labeled vertices respectively , and $|v|$ is the cardinality of the set $N^{-1}(v)$.

5.2. Composition in *M*

We need to define an element of M which corresponds to a tree T_2 glued to a tree T_1 at a labeled vertex $v \in V_{lab}(T_1)$. The trees T_i, $i = 1,2$ correspond to some elements of M. The resulting element will be by definition their operadic composition. It is given by the sum

$$T_1 \circ_v T_2 = \sum_\beta \pm (T_1 \circ_v T_2)_\beta,$$

where the trees $(T_1 \circ_v T_2)_\beta$ are defined below.

First, with the tree T_2 we associate a set $A(T_2) = \bigsqcup_{v \in V_i(T_2)} \{0, ..., |v|\}$. We call it the set of *angles* of T_2. Obviously there is a natural map $\kappa : A(T_2) \to V_i(T_2)$. The path in $\mathbf{R}^2$ which goes from the left to the right and surrounds T_2 defines a complete order on $A(T_2)$. In Figure 7, angles are marked by asterisks.

The datum β above is a (non-strictly) monotonic map $\beta : N^{-1}(v) \to A(T_2)$. We will think of this map as about the way to insert a vertex $w \in N^{-1}(v)$ and the edge (w, v) inside of an angle formed by two edges incoming to $\beta(v)$. Schematically it is shown in Figure 8.

Let $T_\beta = (T_1 \circ_v T_2)_\beta$. Then we define the set of vertices $V(T_\beta)$ as $(V(T_1) \setminus \{v\}) \bigsqcup (V(T_2) \setminus \{root\})$. The map $N = N_{T_\beta}$ is defined as follows: For all $w \in V(T_1) \setminus$

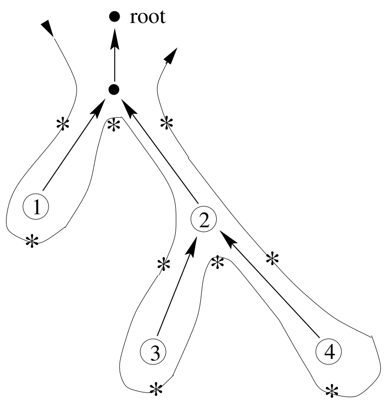

Figure 7. A path from the left to the right passing through all angles.

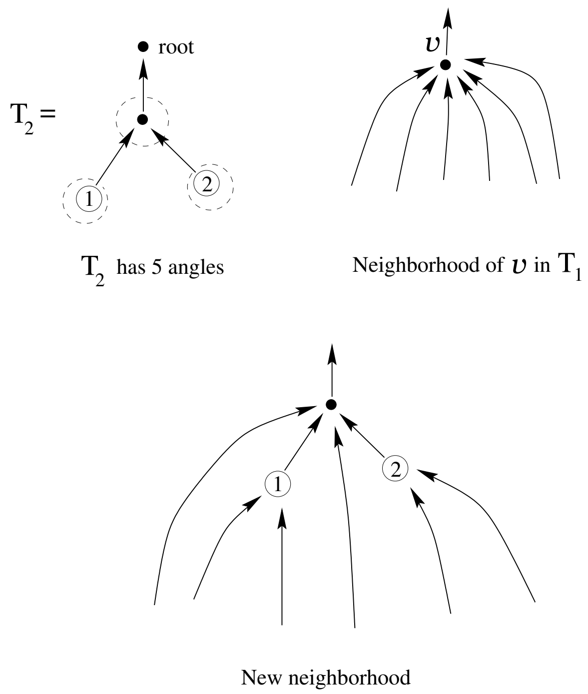

Figure 8. A way to insert tree T_2 in the place of vertex v in tree T_1.

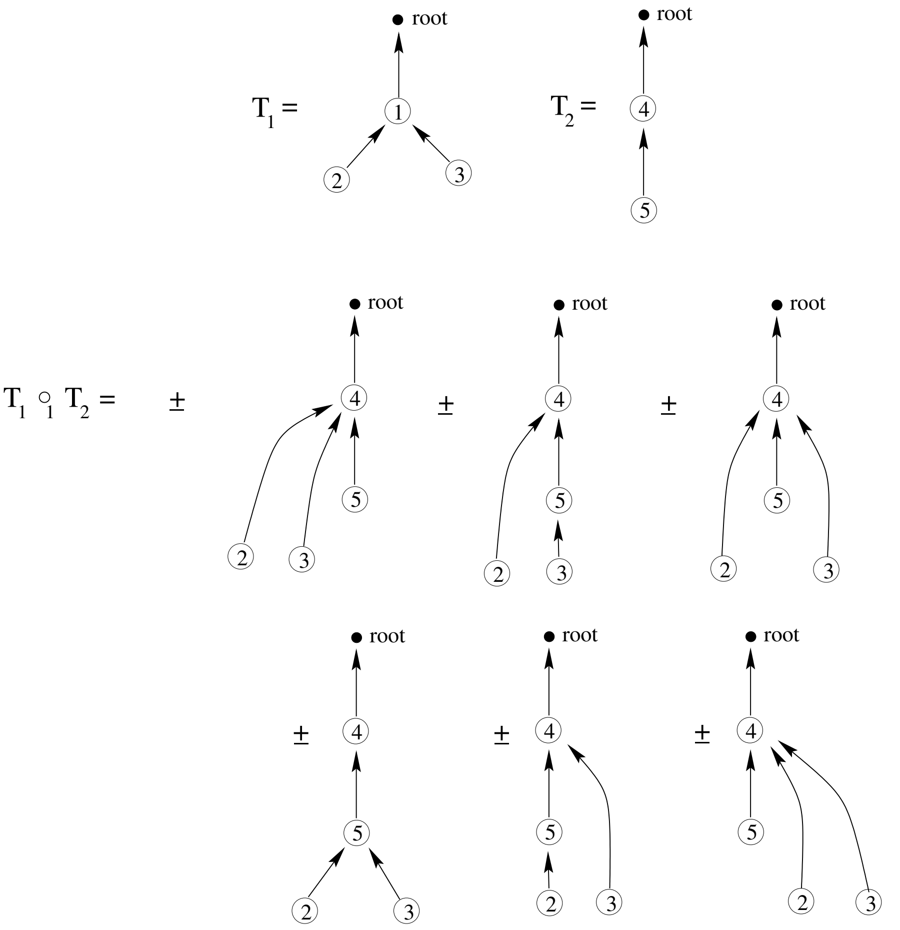

Figure 9. An example of the composition in M.

$\{v\}$ such that $N_{T_1}(w) \neq v$ we put $N(w) = N_{T_1}(w)$ in a self-explained notation. Similarly if $w \in V(T_2), N_{T_2}(w) = root$ we put $N(w) = N_{T_1}(v)$. If $w \in V(T_2), N_{T_2}(w) \neq root_{T_2}$ then we put $N(w) = N_{T_2}(w)$. Let us suppose that $w \in N_{T_1}^{-1}(v)$. Then we define $N(w)$ as $\kappa(\beta(w))$. The root vertex of T_β is the same as for T_1. The labeling and complete orders on the sets $N^{-1}(x)$ are defined in the natural way. Informally speaking, T_β is obtained by removing from T_1 the vertex v together with all incoming edges and vertices, and gluing T_2 to v. Then we use the map β in order to "insert" removed vertices. With such a composition we obtain the structure of a graded operad on M.

We depict an example of the composition in M in Figure 9.

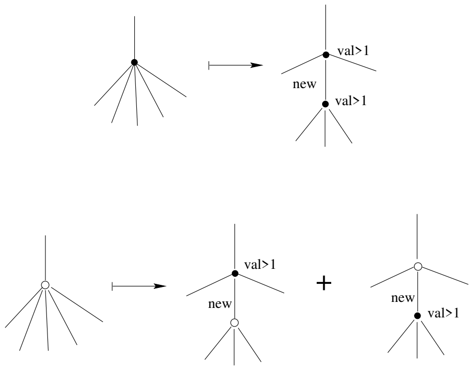

Figure 10. Two types of terms in the formula for differential in M.

5.3. Differential in *M*

For a generator T we define $d_M(T) = \sum_{v \in V_i(T)} d_v(T)$ where each $d_v(T)$ will be of the form

$$d_v(T) = \sum_{i,j} \pm d_{v,i,j}(T).$$

We need to explain the set of indices of summation and each summand. Let us recall that for every vertex $v \in V_i(T), |v| = k$ there is a bijection of sets $\{1,...,k\} \to N^{-1}(v)$ which defines a complete order on the set $N^{-1}(v)$. We will identify an element of $N^{-1}(v)$ with the corresponding number. The indices i, j in the sum above will be half-integers: $1/2 \le i \le j \le k+1/2$. For a pair i, j, we define $d_{v,i,j}(T)$ to be a tree T' such that:

a) $V(T') = (V(T) \setminus \{v\}) \cup \{v^{up}, v^{down}\}$ where $\{v^{up}, v^{down}\}$ are new vertices;

b) $root_{T'} = root_T$;

We put

$$N_{T'}(v^{down}) = v^{up};$$
$$N_{T'}(l) = v^{down} \quad \text{if } i < l < j;$$
$$N_{T'}(l) = v^{up}, \quad \text{if } l < i \text{ or } l > j.$$

For all other vertices u we put $N_{T'}(u) = N_T(u)$.

Complete orders on the sets $N_{T'}^{-1}(u)$ are defined in the natural way. The tree T' has one new edge (v^{down}, v^{up}) which we denote in pictures as "new".

We can depict these definitions as in Figure 10.

The range of summation is defined differently in the case of labeled and non-labeled vertices v. The idea is to keep admissible graphs only.

1) If v is non-labeled then we take sum over $1/2 \le i \le j \le k+1/2$ such that $2 \le j-i$ and $1 \le k-(j-i)$, in $d_{v,i,j}(T)$ both vertices v^{up} and v^{down} are non-labeled.

2) If v is labeled, then $d_v(T) = d_v^{(1)}(T) + d_v^{(2)}(T)$ where the first summand $d_v^{(1)}(T)$ is the sum of $d_{v,i,j}(T)$ with $1 \leq k-(j-i)$ and with the vertex v^{down} appearing with the old label of v, and v^{up} being non-labeled. The second summand $d_v^{(2)}(T)$ is the sum of $d_{v,i,j}(T)$ with $2 \leq j-i$, the vertex v^{down} is non-labeled, and v^{up} has the same label as v in T. Labeling of all vertices different from v^{up} and v^{down} remains the same.

5.4. Action of M on the Hochschild complex

Let (A,m) be an A_∞-algebra. Our next step is to define an action of a dg-operad M on $C^\bullet(A,A)$ so the latter becomes a dg-algebra over M.

Let $T \in M_n, \gamma_i \in C^\bullet(A,A), 1 \leq i \leq n$. We need to define an element $T(\gamma_1,\ldots,\gamma_n) \in C^\bullet(A,A)$. It can be expressed as a sequence of morphisms of graded vector spaces $T(\gamma_1,\ldots,\gamma_n)_N : A^{\otimes k} \to A$, $k \geq 0$.

Let $T_{(k)}$ be a unique (non-admissible) planar tree with only one internal vertex v, which is labeled, and k tails, numbered from the left to the right.

Then we have the composition

$$T_{(k)} \circ_v T = \sum_\beta \pm (T_{(k)} \circ_v T)_\beta$$

defined in the same way as for admissible trees. Here as before $\beta : \{1,\ldots,k\} \to A(T_{(k)} \circ_v T)$ is a monotonic map.

