Variations on deformation quantization

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Abstract. I was asked by the organizers to present some aspects of deformation quantization. Moshé has pursued, for more than 25 years, a research program based on the idea that physics progresses in stages, and one goes from one level of the theory to the next one by a deformation, in the mathematical sense of the word, to be defined in an appropriate category. His study of deformation theory applied to mechanics started in 1974 and led to spectacular developments with the deformation quantization program.

I first met Moshé at a conference in Liège in 1977. A few months later he became my thesis "codirecteur". Since then he has been one of my closest friends, present at all stages of my personal and mathematical life. I miss him....

I have chosen, in this presentation of deformation quantization, to focus on 3 points: the uniqueness – up to equivalence – of a universal star-product (universal in the sense of Kontsevich) on the dual of a Lie algebra, the cohomology classes introduced by Deligne for equivalence classes of differential star-products on a symplectic manifold, and the construction of some convergent star-products on Hermitian symmetric spaces. Those subjects will appear in a promenade through the history of existence and equivalence in deformation quantization.

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1. Introduction

Quantization of a classical system is a way to pass from classical to quantum results.

Classical mechanics, in its Hamiltonian formulation on the motion space, has for framework a symplectic manifold (or more generally a Poisson manifold). Observables are families of smooth functions on that manifold M. The dynamics is defined in terms of a Hamiltonian $H \in C^\infty(M)$ and the time evolution of an observable $f_t \in C^\infty(M \times \mathbb{R})$ is governed by the equation:

$$\frac{d}{dt} f_t = -\{H, f_t\}\,.$$

Quantum mechanics, in its usual Heisenberg's formulation, has for framework a Hilbert space (states are rays in that space). Observables are families of self-adjoint operators on the Hilbert space. The dynamics is defined in terms of a Hamiltonian

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H, which is a self-adjoint operator, and the time evolution of an observable A_t is governed by the equation:

$$\frac{dA_t}{dt} = \frac{i}{\hbar}[H, A_t].$$

A natural suggestion for quantization is a correspondence $Q\colon f \mapsto Q(f)$ mapping a function f to a self-adjoint operator $Q(f)$ on a Hilbert space $\mathscr{H}$ in such a way that $Q(1) = \mathrm{Id}$ and

$$[Q(f), Q(g)] = i\hbar Q(\{f, g\}).$$

There is no such correspondence defined on all smooth functions on M when one puts an irreducibility requirement which is necessary not to violate Heisenberg's principle.

Different mathematical treatments of quantization appeared to deal with this problem:

- Geometric Quantization of Kostant and Souriau. This proceeds in two steps; first prequantization of a symplectic manifold (M, ω) where one builds a Hilbert space and a correspondence Q as above defined on all smooth functions on M but with no irreducibility, then polarization to "cut down the number of variables". One succeeds to quantize only a small class of functions.

- Berezin's quantization where one builds on a particular class of Kähler manifolds a family of associative algebras using a symbolic calculus, i.e., a dequantization procedure.

- Deformation Quantization introduced by Flato, Lichnerowicz, and Sternheimer in [47] and developed in [10] where they "suggest that quantization be understood as a deformation of the structure of the algebra of classical observables rather than a radical change in the nature of the observables."

This deformation approach to quantization is part of a general deformation approach to physics. This was one of the seminal ideas stressed by Moshé [45]: One looks at some level of a theory in physics as a deformation of another level.

Deformation quantization is defined in terms of a star-product which is a formal deformation of the algebraic structure of the space of smooth functions on a Poisson manifold. The associative structure given by the usual product of functions and the Lie structure given by the Poisson bracket are simultaneously deformed.

The plan of this presentation is the following:

- Examples and existence of star-products. After some definitions, I recall the history of the proofs of existence and the history of some constructions of star-products, starting from the Moyal star-product on a vector space endowed with a constant Poisson structure. I give the standard star-product on the dual of a Lie algebra, and recall Kontsevich star-product for any Poisson structure on a vector space. I prove by elementary methods that all universal star-products on the dual of a Lie algebra are essentially equivalent.

- Equivalence of star-products. I first wander through the history of the parametrization of equivalence classes of star-products. I define Deligne's cohomology classes associated to differential star-products on symplectic manifolds and show by Čech methods how this yields an intrinsic parametrization of the equivalence classes of differential star-products and how this allows to study automorphisms of a star-product (in the symplectic framework). Finally, I define a generalized moment map for star-products.

- Convergence of star-products. I study mostly the convergence of a Berezin type star-product on Hermitian symmetric spaces. The construction of such a star-product involves a correspondence between operators (on the Hilbert space given by geometric quantization) and functions (their Berezin symbols). A parameter is introduced in the construction (generalizing the power of the line bundle in geometric quantization). Asymptotic expansion in this parameter on a large algebra of functions yields a deformed product. One proves associativity and convergence of this product.

I would like to thank my friends Georges Pinczon and Daniel Sternheimer who brought many improvements to this presentation.

2. Examples and existence of star-products

Definition 1 *A* ***Poisson bracket*** *defined on the space of smooth functions on a manifold M, is an $\mathbb{R}$-bilinear map on $C^\infty(M)$, $(u,v) \mapsto \{u,v\}$ such that for any $u,v,w \in C^\infty(M)$:*
- $\{u,v\} = -\{v,u\}$;
- $\{\{u,v\},w\} + \{\{v,w\},u\} + \{\{w,u\},v\} = 0$;
- $\{u,vw\} = \{u,v\}w + \{u,w\}v$.

A Poisson bracket is given in terms of a contravariant skew-symmetric 2-tensor P on M, called the ***Poisson tensor****, by*

$$\{u,v\} = P(du \wedge dv).$$

The Jacobi identity for the Poisson bracket Lie algebra is equivalent to the vanishing of the Schouten bracket:

$$[P,P] = 0.$$

(The Schouten bracket is the extension – as a graded derivation for the exterior product – of the bracket of vector fields to skew-symmetric contravariant tensor fields.) A ***Poisson manifold****, denoted (M,P), is a manifold M with a Poisson bracket defined by the Poisson tensor P.*

A particular class of Poisson manifolds, essential in classical mechanics, is the class of **symplectic manifolds**. If (M,ω) is a symplectic manifold (i.e., ω is a closed nondegenerate 2-form on M) and if $u,v \in C^\infty(M)$, the Poisson bracket of u and v is defined by

$$\{u,v\} := X_u(v) = \omega(X_v,X_u),$$

where X_u denotes the Hamiltonian vector field corresponding to the function u, i.e., such that $i(X_u)\omega = du$. In coordinates the components of the corresponding Poisson tensor P^{ij} form the inverse matrix of the components ω_{ij} of ω.

Duals of Lie algebras form the class of linear Poisson manifolds. If $\mathfrak{g}$ is a Lie algebra then its dual $\mathfrak{g}^*$ is endowed with the Poisson tensor P defined by

$$P_\xi(X,Y) := \xi([X,Y])$$

for $X,Y \in \mathfrak{g} = (\mathfrak{g}^*)^* \sim (T_\xi \mathfrak{g}^*)^*$.

Definition 2. **(Bayen et al. [10])** *A **star-product** on (M,P) is a bilinear map*

$$N \times N \to N[[\mu]], \qquad (u,v) \mapsto u * v = u *_\mu v := \sum_{r\geq 0} \mu^r C_r(u,v)$$

where $N = C^\infty(M)$, such that

1) When the map is extended μ-linearly (and continuously in the μ-adic topology) to $N[[\mu]] \times N[[\mu]]$, it is formally associative:

$$(u*v)*w = u*(v*w);$$

2) (a) $C_0(u,v) = uv$, (b) $C_1(u,v) - C_1(v,u) = \{u,v\}$;

3) $1*u = u*1 = u$.

When the C_r's are bidifferential operators on M, one speaks of a **differential star-product**.

Remark 3 A star-product can also be defined not on the whole of $C^\infty(M)$ but on any subspace N of it which is stable under pointwise multiplication and Poisson bracket.

Requiring differentiability of the cochains is essentially the same as requiring them to be local [20].

In (b) we follow Deligne's normalization for C_1: Its skew-symmetric part is $\frac{1}{2}\{\,,\,\}$. In the original definition it was equal to the Poisson bracket. One finds in the literature other normalizations such as $\frac{i}{2}\{\,,\,\}$. All these amount to a rescaling of the parameter μ.

One assumed also the parity condition $C_r(u,v) = (-1)^r C_r(v,u)$ in the earliest definition.

Property (b) above implies that the center of $C^\infty(M)[[\mu]]$, when the latter is viewed as an algebra with multiplication $*$, is a series whose terms Poisson commute with all functions, so is an element of $\mathbb{R}[[\mu]]$ when M is symplectic and connected.

Properties (a) and (b) of Definition 2 imply that the **star-commutator** defined by $[u,v]_* = u*v - v*u$, which obviously makes $C^\infty(M)[[\mu]]$ into a Lie algebra, has the form $[u,v]_* = \mu\{u,v\} + \cdots$ so that repeated bracketing leads to higher and higher order terms. This makes $C^\infty(M)[[\mu]]$ an example of a **pronilpotent Lie algebra**. We denote the **star adjoint representation** by $ad_* u\ (v) = [u,v]_*$.