Every tree $(T_{(k)} \circ_v T)_\beta$ has k tails. For every $v \in V_i((T_{(k)} \circ_v T)_\beta)$ we define a polylinear map $\gamma_v : A^{\otimes |v|} \to A$ in the following way:

1) If v is labeled by j, $1 \leq j \leq n$, we define γ_v as the component of γ_j which belongs to $Hom_{Vect_{\mathbf{Z}}}(A^{\otimes |v|},A)$;

2) If v is non-labeled we define $\gamma_v = m_{|v|}$.

The tree $(T_{(k)} \circ_v T)_\beta$ defines the way to compose operations γ_v into an operation $\gamma_\beta : A^{\otimes k} \to A$. We define $T(\gamma_1,\ldots,\gamma_n)$ to be equal to the sum $\sum_\beta \pm \gamma_\beta$.

We claim that in this way we get on $C^\bullet(A,A)$ a structure of a dg-algebra over M.

This can be checked by a straightforward computation. It is more or less clear from the definitions that $C^\bullet(A,A)$ is an algebra over the graded operad M. Hence the question is about the compatibility with differentials. The latter follows from a more general result (the Theorem 1), and will be proved in Section 6.

5.5. Signs in the minimal operad

Now we would like to discuss the second description of the operad M. It is based on the following result from the theory of Strebel's differentials (see [20]).

Theorem 2 *Let $I = \{z_1,\ldots,z_n\}$ be a finite non-empty subset of the complex line $\mathbf{C}$. Then there exists a unique quadratic differential $\alpha = f(z)(dz)^2$ with $f(z) = \sum_i \frac{a_i}{z-z_i}$, $a_i \in \mathbf{C}, i = 1,\ldots,n$, $\sum_i a_i = 1$, satisfying the following property:*

There exists a unique tree $T \in Tree^{(p)}(I)$ and an embedding $j : T \to \mathbf{CP}^1$, $\{root\} \mapsto \{+i\infty\}$ such that the pair $(\mathbf{CP}^1 \setminus j(T), \alpha)$ is equivalent (as a complex curve with quadratic differential) to the lower half-plane $Im(z) < 0$ equipped with the quadratic differential $i(dz)^2$.

We will call such $\alpha = f(z)(dz)^2$ the Strebel differential associated with the finite set $I \subset \mathbf{C}$.

To every Strebel differential α we can assign canonically the $\{1,...,n\}$-labeled planar tree T. Conversely, having a tree $T \in Tree^{(p)}(n)$ we can look for sequences of pairwise distinct points $(z_1,...,z_n)$ in $\mathbf{C}$ which can appear as the sets of possible poles of the Strebel differential from the Theorem. We remark that the set of all possible $(z_1,...,z_n) \in \mathbf{C}^n \setminus \{diag\}$ form an open cell Str_T. This follows from the fact that this space is a vector bundle of rank 2 over a cell. The latter cell is defined by the lengths of internal edges (lengths are taken with respect to the metric $|\alpha|$ which is well-defined on $\mathbf{C} \setminus I$). Then the embedding of T is fixed up to parallel translations. The latter span the fiber of a vector bundle. The total space of this bundle is the cell $Str_T \subset \mathbf{C}^n \setminus \{diag\}$. It is easy to see that $codim(Str_T) = -deg(T) = -(|E_i(T)| + 2 - 2|V_{lab}(T)|)$.

One can show that there is a finite CW-complex $\Sigma_{(n)} \subset \mathbf{C}^n \setminus \{diag\}$ with the cells Σ_T labeled by $T \in Tree^{(p)}(n)$, and such that Σ_T intersects Str_T transversally and at exactly one point.

Now we can give the second definition of the operad M:

For $n \geq 2$ the complex M_n is defined as the chain complex of $\Sigma_{(n)}$.

Using our agreement about degrees of homological complexes, one can check again that every complex M_n is a finite-dimensional complex concentrated in degrees $\{-(n-1),...,0\}$.

We use the second definition in order to derive the following combinatorial description of the operad M.

One associates to a tree $T \in Tree^{(p)}(n)$ a one-dimensional vector space (in fact an abelian group, so everything can be done over integers). It is defined by the formula $U_T = ((\widetilde{H}_\bullet(\mathbf{R}))^*)^{E_i(T)} \otimes ((\widetilde{H}_\bullet(\mathbf{R}^2)^*) \otimes ((\widetilde{H}_\bullet(\mathbf{R}^2))^{V_{lab}(T)}$.

In the formula $\widetilde{H}_\bullet(X)$ denotes the reduced homology of the one-point compactification of X (Borel-Moore homology).

The graded vector space M_n coincides with the sum of U_T over all trees belonging to $Tree^{(p)}(n)$.

Next step is to interpret various shifts in complexes as tensor products with the reduced homology of vector spaces. For example the shift by 1 is the tensor product with the reduced homology of $\mathbf{R}$.

It is more convenient to use a different notation for the same spaces of reduced homology.

Let $L_1 := \widetilde{H}_\bullet(\mathbf{R})$ and $L_2 := \widetilde{H}_\bullet(\mathbf{R}^2)$. These are 1-dimensional graded vector spaces of pure degrees -1 and -2 correspondingly. For a finite set I we define a vector space

$$M_I := \oplus_{T\in Tree^{(p)}(I)}(L_1^*)^{E_i(T)} \otimes L_2^* \otimes L_2^{\otimes I} = \oplus_{T\in Tree^{(p)}(I)} U_T.$$

Geometrically, the new notation is related to the picture with the Strebel differentials. We think about $\mathbf{R}^2$ as about direct sum of two lines $\mathbf{R}^2 = \mathbf{R}_{hor} \oplus \mathbf{R}_{vert}$ (horizontal and vertical lines). The vertical line is the y-axis in $\mathbf{C} = \mathbf{R}^2$. The vertical line corresponds to L_1 in the notation above, and the horizontal line corresponds to $L_1^* \otimes L_2$ (we think of it as $\mathbf{R}^2/\mathbf{R}_{vert}$).

There is a natural structure of a complex on M_I. We define a linear map $d_I : M_I \to M_I \otimes L_1$ as a sum $d_I = \sum_{T_1\to T} d_{T_1,T}$.

Here $T_1 \to T$ means that T_1 is obtained from T by adding an internal edge. The summand $d_{T_1,T}$ is a linear map from U_T to $U_{T_1} \otimes L_1 = (U_T \otimes L_1^*) \otimes L_1$. It is given by $id_{U_T} \otimes \varepsilon_{L_1}$ where $\varepsilon_{L_1} : \mathbf{1} \to L_1^* \otimes L_1$ is the canonical morphism in the symmetric monoidal category $Vect_{\mathbf{Z}}$.

Lemma 3 *The linear map d_I defines a differential in M_I.*

Proof. We need to prove that $(d_I \otimes id_{L_1}) \circ d_I = 0$. This can be checked directly using the fact that L_1 has degree -1 , so the commutativity constraint acts on $L_1 \otimes L_1$ as $-\sigma$ where σ is the permutation map. ■

The composition maps for the operad M can be naturally described in the new notation. We remark that in order to glue a tree T_2 to a tree T_1 at a vertex v, first we need to define a sequence of trees $\{(T_1 \circ_v T_2)_\beta\}$ (they were also denoted by T_β in the previous description of the gluing). For every $(T_1 \circ_v T_2)_\beta$ the set of internal edges is the disjoint union of the corresponding sets for T_1 and T_2. The set of labeled vertices for $(T_1 \circ_v T_2)_\beta$ is $(V_{lab}(T_1) \cup V_{lab}(T_2)) \setminus \{v\}$. Then we need to define a morphism of vector spaces $U_{T_1} \otimes U_{T_2} \to U_{(T_1\circ_v T_2)_\beta}$.

Equivalently we need to define a morphism of the graded vector space

$$(L_1^*)^{\otimes E_i(T_1)} \otimes L_2^* \otimes L_2^{\otimes I_1} \otimes (L_1^*)^{\otimes E_i(T_2)} \otimes L_2^* \otimes L_2^{\otimes I_2}$$

to the graded vector space

$$(L_1^*)^{\otimes (E_i(T_1)\sqcup E_i(T_2))} \otimes L_2^* \otimes L_2^{\otimes ((I_1\setminus\{v\})\sqcup I_2)}.$$

(The notation is self-explained).

We define the morphism to be the identity on the tensor factors marked by same edges or same elements of the sets $I_k, k = 1,2$. For example $(L_1^*)^{\otimes E_i(T_1)} \otimes (L_1^*)^{\otimes E_i(T_2)}$ is identically mapped to $(L_1^*)^{\otimes (E_i(T_1)\sqcup E_i(T_2))}$.

Hence it is enough to define a morphism $L_2^* \otimes L_2^* \otimes L_2 \to L_2^*$, where first and third tensor factors in the LHS correspond to the tree T_1, middle tensor factor corresponds to the tree T_2 and the space L_2^* in the RHS corresponds to the tree

$(T_1 \circ_v T_2)_\beta$. Then one uses the canonical evaluation map $ev: L_2^* \otimes L_2 \to \mathbf{1}$ in the symmetric monoidal category $Vect_{\mathbf{Z}}$.

5.6. Signs in the Hochschild complex

In this subsection we are going to reformulate the definition of the Hochschild complex using the reduced homology spaces L_1 and L_2. Then one gets the sign agreement with the computations in the previous subsection.

For a graded vector space A we define

$$C = C^\bullet(A,A) = \oplus_I \underline{Hom}(A^{\otimes I}, A) \otimes (L_2^* \otimes L_1)^{\otimes I},$$

where the sum is taken over all non-empty completely ordered finite sets, and $\underline{Hom}$ is the internal Hom in the tensor category $Vect_{\mathbf{Z}}$.

Let us demonstrate how the structures in C can be reformulated by means of this language.

The Gerstenhaber bracket is a map $C \otimes C \to C \otimes (L_2^* \otimes L_1)$. Now the multiplication m can be described as a point of a dg-manifold: $m \in C \otimes L_2$. The bracket $[m,m]$ defines a point of the dg-manifold $C \otimes L_2^* \otimes L_1 \otimes L_2 \otimes L_2 = (C \otimes L_2) \otimes L_1$. If $[m,m] = 0$ then we have a differential $d_m = [m, \cdot] : C \to C \otimes L_1$.

The corresponding structure of DGLA on C can be reformulated as follows: Consider the class $\mathscr{F}$ of *simple forests*. A simple forest F is a finite collection of planar trees $F = \{T_\alpha\}_{\alpha \in \Omega}$ with no internal edges. For every $F \in \mathscr{F}$ we define the graded vector space

$$W_F := \otimes_{\alpha \in \Omega} Hom(A^{\otimes V_t(T_\alpha)}, A) \otimes (L_2^* \otimes L_1)^{\otimes V_t(T_\alpha)} \otimes L_2.$$

We have the natural groupoid structure on $\mathscr{F}$. The correspondence $F \mapsto W_F$ defines a functor from this groupoid to the category $Vect_{\mathbf{Z}}$. The free cocommutative coalgebra cogenerated by $C \otimes L_2$ can be described as a colimit of this functor: $Coalg(C \otimes L_2) = colim_{F \in \mathscr{F}} W_F = \oplus_{F/iso} (W_F)_{Aut F}$.

The coalgebra structure on $Coalg(C \otimes L_2)$ can be described in these terms. Namely $\Delta \circ pr = \sum_{F_1 \subset F} (pr \otimes pr) \circ i(F_1, F_2)$.

Let us explain the notation.

Here Δ is the coproduct on the free cocommutative coalgebra cogenerated by $C \otimes L_2$, pr is the projection to the coinvariants. The sum is taken over all subforests F_1 (unions of some connected components of F), and we fix the splitting $i(F_1, F_2)$: $W_F \simeq W_{F_1} \otimes W_{F_2}$, where F_2 is the complementary forest. We leave to the reader straightforward reformulations of other structures on the Hochschild complex and checking the signs.

Remark 5 For an A_∞-algebra A one can define the opposite algebra A^{op}. In the geometric language of this section it corresponds to the antipodal involution on $\mathbf{R}_{hor}$.

6. Morphism of dg-manifolds

6.1. Generators of the free operad

The main purpose of this section is to prove the Theorem 1. It will be done in subsection 6.2. This subsection is devoted to some technical preparatory material.

Applying the general theory of Section 4 to the case of the operad M we obtain its free resolution P. It is a dg-operad which is free as a graded operad. Its graded components P_n can be described explicitly in terms of the operad M. We know that $P_0 = 0$ and $P_1 = k\cdot 1_P$. One can easily describe the space $G_n = (Free'(M'[1])[-1])_n$ of generators of $P_n, n \geq 2$.

Namely, G_n is a direct sum of 1-dimensional graded vector spaces $W_{\mathbf{T}}$. The sum is taken over the set of equivalence classes of collections $\mathbf{T} = (T, \{T_v\}_{v\in V_i(T)})$ where:

a) $T \in Tree(n), n \geq 2$ such that for every $v \in V_i(T)$ we have $|v| \geq 2$;

b) $T_v \in Tree^{(p)}(N_T^{-1}(v))$.

In order to be consistent with the notation of Section 7, we should also label all internal edges of $\mathbf{T}$ by an additional label *finite*. We omit the labeling since it will not be used in this section.

Clearly the automorphism group of any such $\mathbf{T}$ is trivial. The symmetric group S_n acts freely on G_n by permuting the tails of T. Any $\mathbf{T}$ gives rise to the 1-dimensional vector space $W_{\mathbf{T}} = L_1^{\otimes E_i(T)} \otimes U_{T_v}$, where the space $U_{T'}$ was defined in 5.5. In what follows we will identify $\mathbf{T}$ with the corresponding (up to a sign) generator of G_n.

Notice that if $E_i(T) = \emptyset$ then T has the only internal vertex v and $W_{\mathbf{T}} = U_{T_v}$. Moreover, the set $N^{-1}(v)$ is naturally identified with the set $\{1,...,n\}$, so that we can write $T_v \in Tree^{(p)}(n)$.

There is a natural morphism of S_n-modules $pr_n : G_n \to M_n$, such that $pr_n(W_{\mathbf{T}}) = 0$ if $E_i(T) = \emptyset$, otherwise $pr_n(W_{\mathbf{T}}) = id_{T_v}$ where v is the only internal vertex of T.

A generator $\mathbf{T}$ is pictured by a tree T with n numbered tails, and with generators of M inscribed into all internal vertices of T. If $T_v \in M$ is inscribed into a vertex $v \in V_i(T)$ then the cardinality of $V_{lab}(T_v)$ is equal to the cardinality of $N^{-1}(v)$.

Now we can return to the Theorem 1. We have constructed the minimal operad M. The Hochschild complex $C^\bullet(A,A)$ of an A_∞-algebra A is an algebra over M (if we forget the differentials). Then the free resolution P of M acts on $C^\bullet(A,A)$, so the latter becomes an algebra over the graded operad P (again we forget the differentials).

On the other hand, let A be a graded vector space and $m \in C = C^\bullet(A,A)$, but not necessarily $[m,m] = 0$. Then the constructions of Section 5 give rise to a sequence of elements $\rho(m) = (\rho(m)_n)_{n\geq 1}$ of $\underline{Hom}(M_n \otimes_{S_n} C^{\otimes n}, C)$ (no conditions on m were imposed by the construction). The following lemma is easy to prove.

Lemma 4 *The sequence $\rho(m)$ defines a structure of a graded M-operad on C.*

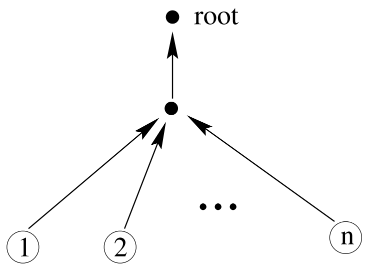

Figure 11. The unique abstract tree which is not killed by the projection from G_n to the space of generators.

Compositions $\gamma(m)_n = \rho(m)_n \circ pr_n$ for $n \geq 2$ give rise to a sequence of S_n-equivariant maps $G_n \otimes_{S_n} C \to C$. Since the graded operad P is freely generated by $G = (G_n)_{n\geq 2}$, these compositions define a point in $\mathscr{M}(P,C)$. We denote the sequence $(\gamma(m)_n)_{n\geq 2}$ by $\gamma(m)$. Thus we can write $\gamma(m) = \rho(m) \cdot pr$ where $pr = (pr_n)_{n\geq 2}$.

We can define an element $d_m \in \underline{Hom}(C,C)[1]$ in the natural way: $d_m = [m,\cdot]$. In general $d_m^2 \neq 0$. In any case it can be naturally extended to a map

$$f : \mathscr{M}_{cat}(\mathscr{A}_\infty, A) \to \mathscr{M}(P,C),$$

such that $f(m) = (d_m, \gamma(m))$.

We claim that this map is a morphism of dg-manifolds and it satisfies the condition $p \cdot f = \nu$ of the Theorem 1. It will be proved in the next subsection.

6.2. Proof of the theorem

In the course of the proof we will not pay much attention to the signs. The reason for that was explained in the previous section. Namely, our second description of the operad M (with reduced homology) gives automatically the agreement of signs.

Proof of the theorem will occupy the rest of this subsection. We start with some general considerations.

Let us recall that points of the moduli space $Y = \mathscr{M}(P,C)$ parametrize pairs (d_C, γ) where $d_C : C \to C \otimes L_1$ is a morphism in $Vect_{\mathbf{Z}}$, and γ is an action of the space of generators $G = (G_n)_{n\geq 2}$ on C. Having the action $\gamma(m) = \rho(m) \cdot pr$ of P on C we would like to compute the odd vector field d_Y at the point $(d_m, \gamma(m))$. Clearly $d_Y(d_C, \gamma) = (d_C^2, \bar{\gamma})$ where $\bar{\gamma} = (\bar{\gamma}_2, \bar{\gamma}_3, \ldots)$ and $\bar{\gamma}_n \in \underline{Hom}(G_n \otimes_{S_n} C^{\otimes n}, C \otimes L_1)$.

We need to compute $\bar{\gamma}$. One can extend d_C to $\underline{Hom}(C^{\otimes n}, C)$ using the Leibniz rule. We denote this extension by $d_C^{(n)}$. To write down $\bar{\gamma}$ we need to know the images of the generators of P under all γ_n. We know that the maps pr_n send to zero all of them except the trees with one non-labeled vertex and n numbered tail vertices. The latter is shown in Figure 11.

Then the direct computation together the previous lemma show that the following result holds.

Lemma 5 *The component $\bar{\gamma}_n$ of $d_Y(f(m))$ is equal to*

$$(d_m^{(n)} \cdot \rho(m)_n + \rho(m)_n \cdot d_{M_n}) \cdot pr_n,$$

where d_{M_n} is the differential d_M being restricted to M_n.

Proof. We sketch the proof. First we observe that all components of $\bar{\gamma}_n$, which corresponds to the generators of G_n with more than two internal edges, vanish. Using the previous lemma one can show that the generators with one internal edge give no input as well. After that one can make computations with elements of the operad M only. Then a direct computation proves the lemma. ■

We would like to prove that the map $f : X = \mathscr{M}_{cat}(\mathscr{A}_\infty, A) \to Y = \mathscr{M}(P, C)$ is a dg-map, i.e., it transforms the odd vector field d_X into the odd vector field d_Y. Vector field on X is given by $\dot{m} = d_X(m) = \frac{1}{2}[m, m]$.

The image of the map f belongs to the vector subspace

$$\underline{Hom}(C, C)[1] \oplus \underline{Hom}(M'(C), C)$$

of Y. Therefore the image $\dot{f}(m) = f_*(d_X(m))$ of the tangent vector $d_X(m)$ belongs to the same vector space. Thus, we have to show that the second component $\dot{\gamma}$ of $\dot{f}(m)$ is equal to $\overline{\gamma}$.

Its first component is equal to $\frac{1}{2}[\dot{m}, \cdot]$ which is the same as $\frac{1}{2}[[m, m], \cdot] = d_m^2$. Notice that $\overline{\gamma} - \dot{\gamma}$ can be considered as an action of an action of M' on C. We decompose it into the sum of terms corresponding to planar trees from $Tree^{(p)}$. For such a tree T the component of $\overline{\gamma} - \dot{\gamma}$ is a sum of four terms described below:

A. These terms correspond to the differential d_M of the operad M. We can schematically write them as $(d_M T)((c_i), m)$. We are inserting $c_i \in C$ in labeled vertices and m in non-labeled vertices.

B. These terms can be schematically written as $\sum_i T([m, c_i], (c_j)_{j \neq i}, m)$. We are inserting m in non-labeled vertices, elements c_j and $[m, c_i]$ in the corresponding labeled vertices.

C. These terms can be schematically written as $[m, T((c_i), m)]$ in the notation above. These terms appear when we apply the differential (= commutator with m) to the tree with c_i inserted in labeled vertices and m inserted in non-labeled vertices.

D. These terms correspond to $-\dot{\gamma}$. Each of them consists of the replacement of m in one non-labeled vertex v by $\dot{m}$. The latter can be in turn replaced by $\frac{1}{2}[m, m]$. We can schematically write the resulting sum as $\sum_{v \in V_{nonl}(T)} T((c_i), \frac{1}{2}[m, m])$. We are inserting the element m in all non-labeled vertices of T except v, the element $\frac{1}{2}[m, m]$ in the non-labeled vertex v and c_i to all labeled vertices.

Notice that although a tree T is always admissible, planar trees appearing in the decomposition of $\dot{\gamma}$ are not necessarily admissible. This means that valencies of some non-labeled vertices can be either 0 or 1.

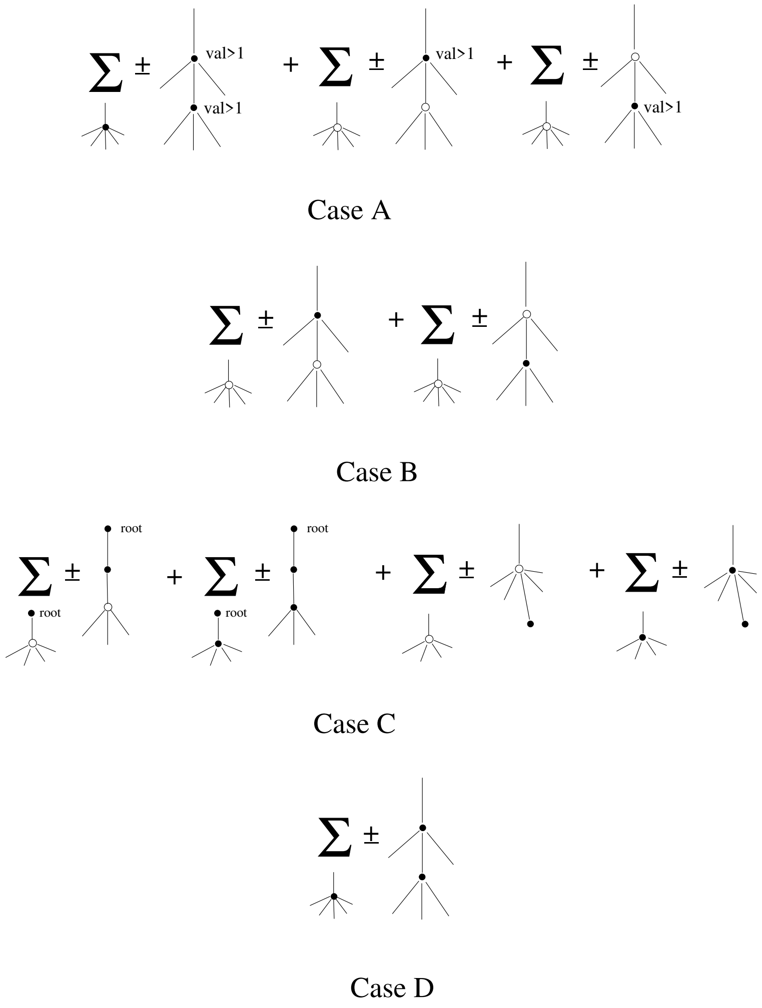

Figure 12. Four terms in the decomposition of $\overline{\gamma} - \dot{\gamma}$.