2.1. The Moyal star-product on $\mathbb{R}^n$

The simplest example of a deformation quantization is the Moyal product for the Poisson structure P on a vector space $V = \mathbb{R}^m$ with constant coefficients:

$$P = \sum_{i,j} P^{ij} \partial_i \wedge \partial_j, \quad P^{ij} = -P^{ji} \in \mathbb{R},$$

where $\partial_i = \partial/\partial x^i$ is the partial derivative in the direction of the coordinate x^i, $i = 1,\dots,n$. The formula for the **Moyal product** is

$$(u *_M v)(z) = \exp\left(\frac{\mu}{2} P^{rs} \partial_{x^r} \partial_{y^s}\right)(u(x)v(y))\Big|_{x=y=z}. \tag{2.1}$$

When P is nondegenerate (so $V = \mathbb{R}^{2n}$), the space of formal power series of polynomials on V with Moyal product is called the Weyl algebra $W = (S(V)[[\mu]], *_M)$.

This example comes from the composition of operators via Weyl's quantization. Weyl's correspondence associates to certain functions f on $\mathbb{R}^{2n}$ an operator $W(f)$ on $L^2(\mathbb{R}^n)$

$$W(f) = \int \tilde{f}(\xi,\eta) \exp\left(\frac{i(\xi P + \eta Q)}{\hbar}\right) d\xi\, d\eta$$

where $\tilde{f}$ is the inverse Fourier transform of f. Then

$$W(f) \circ W(g) = W(f *_M g) \quad (\mu = i\hbar).$$

In fact, Moyal had used in 1949 the deformed bracket which corresponds to the commutator of operators to study quantum statistical mechanics. The Moyal product first appeared in Groenewold.

In 1974, Flato, Lichnerowicz, and Sternheimer [46] studied deformations of the Lie algebra structure defined by the Poisson bracket on the algebra N of smooth functions on a symplectic manifold; they studied 1-differential deformations because the relevant cohomology, i.e., the 1-differential Chevalley cohomology of the Lie algebra N with values in N for the adjoint representation, was known [61].

In 1975, Vey [86] pursued their work in the differential context: He constructed a differential deformation on $M = \mathbb{R}^{2n}$ which turns out to be the Moyal bracket

$\{u,v\}_M := \frac{1}{\mu}(u *_M v - v *_M u)$ and he proved that there exists a differential deformation on a symplectic manifold when its third de Rham cohomology space is trivial ($b_3(M) := \dim H^3(M;\mathbb{R}) = 0$). He also reconstructed the Moyal product on $\mathbb{R}^{2n}$.

This opened the path to deformation quantization presented in 1976 by Flato, Lichnerowicz, and Sternheimer [47].

In 1978, in their seminal paper about deformation quantization [10], Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer proved that the Moyal star-product can be defined on any symplectic manifold (M,ω) which admits a symplectic connection ∇ (i.e., a linear connection such that $\nabla\omega = 0$ and the torsion of ∇ vanishes) with no curvature. They also built star-products on some quotient of $\mathbb{R}^{2n}$.

In 1979, Neroslavsky and Vlassov [68] proved with Lichnerowicz that on any symplectic manifold with $b_3(M) = 0$, there exists a differential star-product.

2.2. Hochschild cohomology

The study of star-products on a manifold M used Gerstenhaber theory of deformations [49] of associative algebras. This uses the Hochschild cohomology of the algebra, here $C^\infty(M)$ with values in $C^\infty(M)$, where p-cochains are p-linear maps from $(C^\infty(M))^p$ to $C^\infty(M)$ and where the **Hochschild coboundary operator** maps the p-cochain C to the $p+1$-cochain

$$(\partial C)(u_0,\dots,u_p) = u_0 C(u_1,\dots,u_p) \\ + \sum_{r=1}^{p}(-1)^r C(u_0,\dots,u_{r-1}u_r,\dots,u_p) + (-1)^{p+1}C(u_0,\dots,u_{p-1})u_p.$$

For differential star-products, we consider differential cochains, i.e., given by differential operators in each argument. The associativity condition for a star-product at order k in the parameter μ reads

$$(\partial C_k)(u,v,w) = \sum_{\substack{r+s=k \\ r,s>0}} (\ C_r(C_s(u,v),w) - C_r(u,C_s(v,w))\ ).$$

If one has cochains C_j, $j<k$, such that the star-product they define is associative up to order $k-1$, then the right-hand side above is a cocycle (∂(RHS)$=0$) and one can extend the star-product to order k if it is a coboundary (RHS$=\partial(C_k)$).

Denoting by m the usual multiplication of functions, and writing $* = m + C$ where C is a formal series of multidifferential operators ($C \in D_{poly}(M)[[\mu]]$) the associativity also reads $\partial C = [C,C]$ where the bracket on the right-hand side is the graded Lie algebra bracket on $D_{poly}(M)[[\mu]]$.

Theorem 4. (Vey [86]) *Every differential p-cocycle C on a manifold M is the sum of the coboundary of a differential (p-1)-cochain and a* 1*-differential skew-symmetric p-cocycle A:*

$$C = \partial B + A.$$

In particular, a cocycle is a coboundary if and only if its total skew-symmetrization, which is automatically 1*-differential in each argument, vanishes. Also*

$$H^p_{\mathrm{diff}}(C^\infty(M), C^\infty(M)) = \Gamma(\Lambda^p TM).$$

Furthermore ([24]),given a connection ∇ *on* M*,* B *can be defined from* C *by universal formulas.*

By universal, we mean the following: Any p-differential operator D of order maximum k in each argument can be written

$$D(u_1, \dots, u_p) = \sum_{|\alpha_1|<k,\dots,|\alpha_p|<k} D^{\alpha_1 \cdots \alpha_p}_{|\alpha_1|,\dots,|\alpha_p|} \nabla_{\alpha_1} u_1 \cdots \nabla_{\alpha_p} u_p$$

where α's are multi-indices, $D_{|\alpha_1|,\dots,|\alpha_p|}$ are tensors (symmetric in each of the p groups of indices) and $\nabla_\alpha u = (\nabla \cdots (\nabla u))(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_q}})$ when $\alpha = (i_1, \dots, i_q)$. We claim that there is a B such that the tensors defining B are universally defined as linear combinations of the tensors defining C, universally meaning in a way which is independent of the form of C. Note that requiring differentiability of the cochains is essentially the same as requiring them to be local [20]. (An elementary proof of the above theorem can be found in [54].)

Remark 5 Behind Theorem 4 above, there are the following stronger results about Hochschild cohomology:

Theorem 6 *Let* $\mathscr{A} = C^\infty(M)$*, let* $\mathscr{C}(\mathscr{A})$ *be the space of continuous cochains and* $\mathscr{C}_{diff}(\mathscr{A})$ *the space of differential cochains. Then*

1) $\Gamma(\Lambda^p TM) \subset H^p(C^\infty(M), C^\infty(M))$*;*

2) the inclusions $\Gamma(\Lambda^p TM) \subset \mathscr{C}_{diff}(\mathscr{A}) \subset \mathscr{C}(\mathscr{A})$ *induce isomorphisms in cohomology.*

Point 1 follows from the fact that any cochain which is 1-differential in each argument is a cocycle and that the skew-symmetric part of a coboundary always vanishes. The fact that the inclusion $\Gamma(\Lambda TM) \subset \mathscr{C}_{diff}(\mathscr{A})$ induces an isomorphism in cohomology is proven by Vey [86]; it gives theorem 4. The general result about continuous cochains is due to Connes [29]. Another proof of Connes result was given by Nadaud in [67]. In the somewhat pathological case of completely general cochains the full cohomology does not seem to be known.

Examples of star-products were built using inductively the explicit formulas for coboundaries on locally symmetric symplectic manifolds in [24]; the cochains are given by universal expressions in the symplectic 2-form, its inverse (i.e., the Poisson tensor) and the curvature tensor of the symmetric symplectic connection.

We also showed that assuming a homogeneity condition on the cochains proves the existence of a star-product on the cotangent bundle to any parallelizable manifold [22] and gave explicit formulas for the cotangent bundle to a Lie group. The vertical part of this gives a deformation quantization of any linear Poisson manifold, i.e., any dual of a Lie algebra [52].

2.3. The standard *-product on $\mathfrak{g}^*$

Let $\mathfrak{g}^*$ be the dual of a Lie algebra $\mathfrak{g}$. The algebra of polynomials on $\mathfrak{g}^*$ is identified with the symmetric algebra $S(\mathfrak{g})$. One defines a new associative law on this algebra by a transfer of the product $\circ$ in the universal enveloping algebra $U(\mathfrak{g})$, via the bijection between $S(\mathfrak{g})$ and $U(\mathfrak{g})$ given by the total symmetrization σ:

$$\sigma : S(\mathfrak{g}) \to U(\mathfrak{g}), \quad X_1 \cdots X_k \mapsto \frac{1}{k!} \sum_{\rho \in S_k} X_{\rho(1)} \circ \cdots \circ X_{\rho(k)}.$$

Then $U(\mathfrak{g}) = \oplus_{n \geq 0} U_n$ where $U_n := \sigma(S^n(\mathfrak{g}))$ and we decompose an element $u \in U(\mathfrak{g})$ accordingly $u = \sum u_n$. We define for $P \in S^p(\mathfrak{g})$ and $Q \in S^q(\mathfrak{g})$

$$P * Q = \sum_{n \geq 0} (\mu)^n \sigma^{-1}((\sigma(P) \circ \sigma(Q))_{p+q-n}). \tag{2.2}$$

This yields a differential star-product on $\mathfrak{g}^*$ [52]. Using Vergne's result on the multiplication in $U(\mathfrak{g})$, this star-product is characterized by

$$\begin{aligned} X * X_1 \ldots X_k &= XX_1 \cdots X_k \\ &+ \sum_{j=1}^{k} \frac{(-1)^j}{j!} \mu^j Bj[[[X, X_{r_1}], \cdots], X_{r_j}] X_1 \cdots \widehat{X_{r_1}} \cdots \widehat{X_{r_j}} \cdots X_k, \end{aligned}$$

where B_j are the Bernouilli numbers. This star-product can be written with an integral formula (for $\mu = 2\pi i$) [37]:

$$u * v(\xi) = \int_{\mathfrak{g} \times \mathfrak{g}} \hat{u}(X) \hat{v}(Y) e^{2i\pi\langle \xi, CBH(X,Y) \rangle} dX dY,$$

where $\hat{u}(X) = \int_{\mathfrak{g}^*} u(\eta) e^{-2i\pi\langle \eta, X \rangle}$ and CBH denotes Campbell-Baker-Hausdorff formula for the product of elements in the group in a logarithmic chart: $\exp X \exp Y = \exp CBH(X,Y)$, $\forall X, Y \in \mathfrak{g}$.