In order to depict all four cases we use the following notation: Composition of an operation α sitting in a vertex v with m produces a new tree with a new non-labeled vertex w, as well as a new edge (w,v) where $N(w) = v$. Similarly, a composition of m with α produces a new tree with a non-labeled vertex w, as well as a new edge (v,w) such that $N(v) = w$. The commutator $[\alpha, m]$ corresponds to the difference of the above-mentioned trees. This agreement will be used also in the case when $\alpha = m$ thus giving the way to depict trees with $\frac{1}{2}[m,m]$ inserted. In Figure 12, we show neighborhoods of vertices where the original tree changes.

Then we split these terms in the following way $A = A_1 + A_2 + A_3$, where:

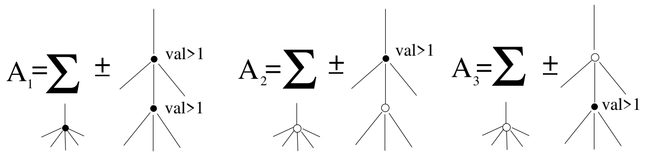

Figure 13. Decomposition of terms of type A.

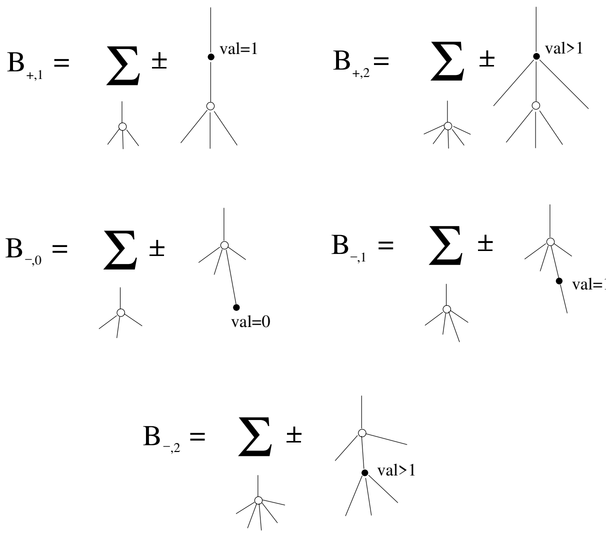

Figure 14. Decomposition of terms of type B.

a) The terms A_1 correspond to the two adjoint non-labeled vertices of valency ≥ 2 with m inserted in each;

b) The terms A_2 correspond to the labeled vertex adjoint to a non-labeled one of valency ≥ 2 with m inserted in the latter;

c) The terms A_3 correspond to the non-labeled vertex of valency ≥ 2 with m inserted in it adjoint to a labeled vertex.

These cases can be depicted as in Figure 13.

Similarly we split the terms B as follows: $B = B_{+,1} + B_{+,2} + B_{-,0} + B_{-,1} + B_{-,2}$, where individual summands are depicted in Figure 14.

We split the terms C such as follows: $C = C'_{root} + C''_{root} + C_\circ + C_\bullet$ where individual summands are depicted in Figure 15.

We split the terms D such as follows: $D = D_0 + D_{1,A} + D_{1,B} + D_2$ where individual summands are depicted in Figure 16. We see that $A_1 + D_2 = 0$, $A_2 + B_{+,2} = 0$, $A_3 + B_{-,2} = 0$, $B_{-,0} + C_\circ = 0$, $D_0 + C_\bullet = 0$.

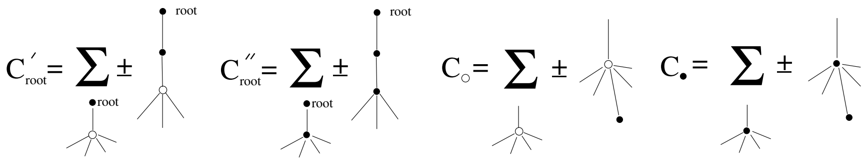

Figure 15. Decomposition of terms of type C.

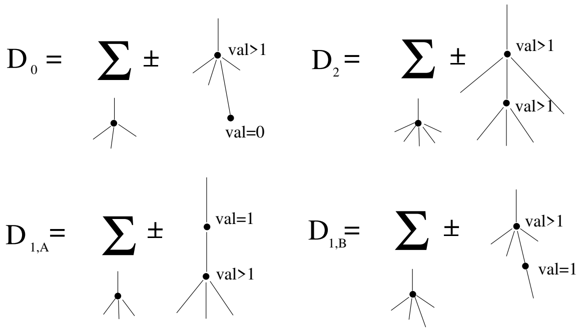

Figure 16. Decomposition of terms of type D.

Furthermore we can split each of the remaining terms into summands $B_{+,1} = B'_{+,1} + B''_{+,1} + B'''_{+,1}, B_{-,1} = B'_{-,1} + B''_{-,1}, D_{1,A} = D'_{1,A} + D''_{1,A} + D'''_{1,A}, D_{1,B} = D'_{1,B} + D''_{1,B}$. This splitting is depicted in Figure 17.

We combine these summands into six groups corresponding to the six types of edges depicted in Figure 18.

We see that $B'_{+,1} + C'_{root} = 0, C''_{root} + D'_{1,A} = 0, B''_{+,1} + B'_{-,1} = 0$, $B''_{-,1} + D''_{1,A} = 0, B'''_{+,1} + D'_{1,B} = 0, D'''_{1,A} + D''_{1,B} = 0$.

Then we conclude that $B_{+,1} + C'_{root} + C''_{root} + D_{1,A} + B_{-,1} + D_{1,B} = 0$. Thus $A + B + C + D = 0$. This means that the map $f : X \to Y$ is a dg-map. Obviously it is $GL(A)$-equivariant and satisfies the condition $pf = \nu$.

This concludes the proof of Theorem 1.

6.3. Remark about a generalization

Let A be an A_∞-algebra, and $C = C^\bullet(A,A)$ be its Hochschild complex. Admissible planar trees with n labeled vertices give rise to operations $C^{\otimes n} \to C$, $n \geq 1$. Analogously, planar trees with n labeled vertices and m tails, $n \geq 0$, $m \geq 1$ give rise to operations $C^{\otimes n} \otimes A^{\otimes m} \to A$.

Let us restrict ourselves to such trees that $|v| \geq 2$ for all non-labeled internal vertices v. Thus we obtain a colored operad $M^{(2)}$ with two colors *(Alg, Hoch)* acting on the set of pairs $(A, C^\bullet(A,A))$, where A is an A_∞-algebra. Clearly $M^{(2)}$ contains as suboperads both $\mathscr{A}_\infty$ and M. Presumably a result analogous to Theorem 1 holds for the colored operad $M^{(2)}$.

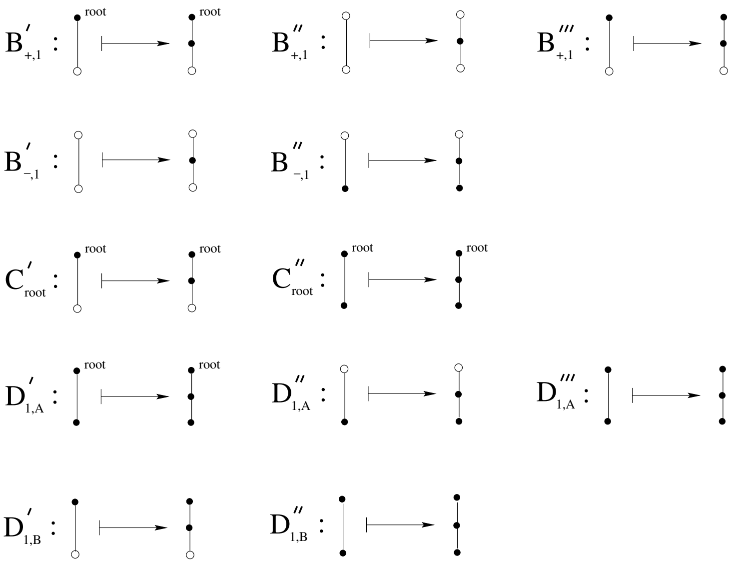

Figure 17. Further decompositions of terms $B_{+,1}, B_{-,1}, D_{1,A}, D_{1,B}$.

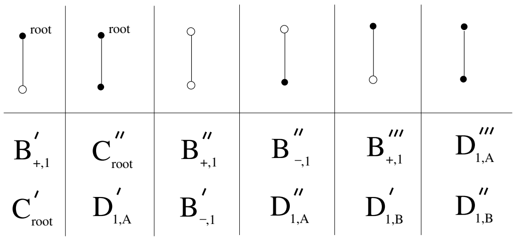

Figure 18. The rest, decomposed into six pairs of terms. In every pair terms cancel each other.

7. The Deligne conjecture

7.1. Preliminaries

We are going to prove the following result.

Theorem 3 *Let P be the free resolution of the minimal operad M constructed in Section 6. Then there is a homomorphism of dg-operads $P \to Chains(FM_2)$ which induces an isomorphism on cohomology (i.e., it is a quasi-isomorphism of dg-operads). Here $Chains(FM_2)$ is the chain operad for the Fulton-MacPherson operad of configurations of points in $\mathbf{R}^2$ (see Section 7.2 below).*

We will construct a homomorphism of the dg-operads which induces a quasi-isomorphism of the chain complexes. Such a homomorphism is not defined canonically. Different choices are naturally parametrized by a contractible topological space.

Using the fact that the operad $Chains(FM_2)$ is free as an operad (not as dg-operad), one can invert the quasi-isomorphism mentioned in the theorem. Since the Hochschild complex is a P-algebra, we obtain the following Corollary known as the Deligne conjecture (see for example [14], [22], [18]).

Corollary 1 *a) The Hochschild complex $C^\bullet(A,A)$ of an A_∞-algebra A can be equipped with a structure of an algebra over the operad $Chains(FM_2)$.*
b) The corresponding structure of a $H^\bullet(Chains(FM_2))$-algebra on the Hochschild cohomology $H^\bullet(A,A)$ coincides with the standard structure of a Gerstenhaber algebra on the Hochschild cohomology of an A_∞-algebra.

Same results remain true with $Chains(FM_2)$ being replaced by the operad $Chains(E_2)$ of chains on the little disc operad E_2 (see Section 7.2 for the definition).

Remark 6 It is easy to see that the operad P can be defined over $\mathbf{Z}$. It follows that such an operad acts on the Hochschild complex of an A_∞-algebra defined over a field of arbitrary characteristic. The quasi-isomorphism $P \to Chains(FM_2)$ can be also defined over the ring of integers (see also [18]).

The proofs of the Theorem and the Corollary will occupy the rest of the section.

We are going to use the following strategy.

Let $\mathbf{T}$ be a "meta-tree" corresponding to a generator of P (see Section 6 and Definition 17 below).

1) To every $\mathbf{T}$ we are going to associate a contractible closed subspace $X_{\mathbf{T}}$ in the Fulton-MacPherson compactification of the configuration space of points in $\mathbf{R}^2$ modulo shifts and dilatations.

2) The collection of subspaces $(X)_{\mathbf{T}}$ will satisfy the following properties:
a) The correspondence $\mathbf{T} \mapsto X_{\mathbf{T}}$ is S_n-equivariant;
b) if a tree $\mathbf{T}'$ appears as a summand in the formula for $d_P(\mathbf{T})$ then $X_{\mathbf{T}'} \subset X_{\mathbf{T}}$;
c) if a composition of trees $\mathbf{T}_1 \circ \mathbf{T}_2 \in P$ appears as a summand in the formula for $d_P(\mathbf{T})$ then the operadic composition $X_{\mathbf{T}_1} \circ X_{\mathbf{T}_2}$ belongs to the stratum $X_{\mathbf{T}}$. The latter composition of the strata has meaning because the Fulton-MacPherson compactifications form a topological operad.

3) For any generator $\mathbf{T}$ of the operad P we will choose inductively chains $\gamma_{\mathbf{T}} \in Chains(FM_2)$ such that $Supp(\gamma_{\mathbf{T}}) \subset X_{\mathbf{T}}$, where $Supp$ means the support of a chain. In this way we obtain a homomorphism of dg-operads $P \to Chains(FM_2)$.

4) This homomorphism is a quasi-isomorphism. This will follow from the fact that P is quasi-isomorphic to the minimal operad M, and on the level of chain complexes (not operads) every M_n is quasi-isomorphic to $Chains(FM_2(n))$.

Remark 7 The theorem and its proof seem to admit a generalization to the case of higher dimensions.

7.2. Little discs operad and Fulton-MacPherson operad

We recall here the definitions of both operads following [14].

We fix the dimension $d \geq 1$. Let us denote by G_d the $(d+1)$-dimensional Lie group acting on $\mathbf{R}^d$ by affine transformations $u \mapsto \lambda u + v$, where $\lambda > 0$ is a real number and $v \in \mathbf{R}^d$ is a vector. This group acts simply transitively on the space of closed discs in $\mathbf{R}^d$ (in the usual Euclidean metric). The disc with center v and with radius λ is obtained from the standard disc

$$D_0 := \{(x_1, \ldots, x_d) \in \mathbf{R}^d | x_1^2 + \cdots + x_d^2 \leq 1\}$$

by a transformation from G_d with parameters (λ, v).

Definition 15 *The little discs operad $E_d = \{E_d(n)\}_{n \geq 0}$ is a topological operad defined as follows:*
1) $E_d(0) = \emptyset$,
2) $E_d(1) = \text{point} = \{\mathrm{id}_{E_d}\}$,
3) for $n \geq 2$ the space $E_d(n)$ is the space of configurations of n disjoint discs $(D_i)_{1 \leq i \leq n}$ inside the standard disc D_0.

The composition $E_d(k) \times E_d(n_1) \times \cdots \times E_d(n_k) \to E_d(n_1 + \cdots + n_k)$ is obtained by applying elements from G_d associated with discs $(D_i)_{1 \leq i \leq k}$ in the configuration in $E_d(k)$ to configurations in all $E_d(n_i)$, $i = 1, \ldots, k$ and putting the resulting configurations together. The action of the symmetric group S_n on $E_d(n)$ is given by renumeration of indices of discs $(D_i)_{1 \leq i \leq n}$.

The space $E_d(n)$ is homotopy equivalent to the configuration space of n pairwise distinct points in $\mathbf{R}^d$.

There is an obvious continuous map $E_d(n) \to \mathsf{Conf}_n(Int(D_0))$ which associates to a collection of disjoint discs the collection of their centers. This map induces a homotopy equivalence because its fibers are contractible.

The little discs operad and homotopy equivalent little cubes operad were introduced in topology by J. P. May in order to describe homotopy types of iterated loop spaces.

The Fulton-MacPherson operad defined below is homotopy equivalent to the little discs operad.

For $n \geq 2$ we denote by $\tilde{E}_d(n)$ the quotient space of the configuration space of n points in $\mathbf{R}^d$

$$\mathsf{Conf}_n(\mathbf{R}^d) := \{(x_1, \ldots, x_n) \in (\mathbf{R}^d)^n | x_i \neq x_j \text{ for any } i \neq j\}$$

by the action of the group G_d. The space $\tilde{E}_d(n)$ is a smooth manifold of dimension $d(n-1)-1$. For $n = 2$, the space $\tilde{E}_d(n)$ coincides with the $(d-1)$-dimensional sphere S^{d-1}. There is an obvious free action of S_n on $\tilde{E}_d(n)$. We define the spaces $\tilde{E}_d(0)$ and $\tilde{E}_d(1)$ to be empty. The collection of spaces $\tilde{E}_d(n)$ does not form an operad because there is no identity element, and compositions are not defined.

Now we are ready to define the operad $FM_d = \{FM_d(n)\}_{n\geq 0}$. The components of the operad FM_d are:

1) $FM_d(0) := \emptyset$,

2) $FM_d(1) =$point,

3) $FM_d(2) = \tilde{E}_d(2) = S^{d-1}$,

4) for $n \geq 3$ the space $FM_d(n)$ is a manifold with corners, its interior is $\tilde{E}_d(n)$, and all boundary strata are certain products of copies of $\tilde{E}_d(n')$ for $n' < n$.

The spaces $FM_d(n)$, $n \geq 2$ can be defined explicitly.

Definition 16 *For $n \geq 2$, the manifold with corners $FM_d(n)$ is the closure of the image of $\tilde{E}_d(n)$ in the compact manifold $\left(S^{d-1}\right)^{n(n-1)/2} \times [0,+\infty]^{n(n-1)(n-2)}$ under the map $G_d \cdot (x_1,\dots,x_n) \mapsto (\left(\frac{x_j-x_i}{|x_j-x_i|}\right)_{1\leq i<j\leq n}, \frac{|x_i-x_j|}{|x_i-x_k|})$ where i,j,k are pairwise distinct indices.*

One can define the natural structure of operad on the collection of spaces $FM_d(n)$. We skip here the obvious definition.

It is easy to check that in this way we obtain a topological operad (in fact an operad in the category of real compact piecewise algebraic sets defined in the Appendix). We call it the *Fulton-MacPherson operad* and denote by FM_d.

Set-theoretically, the operad FM_d is the same as the free operad generated by the collection of sets $(\tilde{E}_d(n))_{n\geq 0}$ endowed with the S_n-actions described above.

Using piecewise algebraic chains from the Appendix we define dg-operads $Chains(E_d)$ and $Chains(FM_d)$ (they are operads in the symmetric monoidal category of complexes of abelian groups). Since we are working over the ground field k of characteristic zero, the dg-operads will be complexes of k-vector spaces.

Notice that when we write $Chains(Z)$ we mean the *cohomological* complex with graded components $Chains_i(Z)[i]$, $i \geq 0$ concentrated in *negative* degrees.

The complex P_n, $n \geq 2$ is concentrated in degrees $[-2n-3,...,0]$. The same is true for the complex $Chains(FM_2(n))$ (we use piecewise algebraic chains) because $dim_{\mathbf{R}} FM_2(n) = 2n-3$.

For every $n \geq 2$, the space P_n has a canonical (up to signs) basis, called the *standard basis*, with elements labeled by certain combinatorial objects, which we will call *meta-trees*.

Definition 17 *A meta-tree* $\mathbf{T}$ *with n tails is given by the following data:*

a) An abstract tree $T \in Tree(n)$ together with a marking $E_i(T) \to \{$finite,infinite$\}$;

b) For every internal vertex v of T we have: $|v| \geq 2$;

c) To every $v \in V_i(T)$ we assign an admissible labeled tree T_v together with a bijection $V_{lab}(T_v) \to N_{T_v}^{-1}(v)$.

We denote by $MT(n)$ the set of isomorphism classes of meta-trees with n tails.

Conjecture 1 *There is a piecewise algebraic (see Appendix) cell decomposition of the spaces $FM_2(n), n \geq 2$, with the cells $\sigma_{\mathbf{T}}$ labeled by $MT(n)$ such that:*

1) The correspondence $\mathbf{T} \to \sigma_{\mathbf{T}}$ is S_n-equivariant;

2) The operadic composition of any two cells is again a cell;

3) There is a morphism of dg-operads $Chains_{(\sigma_{\mathbf{T}})}(FM_2) \to P$ such that every cell $\sigma_{\mathbf{T}}$ is mapped (up to a sign) to the corresponding element of the standard basis of P. Here $Chains_{(\sigma_{\mathbf{T}})}(FM_2)$ denotes the chain subcomplex of $Chains(FM_2)$ formed by k-linear combinations of cells $\sigma_{\mathbf{T}}$;

4) For any $n \geq 2$ the cell decomposition of $FM_2(n)$ formed by cells $(\sigma_{\mathbf{T}}), \mathbf{T} \in MT(n)$ is regular, i.e. the closure of every cell is homeomorphic to a closed ball.

This conjecture implies the Deligne conjecture.

Let us now introduce a partial order on the set $MT(n)$, $n \geq 2$.

Definition 18 *Let $\mathbf{T}, \mathbf{T}'$ be meta-trees with n tails. We say that $\mathbf{T} < \mathbf{T}'$ if there exists a sequence of meta-trees $(\mathbf{T}_0, ..., \mathbf{T}_m)$ such that $\mathbf{T}_0 = \mathbf{T}, \mathbf{T}_m = \mathbf{T}'$ and for any i such that $0 \leq i \leq m-1$ we have: $\mathbf{T}_i$ appears as a summand in the decomposition of $d_P \mathbf{T}_{i+1}$ with respect to the standard basis of P_n.*

It follows from the condition 4) of Conjecture 1, that the nerve of the partially ordered set $(MT(n), <)$ is homeomorphic to $FM_2(n)$. The following conjecture also follows from Conjecture 1.

Conjecture 2 *The nerve of $(MT(n), <)$ is a PL-manifold with the boundary. The above-mentioned homeomorphism with $FM_2(n)$ is a homeomorphism of PL-manifolds with boundaries.*

We checked this conjecture for small n.

7.3. Partial orders induced by trees

We recall here the structure of the space G_n of generators of the components P_n, $n \geq 1$ of the free operad P (see Section 6).

There is a basis of G_n elements of which are (up to signs) parametrized by meta-trees $\mathbf{T} = (T, \{T_v\}_{v \in V_i(T)})$ such that all internal edges of T are finite. The degree of the generator $\mathbf{T} = (T, \{T_v\}_{v \in V_i(T)})$ is equal to $deg(\mathbf{T}) = \sum_{v \in V_i(T)} deg(T_v) - |E_i(T)|$.

Let $\mathbf{T}$ be a generator of P_n. We are going to introduce on the set $\{1, 2, ..., n\}$ two partial orders $<_{h,\mathbf{T}}$ and $<_{v,\mathbf{T}}$ (called *horizontal* and *vertical*). Although these orders will depend on $\mathbf{T}$, we will skip $\mathbf{T}$ from the notation if it does not lead to a confusion.

Let $i, j \in \{1, ..., n\}$, $i \neq j$. We have two tail vertices of T labeled by i and j respectively. Then there exists a unique internal vertex v of T satisfying the following properties:

a) $N_T^k(i) = N_T^l(j) = v$ where k, l are positive integers;

b) the vertex v is minimal among those satisfying a) (which means that k,l are both minimal in a).

Then there exist unique labeled vertices $x,y \in T_v$ such that $N_T^{k-1}(i) = x$ and $N_T^{l-1}(j) = y$. Since T_v is a planar tree, we can compare x and y with respect to exactly one of the following partial orders: "x is to the left of y in T_v" or "x is above y in T_v". We will call them the "horizontal" and "vertical" order respectively.