We call this the **standard (or CBH) star-product** on the dual of a Lie algebra.

An important property of this star-product is its covariance, i.e., the fact that

$$X * Y - Y * X = \mu[X, Y], \quad \forall X, Y \in \mathfrak{g}.$$

In general, when there is a Hamiltonian action of a connected Lie group G on a manifold M, denoting by λ_X the function on M corresponding to $X \in \mathfrak{g}$, a star-product on M is **covariant** under G if $\lambda_X * \lambda_Y - \lambda_Y * \lambda_X = \mu \lambda_{\{X,Y\}}$, $\forall X, Y \in \mathfrak{g}$.

Arnal, Cortet, Molin, and Pinczon have proven in [7] that for any covariant star-product there is a representation of G into the automorphisms of the star-product. We shall come back to this notion of covariance in §3.2.

A program to study representations through deformation quantization methods was introduced in 1978 by Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer in [10, 82] and in Fronsdal [48]. The notion of covariant star-products is essential for this. The program involves a beautiful notion of adapted Fourier transform (or $*$ exponential). I shall not speak about this aspect of deformation quantization; let me just mention that the program to study representations through deformations was carried first in the case of nilpotent Lie groups in the mid-eighties by Arnal and Cortet [5] and that the adapted Fourier transform they found [6] gives a Kirillov type formula for the character of the representation and a localization of the (packets of) coefficients of representations on the corresponding orbits in the dual of the Lie algebra. This theory of star-representations was developed for all representations of compact groups and exponential groups, and for some series of representations of semi-simple groups.

The standard star-product on $\mathfrak{g}^*$ does not restrict to orbits (except for the Heisenberg group) so other algebraic constructions of star-products on $S(\mathfrak{g})$ were considered (with Michel Cahen in [23], with Cahen and Arnal in [4], by Arnal, Ludwig, and Masmoudi in [8] and more recently by Fioresi and Lledo in [44]). For instance, when $\mathfrak{g}$ is semisimple, if $\mathscr{H}$ is the space of harmonic polynomials and if $I_1, \ldots, I_r$ are generators of the space of invariant polynomials, then any polynomial $P \in S(\mathfrak{g})$ can be uniquely written as a sum $P = \sum_{a_1 \ldots a_r} I_1^{a_1} \ldots I_r^{a_r} h_{a_1 \ldots a_r}$ where $h_{a_1 \ldots a_r} \in \mathscr{H}$. One considers the isomorphism σ' between $S(\mathfrak{g})$ and $U(\mathfrak{g})$ induced by this decomposition

$$\sigma'(P) = \sum_{a_1 \ldots a_r} (\sigma(I_1)\circ)^{a_1} \cdots (\sigma(I_r)\circ)^{a_r} \circ \sigma(h_{a_1 \ldots a_r}).$$

This gives a star-product on $S(\mathfrak{g})$ which is not defined by differential operators. In fact, with Cahen and Rawnsley, we proved [27] that if $\mathfrak{g}$ is semisimple, there is no differential star-product on any neighbourhood of 0 in $\mathfrak{g}^*$ such that $C * u = Cu$ for the quadratic invariant polynomial $C \in S(\mathfrak{g})$ and $\forall u \in S(\mathfrak{g})$ (thus no differential star-product which is tangential to the orbits).

In 1983, De Wilde and Lecomte proved [33] that on any symplectic manifold there exists a differential star-product. This was obtained by imagining a very clever generalization of a homogeneity condition consisting here in building at the same time the star-product and a set of local special derivations of it. A very nice presentation of this proof appears in [32]. Their technique works to prove the existence of a differential star-product on a regular Poisson manifold [64].

In 1985, but appearing only in the West in the nineties [40], Fedosov gave a recursive construction of a star-product on a symplectic manifold (M, ω) by:

– Considering the Weyl bundle $\mathscr{W}$ which is the bundle on M in associative Weyl algebras W associated to the principal bundle of symplectic frames;

– Building, from a symplectic connection ∇ on M, a covariant derivative on $\Gamma(\mathscr{W})$, $\partial = \partial^{\nabla} + [r, .]$ $(r \in \Gamma(T^*M \otimes \mathscr{W}))$, such that $\partial \circ \partial = 0$;

– Identifying $C^\infty(M)[[\mu]]$ with $\{s \in \Gamma(\mathscr{W}) \mid \partial s = 0\}$ and transferring the associative pointwise product of sections to an associative product on $C^\infty(M)[[\mu]]$.

In 1994, he extended this result to give a recursive construction in the context of regular Poisson manifold [41].

Independently, also using the framework of Weyl bundles, Omori, Maeda and Yoshioka [72] gave an alternative proof of existence of a differential star-product on a symplectic manifold by gluing local Moyal star-products.

In 1986, Drinfeld [37] proposed a program of quantization for Poisson-Lie groups. One way to consider the problem is to study deformations of the Hopf algebra $U(\mathfrak{g})$ of a finite-dimensional Lie bialgebra $(\mathfrak{g}, p)$, where one deforms the product and the coproduct, the deformation of the coproduct being driven by the cocycle p. By duality this is the quantization of formal Poisson-Lie groups. For the standard structures on semisimple Lie algebras, this was solved by Drinfeld; it is the construction of classical **quantum groups**. Drinfeld showed that the quantization is preferred, i.e., the product in $U(\mathfrak{g})$ is unchanged (or, by duality, the coproduct on formal series – which are the functions on the formal Poisson-Lie group – is unchanged). Etingof and Kazhdan [38] proved that one can deform $U(\mathfrak{g})$ for any Lie bialgebra $(\mathfrak{g}, p)$, so one can always quantize formal Poisson-Lie groups. They proved that the deformation is differential. Pinczon [77] proved that any preferred quantization of a formal Poisson-Lie group is differential.

Quantum groups became very popular and had fundamental applications. It is not my goal to describe those developments but they brought new interest in the question of deformation quantization of general Poisson manifolds.

Indeed, another way to consider the problem of quantization of a Poisson-Lie group (G, P) is to study deformations of the Hopf algebra $C^\infty(G)$, where the deformation of the product of functions is driven by P.

In the papers [17] by Bonneau, Flato, Gerstenhaber, and Pinczon and [16] by Bidegain and Pinczon, the duality between those two approaches is proven in the framework of topological deformations. In particular, any classical quantum group gives, by topological duality, a differential deformation of $C^\infty(G)$. In [15], Bidegain and Pinczon have built a differential preferred deformation of $C^\infty(G)$ for any Poisson-Lie group (G, P) with G semisimple. They also show that any preferred deformation of $C^\infty(G)$ is automatically differential. Then Etingof and Kazhdan [39] proved that there exists a differential deformation of $C^\infty(G)$ for any Poisson-Lie group (G, P), deformation which is preferred in the quasi-triangular case.

The existence of a star-product on other classes of Poisson manifolds was studied by various authors (Omori-Maeda-Yoshioka [75], Tamarkin [84], Asin [9].)

In 1997, Kontsevich [59] gave a proof of the existence of a star-product on any Poisson manifold and gave an explicit formula for a star-product for any Poisson structure on $V = \mathbb{R}^m$. This appeared as a consequence of the proof of his formality theorem. Tamarkin [85] gave a version of the proof in the framework of the theory of operads.

2.4. Kontsevich star-product on V

Let M be a domain in $V = \mathbb{R}^m$ and P be any Poisson structure on M. Kontsevich builds a star-product on (M,P), where the star-product of two functions u and v is given in terms of some universal polydifferential operators applied to the coefficients of the bivector field P and to the functions u,v; the formula is invariant under affine transformations of $\mathbb{R}^m$ and the description of the k^{th} cochain uses a special class G_k of oriented labelled graphs.

An (oriented) graph Γ is a pair (V_Γ, E_Γ) of two finite sets such that E_Γ is a subset of $V_\Gamma \times V_\Gamma$; elements of V_Γ are vertices of Γ, elements of E_Γ are edges of Γ. If $e = (v_1, v_2) \in E_\Gamma \subseteq V_\Gamma \times V_\Gamma$ is an edge it is said that e starts at v_1 and ends at v_2.

A graph Γ belongs to G_k if Γ has $k+2$ vertices $V_\Gamma = \{1,\dots,k\} \sqcup \{L,R\}$ and $2k$ labelled edges (with no multiple edges and no edge of the form (v,v) for $v \in V_\Gamma$), $E_\Gamma = \{e_1^1, e_1^2, e_2^1, e_2^2, \dots, e_k^1, e_k^2\}$ where e_j^1 and e_j^2 start at j.