Let us describe the orders in T_v more precisely. We say that x is *above* y (or y is *below* x) in T_v if there exists a positive integer a such that $N_{T_v}^a(y) = x$.

We say that x is *to the left* of y (or y is *to the right* of x) if there exist positive integers a,b such that $N_{T_v}^a(x) = N_{T_v}^b(y) = w$, but $N_{T_v}^{a-1}(x) \neq N_{T_v}^{b-1}(y)$ and $N_{T_v}^{a-1}(x)$ precedes $N_{T_v}^{b-1}(y)$ in $N_{T_v}^{-1}(w)$ with respect to the order on the latter set given by the planar structure on T_v.

Thus, we have defined two partial orders on vertices of $(T_v)_{v \in V_i(T)}$. They induce partial orders $<_{h,\mathbf{T}}$ (horizontal) and $<_{v,\mathbf{T}}$ (vertical) on the set $\{1,...,n\}$. We have identified the latter with the set of tails of T. Namely, we say that $i <_{h,\mathbf{T}} j$ if (in the above notation) the vertex x is to the left of y. We say that $i <_{v,\mathbf{T}} j$ if y is above x (equivalently we say that x is below y).

The following lemma is easy to prove.

Lemma 6 *Both horizontal and vertical orders are indeed partial orders (i.e., they satisfy all the axioms of orders).*

Definition 19 *Let S be a set, $<_1$ and $<_2$ be partial orders on S. We will call them complementary if any two elements of S can be compared with respect to exactly one of them. This means that for any two elements $i,j \in S$, $i \neq j$ exactly one of the following properties holds: $i <_1 j$, $j <_1 i$, $i <_2 j$, $j <_2 i$.*

Suppose that we have two partial orders as in Definition 19. We define on S two new pre-orders $<_{1+2}$ and $<_{1-2}$ as follows:

(i) $x <_{1+2} y$ if $x <_1 y$ or $x <_2 y$;

(ii) $x <_{1-2} y$ if $x <_1 y$ or $y <_2 x$.

Notice that we can reconstruct $<_1$ and $<_2$ from these new orders. For example $x <_1 y$ is equivalent to the conjunction: $(x <_{1+2} y)$ and $(x <_{1-2} y)$.

Proposition 5 *Formulas (i) and (ii) define complete orders on the set S.*

Proof. Straightforward. ■

Using this result one can easily prove the following one.

Proposition 6 *Let $<_1$ and $<_2$ be a pair of two complementary orders given on a finite set S. Then there exists a unique element $s_0 \in S$ such that for any $i \in S, i \neq s_0$ we have either $s_0 <_1 i$ or $s_0 <_2 i$.*

Proof. The element s_0 is minimal with respect to the complete order $<_{1+2}$. ∎

Summarizing, we can say that on the set of tails of a generator of P_n we have defined two complementary partial orders (or, equivalently, two complete orders). These orders will be used below when we will construct the closed sets $X_{\mathbf{T}}$.

7.4. Closed sets $X_{\mathbf{T}}$

Let us recall that for any $n \geq 2$, $1 \leq i, j \leq n$, $i \neq j$ we have a natural projection $p_{i,j} \colon FM_2(n) \to FM_2(2)$ (forgetting all points in $(x_1, ..., x_n)$ except x_i and x_j). As a topological space $FM_2(2)$ is identified to the unit circle $S^1 \subset \mathbf{R}^2$ via the map $G_2 \cdot (x_1, x_2) \mapsto \frac{x_2 - x_1}{|x_2 - x_1|}$.

We denote by $S^1_{+,v} \subset FM_2$ the closed upper-half circle, and by $S^1_{+,h} \subset S^1$ the one-element subset consisting of the point $\{(1,0)\}$. Then $S^1_{+,v}$ corresponds to the configurations $(x_1, x_2) \in Conf_2(\mathbf{R}^2)$ such that if we put $x_2 - x_1 = re^{i\alpha}, 0 \leq \alpha < 2\pi$ then $\alpha \in [0, \pi]$. Similarly the subset $S^1_{+,h}$ corresponds to the configurations (x_1, x_2) such that both x_j, $j = 1, 2$ belong to the same horizontal line, and x_1 is positioned to the left of x_2.

Suppose that we are given two complementary orders $<_h$ and $<_v$ on the set $\{1, ..., n\}, n \geq 2$. Then we define the following subset of $FM_2(n)$:

$$X_{<_h, <_v} = \{x \in FM_2(n) | i <_h j \Rightarrow p_{i,j}(x) \in S^1_{+,h}, i <_v j \Rightarrow p_{i,j}(x) \in S^1_{+,v}\}.$$

For a generator $\mathbf{T} \in P_n$ we define $X_{\mathbf{T}}$ as $X_{<_{h,\mathbf{T}}, <_{v,\mathbf{T}}}$.

First of all we would like to prove that $X_{\mathbf{T}}$ is contractible. This is a special case of a more general statement.

Proposition 7 *For any pair of complementary orders $<_h$ and $<_v$ given on the set $\{1, ..., n\}$, $n \geq 2$ the subspace $X_{<_h, <_v}$ is non-empty and contractible.*

Proof. It can be done by induction. For $n = 2$ the result is clear, since subsets $S^1_{+,v}$ and $S^1_{+,h}$ are contractible. Suppose that the Proposition is true for $n - 1$ points. Let us take the element i_0 which is minimal in $I_n = \{1, ..., n\}$ with respect to $<_{h+v}$. We have already proved that it exists. Let us consider the induced complementary partial orders $<_{h'}, <_{v'}$ on the set $I_n \setminus \{i_0\}$. They define the subset $X_{<_{h'}, <_{v'}} \subset FM_2(n-1)$ which is non-empty and contractible by induction. Then the result is a corollary of the following observation: the fibers of the natural projection $\pi : X_{<_h, <_v} \to X_{<_{h'}, <_{v'}}$ are contractible. Basically it follows from the fact that the point x_{i_0} is either left or below of all the points $x_i, 1 \leq i \leq n, i \neq i_0$. Let us prove that the fibers of π are non-empty. Indeed, for any $x' \in X_{<_{h'}, <_{v'}}$ the operadic composition $(-1, 0) \circ_j x'$ belongs to the fiber $\pi^{-1}(x')$. Here $(-1, 0) \in S^1$ is considered as a point in $FM_2(\{i_0, j\})$, and j is an auxiliary index.

We leave to the reader the proof of contractibility of the fibers of π. ∎

Remark 8 a) One can prove that all homotopies can be taken in the category of piecewise algebraic sets.
b) It follows from the construction that the map $\mathbf{T} \mapsto X_{\mathbf{T}}$ is S_n-equivariant.

Now we would like to explain Property 2 of $X_{\mathbf{T}}$ stated in Section 7.1. First of all, the S_n-equivariance is obvious.

Let $\mathbf{T} \in G$ be a generator of P. We recall that the differential in P can be written schematically (up to signs) as

$$d_P(\mathbf{T}) = \sum_{v \in V_i(T), l} \mathbf{T}_{v,l} + \sum_{\alpha \in E_{infinite}(T)} \mathbf{T}'_\alpha \circ \mathbf{T}''_\alpha + \sum_{\alpha \in E_{finite}(T), j} \mathbf{T}_{\alpha,j}.$$

Here meta-trees $\mathbf{T}_{v,l} \in G$ arise from the application of the differential of M to the tree T_v inscribed into the vertex $v \in V_i(T)$. This differential was described in Section 5. Index l runs over all possible insertions of a new edge.

The second summand corresponds to the tree, obtained from T by making a finite edge α into an infinite edge. The result is a composition of two generators of P which we denote by $\mathbf{T}'_\alpha$ and $\mathbf{T}''_\alpha$.

The last sum corresponds to the operation "contract a finite edge α" in T. Then the planar trees from M inscribed into the vertices which are endpoints of α must be composed and inscribed into the new vertex. The result is a sum of generators $\mathbf{T}_{\alpha,j}$ of P.

Proposition 8 *In the above notation we have $X_{\mathbf{T}_{v,l}}, X_{\mathbf{T}'_\alpha} \circ X_{\mathbf{T}''_\alpha}, X_{\mathbf{T}_{\alpha,j}}$ belong to $X_{\mathbf{T}}$.*

Proof. Straightforward check which uses the fact that $S^1_{+,h} \in S^1_{+,v}$. ■

7.5. Morphism $P \to Chains(FM_2)$

We would like to construct the chains $\gamma_{\mathbf{T}} \in Chains(FM_2)$ where $\mathbf{T}$ runs through the set of generators of P. Let us explain the idea.

We will construct $\gamma_{\mathbf{T}}$ by induction in the degree of $\mathbf{T}$.

a) Let us assume that $deg(\mathbf{T}) = 0$. Then T has the only internal vertex v and the corresponding planar tree T_v is a binary tree. The corresponding element of the operad P is defined up to a sign. We make the following choice: It is just the composition of several copies of the operation $m_2 \in M_2$.

The corresponding chain $\gamma_{\mathbf{T}}$ should have the dimension zero. Therefore it must be a linear combination of points with integer coefficients. We choose $\gamma_{\mathbf{T}}$ to be just one point in $X_{\mathbf{T}}$ with the multiplicity equal to $+1$.

b) Let us assume that $deg(\mathbf{T}) = -1$. Then the formula for $d_P(\mathbf{T})$ contains two summands $\mathbf{T}', \mathbf{T}''$ corresponding to the two binary planar trees. It is not difficult to check that they appear with the opposite coefficients ± 1. According to a) these summands define 0-dimensional chains $\gamma_{\mathbf{T}'}, \gamma_{\mathbf{T}''}$. We define $\gamma_{\mathbf{T}}$ to be the only (up to

a boundary) 1-chain having $\gamma_{\mathbf{T}'} - \gamma_{\mathbf{T}''}$ as the boundary. This 1-chain exists because of the condition imposed on multiplicities of 0-chains.
c) Suppose that we have a generator $\mathbf{T}$, $deg(\mathbf{T}) = -k, k \geq 2$.

Then we can find a chain $\gamma_{\mathbf{T}}$ of degree $-k$ such that

$$\partial(\gamma_{\mathbf{T}}) = \sum_{v,l} \gamma_{\mathbf{T}_{v,l}} + \sum_{\alpha} \gamma_{\mathbf{T}'_\alpha \circ \mathbf{T}''_\alpha} + \sum_{\alpha,j} \gamma_{\mathbf{T}_{\alpha,j}}$$

where ∂ is the boundary operator in the chain complex for FM_2, and rest of the notation is self-explained.

Indeed, the RHS of this formula is known by the induction assumption. We also know that it is a closed chain because $d_P^2 = 0$. Then one can always find a chain $\gamma_{\mathbf{T}}$ with the given boundary. Indeed, $X_{\mathbf{T}}$ is contractible, and we consider chains of negative degrees (zero degree case was considered in a)). The space parametrizing all different choices of $\gamma_{\mathbf{T}}$ is contractible, so our choice is unique in a given homology class.

The map $\mathbf{T} \to \gamma_{\mathbf{T}}$ extends to a homomorphism $\Phi : P \to Chains(FM_2)$ of graded operads in such a way that $\Phi(\mathbf{T}) = \gamma_{\mathbf{T}}$. We have checked that Φ is compatible with the differentials. Therefore it is a homomorphism of dg-operads.

Theorem 4 *The morphism Φ is a quasi-isomorphism of complexes.*

Proof. We will give only a sketch of the proof. The idea is to consider a subcomplex L of the complex P spanned by the trees without internal edges. This subcomplex is isomorphic to the complex of the minimal dg-operad M. Indeed, P can be decomposed into a direct sum of L and a contractible complex (see Proposition 4 in Section 4). Therefore it is enough to prove that the restriction $\Phi|_L : L \to Chains(FM_2)$ is a quasi-isomorphism of complexes.

In order to do that we need to describe chains from the subcomplex $\Phi(L)$. Our constructions were based on certain non-canonical choices of chains $\gamma_{\mathbf{T}}, \mathbf{T} \in G$. Now, for $\mathbf{T} \in L$ we will make specific choices of $\gamma_{\mathbf{T}}$.

Let $\mathbf{T} \in L$ corresponds to a labeled tree $T_v \in M$. We will denote it simply by T_v, where T_v is a labeled planar tree inscribed in the only internal vertex v of $\mathbf{T}$.

We associate with T_v an abstract tree $\widehat{T_v}$ such that $V_t(\widehat{T_v}) = V_i(T_v)$, and the valency of each internal vertex of $\widehat{T_v}$ is at least 2. Internal vertices of $\widehat{T_v}$ correspond to subsets $T_{\leq x_0}$ described below.

Let us choose an internal vertex $x_0 \in V_i(T_v)$ and consider the set of such $x \in V_i(T_v)$ that $N^j_{T_v}(x) = x_0$ for some $j \geq 0$. Then internal vertices of $\widehat{T_v}$ correspond to such sets $T_{\leq x_0}$ for which the cardinality $|T_{\leq x_0}| \geq 2$. Moreover, there is a path in $\widehat{T_v}$ from a tail vertex $v \in V_t(\widehat{T_v})$ to an internal vertex $T_{\leq x_0}$ iff $v \in T_{\leq x_0}$.

It is well-known that any abstract tree $\widehat{T}$ with valencies of internal vertices at least 2 gives rise to a stratum $J(\widehat{T}) \in FM_2(|V_t(\widehat{T})|)$ (see [6]). It is the operadic composition of $FM_2(N^{-1}_{\widehat{T_v}}(u)), u \in V_i(\widehat{T_v})$.

We are going to construct a subspace $X_{\widehat{T}} \subset J(\widehat{T}_v)$ The space $X_{\widehat{T}}$ will be constructed as the operadic composition of certain subspaces

$$X_u \subset FM_2(|N_{\widehat{T}_v}^{-1}(u)|), u \in V_i(\widehat{T}_v).$$

Let $u = T_{\leq x_0}$ be an internal vertex of $\widehat{T}$. The set of edges having u as an endpoint is in one-to-one correspondence with the set $\{x_0\} \sqcup N_{T_v}^{-1}(x_0)$. The subspace X_u consists of configurations of points $G_2 \cdot (p_{x_0}, (p_y)_{y \in N_T^{-1}(x_0)})$ such that $p_{x_0} = (0,1) \in \mathbf{R}^2$, all points p_y belong to the horizontal line $\{(x,0)|x \in \mathbf{R}\} \subset \mathbf{R}^2$, and their order on this line is the same as their order in $N_{T_v}^{-1}(x_0)$.

One can check that:

a) $X_{\widehat{T}_v}$ is an open cell.

b) The natural projection (forgetting map) $FM_2(|V_i(T_v)|) \to FM_2(|V_{lab}(T_v)|)$ maps $X_{\widehat{T}_v}$ onto an open cell $X_{T_v} \subset X_{\mathbf{T}}$.

c) The closure $\overline{X_{T_v}}$ is a manifold with corners (more precisely, a real piecewise algebraic manifold).

d) The boundary of $\overline{X_{T_v}}$ is the union of cells of the same type .

Using a)-d) one checks that the construction above gives rise to a homomorphism of complexes $\chi : L \to Chains(FM_2)$, and moreover, it coincides with the homomorphism $\Phi|_L$. To be more precise, one observes that the construction of Φ was not canonical. We made certain choices when constructed a chain with the prescribed boundary. The point is that one can choose inductively the chains in such a way that the restriction $\Phi|_L$ of the homomorphism $\Phi : P \to Chains(FM_2)$ coincides with χ.

We claim that χ induces a homotopy equivalence. This follows from the following lemma.

Lemma 7 *Let $MT_1(n)$ denotes the subset of $MT(n)$ consisting of meta-trees with only one internal vertex v. Then the natural embedding of the CW-complex $X_n = \cup_{\mathbf{T} \in MT_1(n)} X_{T_v}$ into $FM_2(n)$ is a homotopy equivalence for all $n \geq 2$.*

Proof. It is not difficult to show that X_n is isomorphic to the CW-complex $\Sigma_{(n)}$ constructed in Section 5.5 via Strebel differentials. It follows from the fact that both are regular complexes and posets of their cells are isomorphic. Moreover, both complexes are $K(\pi,1)$ spaces, classifying spaces for the pure braid group of n-strings. Moreover, the map Φ induces an equivalence of the fundamental groupoids. Hence it is a homotopy equivalence. ■

This Lemma concludes the proof of Theorem 2. ■

7.6. Proof of the Deligne conjecture

The results of the previous subsection establish Theorem 2. In this subsection we will prove the Corollary (Deligne conjecture). First of all, we remark that both P and $Chains(FM_2)$ are dg-operads which are free as graded operads. Therefore the quasi-isomorphism morphism of graded operads $\Phi : P \to Chains(FM_2)$ admits a homotopy inverse $\Psi : Chains(FM_2) \to P$ (see for ex. [16]).

We have already proved that the Hochschild complex of an A_∞-algebra is an algebra over the operad P. Indeed, P is a free resolution of M, and the latter dg-operad acts on the Hochschild complex. Using the morphism Ψ we make the Hochschild complex into an algebra over the dg-operad $Chains(FM_2)$. This concludes the proof of the Deligne conjecture.

8. Appendix: Singular chains and differential forms

In this appendix we are going to describe a theory of singular chains which is suitable for working with manifolds with corners. This formalism is helpful in the proof of formality of the operad of chains on the little disc operad (see [14]). We used it in the proof of the Deligne conjecture.

The idea of the theory of singular chains developed below is the following: One can construct chains which produce complexes quasi-isomorphic to the standard complexes of singular chains, and built out of some kind of "piecewise algebraic spaces".

8.1. Spaces

We recall that closed real semialgebraic sets in $\mathbf{R}^n$ are subsets defined by a finite number of polynomial equations and non-strict inequalities. Constructible sets are obtained from closed semialgebraic ones by boolean operations.

We define the category $\mathscr{P}$ of *compact piecewise algebraic spaces* (compact PA-spaces for short) in the following way.

Objects of $\mathscr{P}$ are pairs (X,n), $n = 1,2,...$ such that $X \subset \mathbf{R}^n$ is a compact constructible set (it is the same as a compact real semialgebraic set).

For two objects (X,n) and (Y,m) the space of morphisms $Hom((X,n),(Y,m))$ is formed by continuous maps $f : X \to Y$ such that $graph(f) \subset \mathbf{R}^n \times \mathbf{R}^m$ is constructible.

In the future we will skip the index n in the notation (X,n) if it will not lead to a confusion.

Obviously we have a functor from $\mathscr{P}$ to the category of compact Hausdorff topological spaces. An isomorphism in $\mathscr{P}$ is a morphism which is a homeomorphism of topological spaces.

Let $X \in \mathscr{P}$. Then one can define a sheaf $\mathscr{O}_X$ of piecewise algebraic functions on X. To do this we note first that one can speak about constructible subsets of X. By definition they are constructible sets in the bigger space $\mathbf{R}^n$. Then for any open $U \subset X$ we define $\mathscr{O}_X$ to be an $\mathbf{R}$-algebra of continuous functions f on U such that for any compact constructible $V \subset U$ we have: $graph(f|_V)$ is constructible. It is easy to see that we get a sheaf $\mathscr{O}_X$ of algebras.

Lemma 8 *Morphisms from X to Y in the category $\mathscr{P}$ are in one-to-one correspondence with homomorphisms of algebras $\mathscr{O}(Y) \to \mathscr{O}(X)$.*

Proof. Exercise. ■

Similarly, we can consider the non compact case.

Definition 20 *Piecewise algebraic space (PA-space for short) is a locally compact Hausdorff topological space X, equipped with the sheaf $\mathscr{O}_X$ of $\mathbf{R}$-algebras which is locally isomorphic to $\mathscr{O}_{X'}$ for some compact $X' \subset \mathscr{P}$.*

Clearly, PA-spaces form a category. Compact PA-spaces are exactly objects of $\mathscr{P}$.

We define a d-dimensional PA-manifold with boundary as a PA-space X which is modeled locally on the closed half-space $\mathbf{R}^{d-1} \times \mathbf{R}_{\geq 0}$.

8.2. Singular chains

We would like to define an appropriate version of singular chains for PA-spaces. We will give two equivalent descriptions.

First description.

Let X be a PA-space. We define the group of n-chains $Chains_n(X) := C_n(X, \mathbf{Z})$ as an abelian group generated by equivalence classes of triples (M, or, f) such that:

i) M is a compact PA-manifold of dimension n;

ii) $f : M \to X$ is a morphism in $\mathscr{P}$;

iii) or is an orientation of M.

We need to define an equivalence relation. It is the same as to say when a finite linear combination $\sum_i n_i(M_i, or_i, f_i)$ is equal to zero in $C_n(X, \mathbf{Z})$.

Notice that $Y = \cup_i f_i(M_i)$ carries the structure of a compact PA-space, of dimension $\leq n$. Then there exists a constructible subset $Y_0 \subset Y$ such that $dim(Y \setminus Y_0) \leq n-1$, and for any point $y \in Y_0$, any i and $x_{i,\alpha} \in f_i^{-1}(y)$ there exists a neighborhood $U_{i,\alpha}$ of $x_{i,\alpha}$ such that $Y_0 \cap f_i(U_{i,\alpha})$ is a PA-manifold and the morphism $f_i|_{U_{i,\alpha}}$ is a homeomorphism of $U_{i,\alpha}$ onto its image.

We choose an orientation or_y of Y_0 near the point y. Then the above mentioned linear combination is declared to be zero iff for every point $y \in Y_0$ we have $\sum_{i,\alpha} n_i sgn(i, \alpha, y) = 0$ where $sgn(i, \alpha, y)$ is defined as $f_{i*}|_{U_{i,\alpha}}(or_i) = sgn(i, \alpha, y) or_y$, and sum is taken over all points $x_{i,\alpha}$ such that $f_i(x_{i,\alpha}) = y$. Notice that $sgn(i, \alpha, y)$ always takes values ± 1.

Second description.

We will define $C_n(X) = C_n(X, \mathbf{Z})$ as a quotient by certain equivalence relation of the set of quadruples $(Y, Y_0, or, mult)$ such that:

a) $Y \subset X$ is a compact PA-subspace, $dimY \leq n$, which contains an open dense constructible subset Y_0 without boundary which is a PA-manifold of dimension n;

b) or is an orientation of Y_0;

c) $mult : Y_0 \to \{1,2,...,\}$ is a locally constant map (multiplicity).

The equivalence relation is defined such as follows.

We say that $(Y, Y_0, or, mult)$ is equivalent to $(Y', Y_0', or', mult')$ iff
(i) $Y = Y'$ and there exists an open constructible $Y_0'' \subset Y_0' \cap Y_0''$ such that Y is equal to the closure of Y_0'';
(ii) Restrictions of orientations and multiplicity functions of Y_0 and Y_0' to Y_0'' coincide.

Using the pairing between chains and differential forms (see 8.3), one can show that the first and the second descriptions give canonically isomorphic sets of chains.