To each labelled graph $\Gamma \in G_k$ and each skew-symmetric 2-tensor P is associated a bidifferential operator $C_\Gamma(P)$ on $M \subset \mathbb{R}^m$:

$$\begin{aligned}(C_\Gamma(P))(u,v) :\, = & \sum_{I:E_\Gamma \longrightarrow \{1,\dots,m\}} [\prod_{j=1}^{k} (\prod_{e \in E_\Gamma \mid e=(*,j)} \partial_{I(e)}) \ P^{I(e_j^1)I(e_j^2)}] \\ & \times \ (\prod_{e \in E_\Gamma \mid e=(*,L)} \partial_{I(e)}) \ u \times (\prod_{e \in E_\Gamma \mid e=(*,R)} \partial_{I(e)}) \ v.\end{aligned}$$

A weight $w_\Gamma \in \mathbb{R}$ is associated with each graph $\Gamma \in G_k$. Denote by $\mathscr{H}_k$ the space of configurations of k numbered distinct points in the upper half plane $\mathscr{H} = \{z \in \mathbb{C} \mid Im(z) > 0\}$:

$$\mathscr{H}_k = \{(p_1,\dots,p_k) \mid p_j \in \mathscr{H} \quad p_i \neq p_j \quad \text{for} \quad i \neq j\}.$$

For $p \in \mathscr{H}$, $q \in \mathscr{H} \sqcup \mathbb{R}$ define

$$\phi^h(p,q) = \frac{1}{2i} Log \left(\frac{(q-p)(\overline{q}-p)}{(q-\overline{p})(\overline{q}-\overline{p})} \right).$$

If $\Gamma \in G_k$ is a graph as above, assign a point $p_j \in \mathscr{H}$ to the vertex j for $1 \leq j \leq k$, point $0 \in \mathbb{R} \subset \mathbb{C}$ to the vertex L, and point $1 \in \mathbb{R} \subset \mathbb{C}$ to the vertex R. Every edge $e \in E_\Gamma$ defines an ordered pair (p,q) of points on $\mathscr{H} \sqcup \mathbb{R}$, thus a function $\phi_e^h = \phi^h(p,q)$ on $\mathscr{H}_k$ with values in $\mathbb{R}/2\pi\mathbb{Z}$.

The weight of Γ is defined by

$$w_\Gamma = \frac{1}{k!(2\pi)^{2k}} \int_{\mathscr{H}_k} \wedge_{i=1}^{k} (d\phi^h_{e_i^1} \wedge d\phi^h_{e_i^2}).$$

Theorem 7. **(Kontsevich [59])** *Let P be a Poisson tensor on a domain of $\mathbb{R}^m$. Then*

$$u *_\mu v := \sum_{k=0}^{\infty} \mu^k \sum_{\Gamma\in G_k} w_\Gamma(C_\Gamma(P))(u,v)$$

defines a differential star-product.

Definition 8 *A star-product on $\mathbb{R}^m$ is* **universal** *if it is given, for any Poisson tensor P by*

$$u *_\mu v = uv + \sum_{n>0} \mu^n (C_n(P))(u,v) \quad \textit{with} \quad C_n(P) = \sum_{\Gamma\in G} w'(\Gamma) C_\Gamma(P),$$

where the $w'(\Gamma)$ are scalars which are independent of P.

Proposition 9 *Two universal star-products $*_\mu$ and $*'_\mu$ on $\mathfrak{g}^*$ are always equivalent modulo a change of parameter. This means that there exists a series $T(\mu) = \mathrm{Id} + \sum_{r=1}^{\infty} \mu^r T_r$ of linear universal differential operators and a series $\mu' = \mu + \sum_{n>1} f_n \mu^n$ such that*

$$u *'_{\mu'} v = T^{-1}(\mu)(T(\mu) u *_\mu T(\mu) v).$$

This was first proven by Arnal et al. [3]; we give here an elementary proof of this fact.

Proof. Assume by induction that $*$ and $*'$ coincide at order $n-1$. Associativity relation at order n implies $\partial C_n(P) = \partial C'_n(P)$; thus $C_n(P) = C'_n(P) + \partial E_n(P) + A_n(P)$ where E_n is a universal differential operator and $A_n(P)$ is universal, skew-symmetric and of order $(1,1)$. Clearly, if A_n corresponds to a graph in G_k, it has $2k$ arrows and $k+2$ vertices, but since P is linear on $\mathfrak{g}^*$, at most one arrow can end at any of the first k vertices and since A_n is 1-differential in each argument, exactly one arrow ends at each of the last 2 vertices. Hence $2k \le k+2$ so $k \le 2$; but if $k = 1$ the only graph yields the Poisson bracket of functions so this can be cancelled by a change of parameter and if $k = 2$ the graph corresponds to a symmetric bidifferential operator, hence does not yield an $A_n(P)$. So we can assume that A_n vanishes but then the equivalence through $T(\mu) = \mathrm{Id} + \mu^n E_n(P)$ builds a star-product which is universal and coincide with $*$ at order n.

Remark 10 When a covariance condition or a homogeneity condition ($C_n(sP) = s^n C_n(P)$) is added, clearly there is no need for a change of parameter since the graph in G_1 can only arise in C_1 so two such universal star-products are always equivalent.

In particular, the star-product of Kontsevich and the standard CBH star-product on $\mathfrak{g}^*$, which in general are not the same, are equivalent. The equivalence is given by universal differential operators, hence combinations of wheels which are graphs consisting of $k+1$ points $1,\ldots,k$ and L, and $2k$ arrows: $(1,2),(2,3),\ldots,(k-1,k)$, $(k,1)$ and (j,L), $1 \le j \le k$. Such a wheel clearly vanishes when the Poisson structure corresponds to a nilpotent Lie algebra so there is only one covariant universal star-product on the dual of a nilpotent Lie algebra (this appears in Arnal [2] and Kathotia [58]).

The equivalence between Kontsevich and CBH star-product has been explicitly constructed (see Arnal et al. [3] and Dito [36]) and gives an integral formula for Kontsevich star-product:

$$u * v(\xi) = \int \tilde{u}(X)\tilde{v}(Y)\frac{F(X)F(Y)}{F(CBH(X,Y))}\exp(2\pi i\langle\xi, CBH(X,Y)\rangle)dXdY,$$

where F is some formal function on $\mathfrak{g}$ written as a sum of products of traces of powers of $\mathrm{ad}(X)$.

This star-product has been used recently by Andler, Dvorsky and Sahi [1] to establish a conjecture of Kashiwara and Vergne which, in turn, gives a new proof of Duflo's result on the local solvability of bi-invariant differential operators on a Lie group.

3. Equivalence of star-products

Definition 11 *Two star-products $*$ and $*'$ on (M,P) are said to be* ***equivalent*** *if there is a series*

$$T = \mathrm{Id} + \sum_{r=1}^{\infty} \mu^r T_r,$$

where the T_r are linear operators on $C^\infty(M)$, such that

$$T(f * g) = Tf *' Tg. \tag{3.1}$$

Remark that the T_r automatically vanish on constants since 1 is a unit for $*$ and for $*'$. Using in a similar way linear operators which do not necessarily vanish on constants, one can pass from any associative deformation of the product of functions on a Poisson manifold (M,P) to another such deformation with 1 being a unit.

Definition 12 *A* ***Poisson deformation*** *of the Poisson bracket on a Poisson manifold (M,P) is a Lie algebra deformation of $(C^\infty(M),\{\ ,\ \})$ which is a derivation in each argument, i.e., of the form $\{u,v\}_\mu = P_\mu(du,dv)$ where $P_\mu = P + \sum \mu^k P_k$ is a series of skew-symmetric contravariant 2-tensors on M (such that $[P_\mu, P_\mu] = 0$).*

Two Poisson deformations P_μ and P'_μ of the Poisson bracket P on a Poisson manifold (M,P) are ***equivalent*** *if there exists a formal path in the diffeomorphism group of M, starting at the identity, i.e., a series $T = \exp D = \mathrm{Id} + \sum_j \frac{1}{j!}D^j$ for $D = \sum_{r\ge 1}\mu^r D_r$ where the D_r are vector fields on M, such that*

$$T\{u,v\}_\mu = \{Tu, Tv\}'_\mu$$

where $\{u,v\}_\mu = P_\mu(du,dv)$ and $\{u,v\}'_\mu = P'_\mu(du,dv)$.

In the general theory of deformations, Gerstenhaber [49] showed how equivalence is linked to some second cohomology space.

For symplectic manifolds, Flato, Lichnerowicz, and Sternheimer in 1974 studied 1-differential deformations of the Poisson bracket [47]; it follows from their work, and appears in Lecomte [60], that:

Proposition 13 *The equivalence classes of Poisson deformations of the Poisson bracket P on a symplectic manifold (M,ω) are parametrized by $H^2(M;\mathbb{R})[[\mu]]$.*

Indeed, one first shows that any Poisson deformation P_μ of the Poisson bracket P on a symplectic manifold (M,ω) is of the form P^Ω for a series $\Omega = \omega + \sum_{k\geq 1}\mu^k\omega_k$ where the ω_k are closed 2-forms, and $P^\Omega(du,dv) = -\Omega(X_u^\Omega, X_v^\Omega)$ where $X_u^\Omega = X_u + \mu(\dots) \in \Gamma(TM)[[\mu]]$ is the element defined by $i(X_u^\Omega)\Omega = du$.

> Observe that for any series Ω as above the series of 1-differential 2-cochains P^Ω satisfies $[P^\Omega, P^\Omega] = 0$ because Ω is a closed 2-form, so defines indeed a Poisson deformation of P.
>
> Conversely, given a 1-differential deformation $P_\mu = P + \sum_j \mu^j P_j$, assume it coincides up to order k ($k \geq 0$) with P^{Ω_k} for some $\Omega_k = \omega + \sum_{j\geq 1}^k \mu^j\omega_j$ then $[P_\mu, P_\mu] - [P^{\Omega_k}, P^{\Omega_k}] = 0$ imply at order $k+1$ that $[P, P_{k+1} - (P^{\Omega_k})_{k+1} = 0]$ so that there exists a closed 2-form ω_{k+1} such that $(P_{k+1} - (P^{\Omega_k})_{k+1})(du,dv) = \omega_{k+1}(X_u, X_v)$. Hence P_μ coincides up to order $k+1$ with $P^{\Omega_{k+1}}$ where $\Omega_{k+1} = \omega + \sum_{j\geq 1}^{k+1}\mu^j\omega_j$. By induction, any Poisson deformation of P is of the form P^Ω for a series $\Omega = \omega + \sum_{k\geq 1}\mu^k\omega_k$ where the ω_k are closed 2-forms.

One then shows that two Poisson deformations P^Ω and $P^{\Omega'}$ are equivalent if and only if ω_k and ω'_k are cohomologous for all $k \geq 1$.

> If Ω and Ω' coincide to order $k-1$ and $\omega_k - \omega'_k = dF_k$, then $\{u,v\}_\mu$ and $T^{-1}\{Tu,Tv\}'_\mu$, with $T = \exp D$, $Du = \mu^{k-1}F_k(X_u)$, correspond to forms which coincide to order k so one has equivalence if all forms are cohomologous.
>
> Conversely, if P^Ω and $P^{\Omega'}$ are equivalent and Ω and Ω' coincide to order $k-1$, then the equivalence can be written $T = \exp D$, $Du = \sum_{j\geq k}\mu^j F_j(X_u)$, and $\omega_k - \omega'_k = dF_k$. Indeed if $Tu = u + \mu^r F_r(X_u) + \cdots$, then the relation $P^\Omega(du,dv) = T^{-1}P^{\Omega'}(dTu, dTv)$ at order $r \leq k$ yields $\omega_r - \omega'_r = dF_r$. When $r < k$ this means $dF_r = 0$; locally, on a contractible set U, $F_{r|U} = df_U$. The map $D_r : u \mapsto P^{\Omega'}(df_U, u)$ is globally defined and is a derivation of $\{\ ,\ \}'$ so $T' = \exp -\mu^r D_r \circ T = \mathrm{Id} + \mu^{r+1}F_{r+1} + \cdots$ is still an equivalence. By induction, one gets the result.

In 1978, Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer [11] stressed that different orderings in physics lead to equivalent star-products on $\mathbb{R}^{2n}$. This shows that the notion of mathematical equivalence is not the same as the notion of physical equivalence (i.e., two star-products leading to the same spectrum for each observable); we studied this difference with Cahen, Flato, and Sternheimer in [21]. Bayen et al. also proved in [10] that the Moyal star-product is the only one whose cochains are given by polynomials in the Poisson structure P; this was the first consideration of some universality property to build and classify star-products.

Recall that a star-product $*$ on (M,ω) is called differential if the 2-cochains $C_r(u,v)$ giving it are bidifferential operators. As was observed by Lichnerowicz [63] and Deligne [31]:

Proposition 14 *If $*$ and $*'$ are differential star-products and $T(u)=\sum_{r\geq 0}\mu^r T_r(u)$ is an equivalence so that $T(u*v)=T(u)*'T(v)$ then the T_r are differential operators.*

Proof. Indeed if $T=\mathrm{Id}+\mu^k T_k+\dots$ then $\partial T_k=C'_k-C_k$ is differential so C'_k-C_k is a differential 2-cocycle with vanishing skew-symmetric part but then, using Vey's formula, it is the coboundary of a differential 1-cochain E and T_k-E, being a 1-cocycle, is a vector field so T_k is differential. One then proceeds by induction, considering $T'=(\mathrm{Id}+\mu^k T_k)^{-1}\circ T=\mathrm{Id}+\mu^{k+1}T'_{k+1}+\cdots$ and the two differential star-products $*$ and $*''$, where $u*''v=(\mathrm{Id}+\mu^k T_k)^{-1}((\mathrm{Id}+\mu^k T_k)u*'(\mathrm{Id}+\mu^k T_k)v)$, which are equivalent through T' (i.e., $T'(u*v)=T'(u)*''T'(v)$).

A differential star-product is equivalent to one with linear term in μ given by $\frac{1}{2}\{u,v\}$. Indeed $C_1(u,v)$ is a Hochschild cocycle with antisymmetric part given by $\frac{1}{2}\{u,v\}$ so $C_1=\frac{1}{2}P+\partial B$ for a differential 1-cochain B. Setting $T(u)=u+\mu B(u)$ and $u*'v=T(T^{-1}(u)*T^{-1}(v))$, this equivalent star-product $*'$ has the required form.

In 1979, we proved [50] that all differential deformed brackets on $\mathbb{R}^{2n}$ (or on any symplectic manifold such that $b_2=0$) are equivalent modulo a change of the parameter, and this implies a similar result for star-products; this was proven by direct methods by Lichnerowicz [62]:

Proposition 15 *Let $*$ and $*'$ be two differential star-products on (M,ω) and suppose that $H^2(M;\mathbb{R})=0$. Then there exists a local equivalence $T=\mathrm{Id}+\sum_{k\geq 1}\mu^k T_k$ on $C^\infty(M)[[\mu]]$ such that $u*'v=T(T^{-1}u*T^{-1}v)$ for all $u,v\in C^\infty(M)[[\mu]]$.*

Proof. Let us suppose that, modulo some equivalence, the two star-products $*$ and $*'$ coincide up to order k. Then associativity at order k shows that $C_k-C'_k$ is a Hochschild 2-cocycle and so by (4) can be written as $(C_k-C'_k)(u,v)=(\partial B)(u,v)+A(X_u,X_v)$ for a 2-form A. The total skew-symmetrization of the associativity relation at order $k+1$ shows that A is a closed 2-form. Since the second cohomology vanishes, A is exact, $A=dF$. Transforming by the equivalence defined by $Tu=$

$u + \mu^{k-1} 2F(X_u)$, we can assume that the skew-symmetric part of $C_k - C'_k$ vanishes. Then $C_k - C'_k = \partial B$ where B is a differential operator. Using the equivalence defined by $T = I + \mu^k B$ we can assume that the star-products coincide, modulo an equivalence, up to order $k+1$ and the result follows from induction since two star-products always agree in their leading term.

It followed from the above proof and results similar to [50] (i.e., two star-products which are equivalent and coincide at order k differ at order $k+1$ by a Hochschild 2-cocycle whose skew-symmetric part corresponds to an exact 2-form) that at each step in μ, equivalence classes of differential star-products on a symplectic manifold (M,ω) are parametrized by $H^2(M;\mathbb{R})$, if all such deformations exist. The general existence was proven by De Wilde and Lecomte. At that time, one assumed the parity condition $C_n(u,v) = (-1)^n C_n(v,u)$, so equivalence classes of such differential star-products were parametrized by series $H^2(M;\mathbb{R})[[\mu^2]]$. The parametrization was not canonical.

In 1994, Fedosov proved that his recursive construction works in a more general setting: Given any series of closed 2-forms on a symplectic manifold (M,ω), he could build a connection on the Weyl bundle whose curvature is linked to that series and a star-product whose equivalence class only depends on the element in $H^2(M;\mathbb{R})[[\mu]]$ corresponding to that series of forms.

In 1995, Nest and Tsygan [69], then Deligne [31] and Bertelson [13] proved that any differential star-product on a symplectic manifold (M,ω) is equivalent to a Fedosov star-product and that its equivalence class is parametrized by the corresponding element in $H^2(M;\mathbb{R})[[\mu]]$.

In 1997, Kontsevich [59] proved that the coincidence of the set of equivalence classes of star and Poisson deformations is true for general Poisson manifolds:

Theorem 16 *The set of equivalence classes of differential star-products on a Poisson manifold (M,P) can be naturally identified with the set of equivalence classes of Poisson deformations of P:*

$$P_\mu = P\mu + P_2\mu^2 + \cdots \in \Gamma(X, \wedge^2 T_X)[[\mu]], \quad [P_\mu, P_\mu] = 0.$$

Remark that all results concerning parametrization of equivalence classes of differential star-products are still valid for star-products defined by local cochains or for star-products defined by continuous cochains ([53], Pinczon [77]). Parametrization of equivalence classes of special star-products have been obtained: star-products with separation of variables (by Karabegov [56]), invariant star-products on a symplectic manifold when there exists an invariant symplectic connection (with Bertelson and Bieliavsky [14]), algebraic star-products (Chloup [28], Kontsevich [59]).