One can also check the following properties of singular chains:

1) There is a naturally defined differential $\partial : C_n(X,\mathbf{Z}) \to C_{n-1}(X,\mathbf{Z}), \partial^2 = 0$ (in Description 1 it is well-defined by the formula $\partial(M, or, f) = (\partial M, \partial(or), f|_{\partial M})$).

2) The correspondence $X \to C_\bullet(X,\mathbf{Z})$ is a functor from the category of PA-spaces to the category of abelian groups;

3) $C_i(X,\mathbf{Z})$ vanishes for $i < 0$ and $i > dimX$;

4) If (X, or) is a compact oriented PA-manifold, then there exists a canonically defined chain $[X] \in C_n(X,\mathbf{Z})$. In Description 1, it is defined by the formula $[X] = (X, or, id_X)$;

5) For any finite collection of PA-spaces $(X_i)_{i\in I}$ there is a natural homomorphism of complexes of abelian groups $\otimes_{i\in I} C_\bullet(X_i,\mathbf{Z}) \to C_\bullet(\prod_{i\in I} X_i, \mathbf{Z})$;

6) There is a naturally defined soft sheaf of complexes $\underline{C}_\bullet^{closed}$ on X such that for a compact X the abelian group $\Gamma(X, \underline{C}_\bullet^{closed})$ coincides with $C_\bullet(X,\mathbf{Z})$.

7) The homology of $C_\bullet(X,\mathbf{Z})$ is naturally isomorphic to the usual singular homology $H_\bullet(X,\mathbf{Z})$.

For non-compact PA-spaces we define locally finite chains as global sections of the sheaf $\underline{C}_\bullet^{closed}$. Then the homology of $C_\bullet^{closed}(X,\mathbf{Z})$ is isomorphic to the usual singular homology with locally compact support $H_\bullet^{closed}(X,\mathbf{Z})$.

Remark 9 a) Mayer-Vietoris sequence trivializes for PA chains: If $X = Y_1 \cup Y_2$ is a union of locally closed constructible subsets, then the corresponding short sequence of abelian groups

$$0 \to C_n(Y_1 \cap Y_2, \mathbf{Z}) \to C_n(Y_1,\mathbf{Z}) \oplus C_n(Y_2,\mathbf{Z}) \to C_n(X,\mathbf{Z}) \to 0$$

is exact.
b) The Property 5 is very convenient in order to formulate the Deligne conjecture. It allows to avoid Eilenberg-Zilber theorem.

c) Property 4 seems to be useful for the higher-dimensional generalization of the Deligne conjecture (see [14]). Namely, if A is a d-algebra (i.e., an algebra over the dg-operad $Chains(FM_d)$), then $A[d-1]$ carries a natural structure of an L_∞-algebra. More precisely, for any $n \geq 2$ the fundamental cycle $[FM_d(n)]$ gives the n-th higher bracket on $A[d-1]$.

8.3. Differential forms: First approximation

Let X be a PA-space. We would like to define an appropriate notion of differential form on X. In this subsection we construct a first approximation to the future algebra of differential forms.

Definition 21 *The sheaf $\Omega^k_{X,min}$ is locally defined as a vector subsubspace in $Hom(C_k(X,\mathbf{Z}),\mathbf{R})$ generated by functionals $l = (f_0, f_1, ..., f_k), f_i \in \mathcal{O}_X$ such that*

$$l(M, or, \phi) = \int_{M_0} \phi^*(f_0 df_1 \wedge df_2 ... \wedge df_k).$$

Here $M_0 = M$ is a dense open constructible subset. We can assume that M is a subspace of some $\mathbf{R}^N$, and all functions f_i are smooth on a smooth dense open submanifold $M_0 \subset M$.

Proposition 9 *The functional l from the definition above is well-defined (i.e., the integral absolutely converges).*

Proof. Proof. We can use a sequence of functions $(f_0, f_1, ..., f_k)$ in order to map M to $\mathbf{R}^{k+1}$. Then we obtain a singular chain $\gamma \in C_k(\mathbf{R}^{k+1})$. The support of γ is a compact constructible subset of dimension $\leq k$. Moreover, the volume of γ with respect to the induced metric is finite. It follows that $\int_\gamma \omega$ absolutely converges for an arbitrary smooth differential k-form ω. In particular the integral in question converges. ∎

It is easy to check that the differential $d : \Omega^k_{X,min} \to \Omega^{k+1}_{X,min}$ is well-defined (as the adjoint to the boundary operator on the chains). Moreover the following Stokes formula holds.

Lemma 9 *In the previous notation we have:*

$$(1, f_0, ..., f_k)(M, or, \phi) = (f_0, ..., f_k)(\partial M, or|_{\partial M}, \phi|_{\partial M}).$$

Hence we obtain a complex of sheaves $\Omega^k_{X,min}$ for every $k \geq 0$. Our (first approximation to) k-forms are sections of these sheaves. One can take pull-backs of forms.

Moreover, we can introduce wedge product of forms. Indeed the space of k-forms $\Omega^k_{min}(X)$ is naturally embedded into the direct limit of spaces of smooth

k-forms $\Omega^k_{C^\infty}(X_\alpha)$. Here $X_\alpha \subset X$ are open dense locally constructible subspaces, $X_\alpha = \sqcup_i X_{i,\alpha}$, where $X_{i,\alpha}$ is a PA-manifold. Moreover, it is assumed that we have fixed an embedding $X_{i,\alpha} \to \mathbf{R}^{N_{i,\alpha}}$ identifying $X_{i,\alpha}$ with a C^∞-submanifolds in $\mathbf{R}^{N_{i,\alpha}}$. Using the embeddings we define $\Omega^k_{C^\infty}(X_\alpha)$. Therefore the wedge product given by the formula $(f_0,...,f_k)\wedge(g_0,...,g_n) = (f_0g_0, f_1,...,f_k, g_1,...,g_n)$ is well-defined.

The above defined forms are sections of soft sheaves because one can use piecewise algebraic functions to produce a partition of unity, so the standard proofs work. One can show that Mayer-Vietoris sequence degenerates as in the case with chains above: If $X = Y_1 \cup Y_2$ is a union of locally closed piecewise algebraic subspaces then one has a short exact sequence

$$0 \to \Omega^k_{min}(X) \to \Omega^k_{min}(Y_1) \oplus \Omega^k_{min}(Y_2) \to \Omega^k_{min}(Y_1 \cap Y_2) \to 0.$$

In the next subsection we will extend algebras $\Omega^\bullet_{min}(X)$ adding push-forwards of such forms. The following example illustrates one of the reasons for that.

Example 1 *Let $Y = [0,1]\times[0,1]$, $X = [0,1]$ and $f : Y \to X$ be the map $(x,y) \mapsto t = xy$. Then, outside of $t = 1$, we have a bundle in the category $\mathscr{P}$ (i.e., base, fibers and total space are objects, projection is a morphism in $\mathscr{P}$). Take the form $\omega = xdy$. Then $f_*\omega = t(logt)$ is a continuous function, but does not belong to $\mathscr{O}(X)$.*

8.4. Full algebra of forms

We are going to define a dg-algebra $\Omega^\bullet_{PA}(X)$ which contains $\Omega^\bullet_{min}(X)$ and satisfies Poincaré lemma, etc. Elements of $\Omega^\bullet_{PA}(X)$ will be called PA-forms, although their coefficients are not PA-functions (see Example 1). It is interesting that only 0-forms will be forms with continuous coefficients. Higher order forms can have coefficients in $L_1^{loc}(X)$. But they are still closed under the wedge product.

We start with some preliminaries. Let $f : Y \to X$ be a proper PA-map of PA-spaces. For simplicity we will give the definition below in the case when X is compact.

Definition 22 *Continuous family of PA k-cycles (PA-family of cycles for short) is defined as an element of the abelian group $C_k(Y \to X)$ described below.*

The group $C_k(Y \to X)$ consists of maps $\Phi : X \to C_k(X)$ such that:

a) For any $x \in X$ the set $Supp\,\Phi(x)$ belongs to $f^{-1}(x)$.

b) The set $Z = \cup_{x\in X} Supp\,\Phi(x)$ is constructible with the compact closure $\overline{Z} \subset Y$.

c) There exists a dense constructible $Z_0 \subset \overline{Z}$ such that for any $x \in X$ the intersection $Z_0 \cap f^{-1}(x)$ is a PA-manifold without boundary, dense in $\overline{Z} \cap f^{-1}(x)$. Moreover, the chain $\Phi(x)$ is obtained from some orientation of $Z_0 \cap f^{-1}(x)$ and locally constant multiplicity map $Z_0 \cap f^{-1}(x) \to \mathbf{Z}$ as in the second description of PA-chains.

d) For any $x \in X$ and $z \in f^{-1}(x) \cap Z_0$, and any sequence $(x_i), x_i \to x$ the multiplicity at the point z is the "natural limit" of the multiplicities of $f^{-1}(x_i) \cap Z_0$.

In the example below, X is not compact. It easy to make necessary modifications of our definition, so it will be valid in the non compact case. In that case we get a sheaf on X.

Example 2 *Let $Y = \mathbf{R}^2, X = \mathbf{R}$ and $f : Y \to X$ is the projection $(x_1, x_2) \mapsto x_1$. Let us define $\Phi(x_1)$ to be equal $\{x_1\} + \{-x_1\}$ for $x_1 > 0$ and equal to $2\{0\}$ otherwise. Then this map gives an element of $C_0(Y \to X)$.*

We will briefly describe some properties of PA-families of cycles below.
1) PA-families $C_\bullet(Y \to X) = \{C_k(Y \to X)\}_{k \geq 0}$ form a complex of sheaves (abelian groups in the case when X is compact).
2) There are natural operations $C_k(Y \to X) \otimes C_l(X) \to C_{k+l}(Y)$. Informally we will denote them by $(\gamma, \alpha) \mapsto \gamma \times \alpha$.
3) For any Cartesian square:

$$\begin{array}{ll} Y' & \to Y \\ \downarrow & \quad \downarrow \\ X' & \to X \end{array}$$

one has the natural morphisms

$$C_k(Y \to X) \to C_k(Y' \to X')$$

and

$$C_k(Y \to X) \otimes C_l(X' \to X) \to C_{k+l}(Y' \to X).$$

Now we are ready to define piecewise PA-forms.

Definition 23 *PA-forms of degree k are sections of soft sheaves which are locally given by functionals $l : C_k(X) \to \mathbf{R}$ such that*

$$l(\alpha) = \int_{\gamma \times \alpha} \omega$$

where $\gamma \in C_l(Y \to X), l \geq 0$ for some proper $Y \to X$, and $\omega \in \Omega^{k+l}(Y)_{min}$.

We will denote the space of PA-forms on X by $\Omega^\bullet_{PA}(X)$.

Similarly to the case of $\Omega^\bullet_{min}(X)$ one can see that at the "generic point" the space $\Omega^\bullet_{PA}(X)$ can be naturally embedded into the space of differential forms. In fact we obtain a soft sheaf of dg-algebras. It is closed under pushforwards f_* where $f : Y \to X$ is a locally trivial bundle in the category $\mathscr{P}$ with fibers which are compact oriented PA-manifolds.

The latter fact can be generalized further. One can consider a family $f : Y \to X$ where Y and X are oriented compact PA-manifolds, and all fibers of f have the same dimension $dim(Y) - dim(X)$. Then one get a continuous family of cycles ("fundamental cycle" of $f^{-1}(x), x \in X$) over X. This gives a pushforward of PA-forms for certain maps ("flat morphisms") which are not necessarily fibrations.

Notice that the Poincaré lemma holds for piecewise forms. One can imitate the usual proof based on the integration over rays $(x,t) \mapsto xt$, where $x \in U \subset \mathbf{R}^n$, U is a convex domain containing $0 \in \mathbf{R}^n$, and $t \in [0,1]$.

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