The association of an element in $H^2(M;\mathbb{R})[[\mu]]$ to the equivalence class of a star-product on a symplectic manifold is one way to associate an invariant to a star-product; other such associations are obtained by star version of index theorems and trace functionals on the algebra $(C^\infty(M)[[\mu]],*)$. I shall not develop that aspect here. It was first considered by Connes, Flato, and Sternheimer in [30] where they introduce the notion of closed star-product, i.e., such that

$$\int_M a*b\ \omega^n = \int_M b*a\ \omega^n \pmod{\mu^n}\quad \forall a,b\in C^\infty(M)[[\mu]],$$

and show how their classification is linked to cyclic cohomology.

They obtain, for the cotangent bundle to a compact Riemannian manifold, its Todd class as the "character" associated to the star-product corresponding to normal ordering.

The notion of a trace for star-products and the star version of index theorems have been studied by Fedosov [41, 42, 43], and by Nest and Tsygan [69, 70].

3.1. Deligne's cohomology classes associated to differential star-products on symplectic manifolds

Deligne defines two cohomological classes associated to differential star-products on a symplectic manifold. This leads to an intrinsic way to parametrize the equivalence class of such a differential star-product. Although the question makes sense more generally for Poisson manifolds, Deligne's method depends crucially on the Darboux theorem and the uniqueness of the Moyal star-product on $\mathbb{R}^{2n}$ so the methods do not extend to general Poisson manifolds.

The first class is a relative class; fixing a star-product on the manifold, it intrinsically associates to any equivalence class of star-products an element in $H^2(M;\mathbb{R})[[\mu]]$. This is done in Čech cohomology by looking at the obstruction to gluing local equivalences.

Deligne's second class is built from special local derivations of a star-product. The same derivations played a special role in the first general existence theorem [33] for a star-product on a symplectic manifold. Deligne used some properties of Fedosov's construction and central curvature class to relate his two classes and to see how to characterize an equivalence class of star-products by the derivation related class and some extra data obtained from the second term in the deformation. With John Rawnsley [54], we did this by direct Čech methods which I shall present here.

3.1.1. *The relative class*

Let $*$ and $*'$ be two differential star-products on (M,ω). Let U be a contractible open subset of M and $N_U = C^\infty(U)$. Remark that any differential star-product on M restricts to U and $H^2(M;\mathbb{R})(U) = 0$, hence, by Proposition 15, there exists a local equivalence $T = \mathrm{Id} + \sum_{k\geq 1}\mu^k T_k$ on $N_U[[\mu]]$ so that $u *' v = T(T^{-1}u * T^{-1}v)$ for all $u,v\in N_U[[\mu]]$.

Proposition 17 *Let $*$ be a differential star-product on (M,ω) and suppose that $H^1(M;\mathbb{R})$ vanishes.*

– *Any self-equivalence* $A = \mathrm{Id} + \sum_{k\geq 1}\mu^k A_k$ *of* $*$ *is inner:* $A = \exp \mathrm{ad}_* a$ *for some* $a \in C^\infty(M)[\![\mu]\!]$.

– *Any* μ*-linear derivation of* $*$ *is of the form* $D = \sum_{i\geq 0}\mu^i D_i$ *where each* D_i *corresponds to a symplectic vector field* X_i *and is given on a contractible open set* U *by*

$$D_i u|_U = \frac{1}{\mu}\left(f_i^U * u - u * f_i^U\right)$$

if $X_i u|_U = \{f_i^U, u\}|_U$.

Indeed, one builds a recursively; assuming $A = \mathrm{Id} + \sum_{r\geq k}\mu^r A_r$ and $k \geq 1$, the condition $A(u*v) = Au * Av$ implies at order k in μ that $A_k(uv) + C_k(u,v) = A_k(u)v + uA_k(v) + C_k(u,v)$ so that A_k is a vector field. Taking the skew part of the terms in μ^{k+1} we have that A_k is a derivation of the Poisson bracket. Since $H^1(M;\mathbb{R}) = 0$, one can write $A_k(u) = \{a_{k-1}, u\}$ for some function a_{k-1}. Then $(\exp -\mathrm{ad}_* \mu^{k-1} a_{k-1}) \circ A = \mathrm{Id} + O(\mu^{k+1})$ and the induction proceeds. The proof for μ-linear derivation is similar.

The above results can be applied to the restriction of a differential star-product on (M,ω) to a contractible open set U. Set, as above, $N_U = C^\infty(U)$. If $A = \mathrm{Id} + \sum_{k\geq 1}\mu^k A_k$ is a formal linear operator on $N_U[\![\mu]\!]$ which preserves the differential star-product $*$, then there is $a \in N_U[\![\mu]\!]$ with $A = \exp \mathrm{ad}_* a$. Similarly, any local μ-linear derivation D_U of $*$ on $N_U[\![\mu]\!]$ is essentially inner: $D_U = \frac{1}{\mu}\mathrm{ad}_* d_U$ for some $d_U \in N_U[\![\mu]\!]$.

It is convenient to write the composition of automorphisms of the form $\exp \mathrm{ad}_* a$ in terms of a. In a pronilpotent situation this is done with the **Campbell–Baker–Hausdorff composition** which is denoted by $a \circ_* b$:

$$a \circ_* b = a + \int_0^1 \psi(\exp \mathrm{ad}_* a \circ \exp t\, \mathrm{ad}_* b) b\, dt$$

where

$$\psi(z) = \frac{z\log(z)}{z-1} = \sum_{n\geq 1}\left(\frac{(-1)^n}{n+1} + \frac{(-1)^{n+1}}{n}\right)(z-1)^n.$$

Notice that the formula is well-defined (at any given order in μ, only a finite number of terms arise) and it is given by the usual series

$$a \circ_* b = a + b + \frac{1}{2}[a,b]_* + \frac{1}{12}([a,[a,b]_*]_* + [b,[b,a]_*]_*)\cdots.$$

The following results are standard (N. Bourbaki, Groupes et algèbres de Lie, *Éléments de Mathématique*, Livre 9, Chapitre 2, §6):

– $\circ_*$ is an associative composition law;

– $\exp \mathrm{ad}_*(a \circ_* b) = \exp \mathrm{ad}_* a \circ \exp \mathrm{ad}_* b$;

– $a \circ_* b \circ_* (-a) = \exp(\mathrm{ad}_* a)\, b$;

– $-(a \circ_* b) = (-b) \circ_* (-a)$;

– $\left.\dfrac{d}{dt}\right|_0 (-a) \circ_* (a+tb) = \dfrac{1-\exp(-\mathrm{ad}_* a)}{\mathrm{ad}_* a}(b)$.

Let (M,ω) be a symplectic manifold. We fix a locally finite open cover $\mathscr{U} = \{U_\alpha\}_{\alpha\in I}$ by Darboux coordinate charts such that the U_α and all their non-empty intersections are contractible, and we fix a partition of unity $\{\theta_\alpha\}_{\alpha\in I}$ subordinate to $\mathscr{U}$. Set $N_\alpha = C^\infty(U_\alpha)$, $N_{\alpha\beta} = C^\infty(U_\alpha \cap U_\beta)$, and so on.

Now suppose that $*$ and $*'$ are two differential star-products on (M,ω). We have seen that their restrictions to $N_\alpha[\![\mu]\!]$ are equivalent so there exist formal differential operators $T_\alpha\colon N_\alpha[\![\mu]\!] \to N_\alpha[\![\mu]\!]$ such that

$$T_\alpha(u * v) = T_\alpha(u) *' T_\alpha(v), \qquad u,v \in N_\alpha[\![\mu]\!].$$

On $U_\alpha \cap U_\beta$, $T_\beta^{-1} \circ T_\alpha$ will be a self-equivalence of $*$ on $N_{\alpha\beta}[\![\mu]\!]$ and so there will be elements $t_{\beta\alpha} = -t_{\alpha\beta}$ in $N_{\alpha\beta}[\![\mu]\!]$ with

$$T_\beta^{-1} \circ T_\alpha = \exp \mathrm{ad}_* t_{\beta\alpha}.$$

On $U_\alpha \cap U_\beta \cap U_\gamma$ the element

$$t_{\gamma\beta\alpha} = t_{\alpha\gamma} \circ_* t_{\gamma\beta} \circ_* t_{\beta\alpha}$$

induces the identity automorphism and hence is in the center $\mathbb{R}[\![\mu]\!]$ of $N_{\alpha\beta\gamma}[\![\mu]\!]$. The family of $t_{\gamma\beta\alpha}$ is thus a Čech 2-cocycle for the covering $\mathscr{U}$ with values in $\mathbb{R}[\![\mu]\!]$. The standard arguments show that its class does not depend on the choices made, and is compatible with refinements. Since every open cover has a refinement of the kind considered it follows that $t_{\gamma\beta\alpha}$ determines a unique Čech cohomology class $[t_{\gamma\beta\alpha}] \in H^2(M;\mathbb{R})[\![\mu]\!]$.

Definition 18 $t(*',*) = [t_{\gamma\beta\alpha}] \in H^2(M;\mathbb{R})[\![\mu]\!]$ *is* ***Deligne's relative class***.

It is easy to see, using the fact that the cohomology of the sheaf of smooth functions is trivial:

Theorem 19. (Deligne) *Fixing a differential star-product $*$, the class $t(*',*)$ in $H^2(M;\mathbb{R})[\![\mu]\!]$ depends only on the equivalence class of the differential star-product $*'$, and sets up a bijection between the set of equivalence classes of differential star-products and $H^2(M;\mathbb{R})[\![\mu]\!]$.*

If $*$, $*'$, $*''$ are three differential star-products on (M,ω) then

$$t(*'',*) = t(*'',*') + t(*',*). \tag{3.2}$$

3.1.2. *The derivation related class*
The addition formula above suggests that $t(*',*)$ should be a difference of classes $c(*'), c(*) \in H^2(M;\mathbb{R})[\![\mu]\!]$. Moreover, the class $c(*)$ should determine the star-product $*$ up to equivalence.

Definition 20 *Let U be an open set of M. Say that a derivation D of $(C^\infty(U)[\![\mu]\!],*)$ is* ***μ-Euler*** *if it has the form*

$$D = \mu\frac{\partial}{\partial\mu} + X + D', \tag{3.3}$$

where X is conformally symplectic on U ($\mathscr{L}_X\omega|_U = \omega|_U$) and $D' = \sum_{r\geq 1}\mu^r D'_r$ with the D'_r differential operators on U.

Proposition 21 *Let $*$ be a differential star-product on (M,ω). For each $U_\alpha \in \mathscr{U}$ there exists a μ-Euler derivation $D_\alpha = \mu\frac{\partial}{\partial\mu} + X_\alpha + D'_\alpha$ of the algebra $(N_\alpha[\![\mu]\!],*)$.*

Proof. On an open set in $\mathbb{R}^{2n}$ with the standard symplectic structure Ω, denote the Poisson bracket by P. Let X be a conformal vector field so $\mathscr{L}_X\Omega = \Omega$. The Moyal star-product $*_M$ is given by $u *_M v = uv + \sum_{r\geq 1}(\frac{\mu}{2})^r/r!P^r(u,v)$ and $D = \mu\frac{\partial}{\partial\mu} + X$ is a derivation of $*_M$.

Now (U_α,ω) is symplectomorphic to an open set in $\mathbb{R}^{2n}$ and any differential star-product on this open set is equivalent to $*_M$ so we can pull back D and $*_M$ to U_α by a symplectomorphism to give a star-product $*'$ with a derivation of the form $\mu\frac{\partial}{\partial\mu} + X_\alpha$. If T is an equivalence of $*$ with $*'$ on U_α then $D_\alpha = T^{-1}\circ(\mu\frac{\partial}{\partial\mu} + X_\alpha)\circ T$ is a derivation of the required form.

We take such a collection of derivations D_α given by Proposition 21 and on $U_\alpha \cap U_\beta$ we consider the differences $D_\beta - D_\alpha$. They are derivations of $*$ and the μ derivatives cancel out, so $D_\beta - D_\alpha$ is a μ-linear derivation of $N_{\alpha\beta}[\![\mu]\!]$. Any μ-linear derivation is of the form $\frac{1}{\mu}\operatorname{ad}_* d$, so there are $d_{\beta\alpha} \in N_{\alpha\beta}[\![\mu]\!]$ with

$$D_\beta - D_\alpha = \frac{1}{\mu}\operatorname{ad}_* d_{\beta\alpha}, \tag{3.4}$$

with $d_{\beta\alpha}$ unique up to a central element. On $U_\alpha \cap U_\beta \cap U_\gamma$ the combination $d_{\alpha\gamma} + d_{\gamma\beta} + d_{\beta\alpha}$ must be central and hence defines $d_{\gamma\beta\alpha} \in \mathbb{R}[\![\mu]\!]$. It is easy to see that $d_{\gamma\beta\alpha}$ is a 2-cocycle whose Čech class $[d_{\gamma\beta\alpha}] \in H^2(M;\mathbb{R})[\![\mu]\!]$ does not depend on any of the choices made.

Definition 22 *$d(*) = [d_{\gamma\beta\alpha}] \in H^2(M;\mathbb{R})[\![\mu]\!]$ is* ***Deligne's intrinsic derivation related class****.*

- In fact the class considered by Deligne is actually $\frac{1}{\mu}d(*)$. A purely Čech-theoretic accounts of this class is given in Karabegov [56].
- If $*$ and $*'$ are equivalent then $d(*') = d(*)$;
- If $d(*) = \sum_{r\geq 0}\mu^r d^r(*)$ then $d^0(*) = [\omega]$ under the de Rham isomorphism, and $d^1(*) = 0$;

Consider two differential star-products $*$ and $*'$ on (M,ω) with local equivalences T_α and local μ-Euler derivations D_α for $*$. Then $D'_\alpha = T_\alpha \circ D_\alpha \circ T_\alpha^{-1}$ are local μ-Euler derivations for $*'$. Let $D_\beta - D_\alpha = \frac{1}{\mu}\,\mathrm{ad}_* d_{\beta\alpha}$ and $T_\beta^{-1}\circ T_\alpha = \exp \mathrm{ad}_* t_{\beta\alpha}$ on $U_\alpha \cap U_\beta$. Then $D'_\beta - D'_\alpha = \frac{1}{\mu}\mathrm{ad}_{*'} d'_{\beta\alpha}$ where

$$d'_{\beta\alpha} = T_\beta d_{\beta\alpha} - \mu T_\beta \circ \left(\frac{1-\exp\left(-\,\mathrm{ad}_* t_{\alpha\beta}\right)}{\mathrm{ad}_* t_{\alpha\beta}}\right)\circ D_\alpha t_{\alpha\beta}.$$

In this situation

$$d'_{\gamma\beta\alpha} = T_\alpha(d_{\gamma\beta\alpha} + \mu^2\frac{\partial}{\partial\mu}t_{\gamma\beta\alpha}).$$

This gives a direct proof of:

Theorem 23. **(Deligne)** *The relative class and the intrinsic derivation related classes of two differential star-products $*$ and $*'$ are related by*

$$\mu^2\frac{\partial}{\partial\mu}t(*',*) = d(*') - d(*). \tag{3.5}$$

3.1.3. *The characteristic class*
The formula above shows that the information which is "lost" in $d(*') - d(*)$ corresponds to the zeroth order term in μ of $t(*',*)$.

Remark 24 In [51, 35] it was shown that any bidifferential operator C, vanishing on constants, which is a 2-cocycle for the Chevalley cohomology of $(C^\infty(M),\{\ ,\ \})$ with values in $C^\infty(M)$ associated to the adjoint representation (i.e., such that $S_{u,v,w}[\{u,C(v,w)\} - C(\{u,v\},w)] = 0$ where $S_{u,v,w}$ denotes the sum over cyclic permutations of u,v and w) can be written as

$$C(u,v) = aS^3_\Gamma(u,v) + A(X_u,X_v) + [\{u,Ev\} + \{Eu,v\} - E(\{u,v\})]$$

where $a\in\mathbb{R}$, where S^3_Γ is a bidifferential 2-cocycle introduced in [10] (which vanishes on constants and is never a coboundary and whose symbol is of order 3 in each argument), where A is a closed 2-form on M and where E is a differential operator vanishing on constants. Hence

$$H^2_{\mathrm{Chev,nc}}(C^\infty(M),C^\infty(M)) = \mathbb{R}\oplus H^2(M;\mathbb{R})$$

and we define the # operator as the projection on the second factor relative to this decomposition.

Proposition 25 *Given two differential star-products* $*$ *and* $*'$*, the term of order zero in Deligne's relative class* $t(*',*) = \sum_{r\geq 0} \mu^r t^r(*',*)$ *is given by*

$$t^0(*',*) = -2(C'^{-}_2)^{\#} + 2(C^{-}_2)^{\#}.$$

If $C_1 = \frac{1}{2}\{\ ,\ \}$, then $C^{-}_2(u,v) = A(X_u, X_v)$ where A is a closed 2-form and $(C^{-}_2)^{\#} = [A]$ so it "is" the skew-symmetric part of C_2.

It follows from what we did before that the association to a differential star-product of $(C^{-}_2)^{\#}$ and $d(*)$ completely determines its equivalence class.

Definition 26 *The* ***characteristic class*** *of a differential star-product* $*$ *on* (M,ω) *is the element* $c(*)$ *of the affine space* $\frac{-[\omega]}{\mu} + H^2(M;\mathbb{R})[\![\mu]\!]$ *defined by*

$$c(*)^0 = -2(C^{-}_2)^{\#},$$
$$\frac{\partial}{\partial\mu} c(*)(\mu) = \frac{1}{\mu^2} d(*).$$

Theorem 27 *The characteristic class has the following properties:*

- *The relative class is given by*

$$t(*',*) = c(*') - c(*); \tag{3.6}$$

- *The map C from equivalence classes of star-products on* (M,ω) *to the affine space* $\frac{-[\omega]}{\mu} + H^2(M;\mathbb{R})[\![\mu]\!]$ *mapping* $[*]$ *to* $c(*)$ *is a bijection.*

- *If* $\psi\colon M \to M'$ *is a diffeomorphism and if* $*$ *is a star-product on* (M,ω) *then* $u *' v = (\psi^{-1})^*(\psi^* u * \psi^* v)$ *defines a star-product denoted* $*' = (\psi^{-1})^* *$ *on* (M',ω') *where* $\omega' = (\psi^{-1})^*\omega$*. The characteristic class is natural relative to diffeomorphisms:*

$$c((\psi^{-1})^* *) = (\psi^{-1})^* c(*). \tag{3.7}$$

- *Consider a change of parameter* $f(\mu) = \sum_{r\geq 1} \mu^r f_r$ *where* $f_r \in \mathbb{R}$ *and* $f_1 \neq 0$ *and let* $*'$ *be the star-product obtained from* $*$ *by this change of parameter, i.e.,* $u *' v = u.v + \sum_{r\geq 1} (f(\mu))^r C_r(u,v) = u.v + f_1 \mu C_1(u,v) + \mu^2((f_1)^2 C_2(u,v) + f_2 C_1(u,v)) + \cdots$*. Then* $*'$ *is a differential star-product on* (M,ω') *where* $\omega' = \frac{1}{f_1}\omega$ *and we have equivariance under a change of parameter:*

$$c(*')(\mu) = c(*)(f(\mu)). \tag{3.8}$$

The characteristic class $c(*)$ coincides (cf. Deligne [31] and Neumaier [71]) for Fedosov-type star-products with their characteristic class introduced by Fedosov as the de Rham class of the curvature of the generalized connection used to build them (up to a sign and factors of 2). That characteristic class is also studied by

Weinstein and Xu in [88]. The fact that $d(*)$ and $(C_2^-)^\#$ completely characterize the equivalence class of a star-product is also proven by Čech methods in De Wilde [32].

3.2. Automorphisms of a star-product and generalized moment map

The above proposition allows to study automorphisms of star-products on a symplectic manifold ([78], [54]).

Definition 28 *An* ***isomorphism*** *from a differential star-product $*$ on (M,ω) to a differential star-product $*'$ on (M',ω') is an $\mathbb{R}$-linear bijective map $A\colon C^\infty(M)[[\mu]] \to C^\infty(M')[[\mu]]$, continuous in the μ-adic topology (i.e., $A(\sum_r \mu^r u_r)$ is the limit of $\sum_{r\leq N} A(\mu^r u_r)$), such that*

$$A(u * v) = Au *' Av.$$

Notice that if A is such an isomorphism, then $A(\mu)$ is central for $*'$ so that $A(\mu) = f(\mu)$ where $f(\mu) \in \mathbb{R}[[\mu]]$ is without constant term to get the μ-adic continuity. Let us denote by $*''$ the differential star-product on $(M, \omega_1 = \frac{1}{f_1}\omega)$ obtained by a change of parameter

$$u *''_\mu v = u *_{f(\mu)} v = F(F^{-1}u * F^{-1}v),$$

where $F\colon C^\infty(M)[[\mu]] \to C^\infty(M)[[\mu]]$ is defined by $\sum_r \mu^r u_r \mapsto \sum_r f(\mu)^r u_r$. Define $A' : C^\infty(M)[[\mu]] \to C^\infty(M')[[\mu]]$ by $A = A' \circ F$. Then A' is a μ-linear isomorphism between $*''$ and $*'$:

$$A'(u *'' v) = A'u *' A'v.$$

At order zero in μ this yields $A'_0(u.v) = A'_0 u . A'_0 v$ so that there exists a diffeomorphism $\psi : M' \to M$ with $A'_0 u = \psi^* u$. The skew-symmetric part of the isomorphism relation at order 1 in μ implies that $\psi^*\omega_1 = \omega'$. Let us denote by $*'''$ the differential star-product on (M, ω_1) obtained by pullback via ψ of $*'$:

$$u *''' v = (\psi^{-1})^*(\psi^* u *' \psi^* v)$$

and define $B : C^\infty(M)[[\mu]] \to C^\infty(M)[[\mu]]$ so that $A' = \psi^* \circ B$. Then B is μ-linear, starts with the identity and

$$B(u *'' v) = Bu *''' Bv$$

so that B is an equivalence – in the usual sense – between $*''$ and $*'''$. Hence [54]

Proposition 29 *Any isomorphism between two differential star-products on symplectic manifolds is the combination of a change of parameter and a μ-linear isomorphism. Any μ-linear isomorphism between two star-products $*$ on (M,ω) and $*'$ on (M',ω') is the combination of the action on functions of a symplectomorphism $\psi : M' \to M$ and an equivalence between $*$ and the pullback via ψ of $*'$. In particular, it exists if and only if those two star-products are equivalent, i.e., if and only if $(\psi^{-1})^* c(*') = c(*)$, where here $(\psi^{-1})^*$ denotes the action on the second de Rham cohomology space.*

In particular, two differential star-products $*$ on (M,ω) and $*'$ on (M',ω') are isomorphic if and only if there exist $f(\mu)=\sum_{r\geq 1}\mu^r f_r \in \mathbb{R}[[\mu]]$ with $f_1\neq 0$ and $\psi\colon M'\to M$, a symplectomorphism, such that $(\psi^{-1})^*c(*')(f(\mu))=c(*)(\mu)$. In particular [50]: if $H^2(M;\mathbb{R})=\mathbb{R}[\omega]$ then there is only one star-product up to equivalence and change of parameter.

Omori et al. [73] also show that when reparametrizations are allowed then there is only one star-product on $\mathbb{C}P^n$.

A special case of Proposition 29 gives:

Proposition 30 *A symplectomorphism ψ of a symplectic manifold can be extended to a μ-linear automorphism of a given differential star-product on (M,ω) if and only if $(\psi)^*c(*)=c(*)$.*

Notice that this is always the case if ψ can be connected to the identity by a path of symplectomorphisms (and this result was in Fedosov [41]).

If G is a connected Lie group acting on the symplectic manifold (M,ω) by symplectomorphisms, each element of G can be lifted to an automorphism of a star-product on M. The group G acts on the quantum level if there is a homomorphism ρ from G into the automorphism group of $*$ such that $\rho(g)=g^*+\mu\dots\ \forall g\in G$. At the Lie algebra level, one considers a homomorphism σ from the Lie algebra $\mathfrak{g}$ of G to the algebra of derivations of $*$ such that $\sigma(X)=X^*+\mu\dots\ \forall X\in\mathfrak{g}$ where X^* is the fundamental vector field on M associated to the action of G on M (i.e., $X^*_x=\frac{d}{dt}\exp -tX\cdot x_{|_0}$). Now any local μ-linear derivation of $*$ can be locally written on a contractible set U as $\mathrm{ad}_*(\frac{1}{\mu}f)$ for some $f\in C^\infty(U)[[\mu]]$.

Definition 31 *Given a star-product $*$ on a Poisson manifold (M,P) and given a connected Lie group G acting on M, the star-product gives* **a quantization of the action of the Lie algebra $\mathfrak{g}$** *if there exists a* **generalized moment map** *i.e., a map*

$$\tau:\mathfrak{g}\to\frac{1}{\mu}C^\infty(M)[[\mu]]:X\mapsto\tau(X)$$

such that

$$\tau(X)*\tau(Y)-\tau(Y)*\tau(X)=\tau([X,Y]),\quad \forall X,Y\in\mathfrak{g} \tag{3.9}$$

and

$$\tau(X)*u-u*\tau(X)=X^*u+\mu\cdots,\ \ \forall X\in\mathfrak{g}. \tag{3.10}$$

Remark that the two conditions imply for the term of τ of order -1 in μ that the action of G on M admits a moment map, i.e., there is a map $\lambda:\mathfrak{g}\to C^\infty(M)$ such that $\{\lambda(Y),u\}=Y^*u$ and $\{\lambda(Y),\lambda(Y)\}=\lambda(\{X,Y\})$, and

$$\tau(X)=\frac{1}{\mu}\lambda(X)+\cdots.$$

When one can choose $\tau(X) = \frac{1}{\mu}\lambda(X)$, $\forall X \in \mathfrak{g}$, the notion of a star-product which is a quantization of the action of $\mathfrak{g}$ coincides with the notion of a covariant star-product.

When one can choose τ so that $\tau(X) * u - u * \tau(X) = X^* u \ \ \forall X \in \mathfrak{g}$, it implies that $X^*(u*v) = (X^*u)*v + u*(X^*v)$, so that

$$g^*(u*v) = g^*u * g^*v \ \forall g \in G$$

which are the conditions in Bayen et al [10] for the star-product to be **geometrically invariant**; in that situation, the notion of generalized moment map coincides with the notion of quantum moment map introduced by Xu in [89].

In the general case, when one has found a map τ at order k in μ satisfying condition (3.9) at order k, then one can extend things one order further if a Chevalley 2-cocycle from $\mathfrak{g}$ with values in $C^\infty(M)$ (for the representation of $Y \in \mathfrak{g}$ given by Y^*) is a 2-coboundary. When one deals with geometrically invariant star-products, one can always find a τ such that $\tau(X)*u - u*\tau(X) = X^*u$, $\forall X \in \mathfrak{g}$, if $H^2(\mathfrak{g},\mathbb{R}) = 0$ and $H^1(M;\mathbb{R}) = 0$ as was found in [89].

4. Convergence of star-products

Remark 32 Let (M,P) be a Poisson manifold and let $*$ be a differential star-product on it with 1 acting as the identity. Observe that if there exists a value k of μ such that

$$u*v = \sum_{r=0}^{\infty} \mu^r C_r(u,v)$$

converges (for the pointwise convergence of functions), for all $u,v \in C^\infty(M)$, to $F_k(u,v)$ in such a way that F_k is associative, then $F_k(u,v) = uv$.

So assuming "too much" convergence kills all deformations. On the other hand, in any physical situation, one needs some convergence properties to be able to compute the spectrum of quantum observables in terms of a star-product as was done for some observables already in Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer [11].

Now consider the example of the Moyal star-product on the symplectic vector space $(\mathbb{R}^{2n},\omega)$. The formal formula for the Moyal star-product

$$(u *_M v)(z) = \exp\left(\frac{\mu}{2} P^{rs} \partial_{x^r} \partial_{y^s}\right) (u(x)v(y))\Big|_{x=y=z}$$

obviously converges when u and v are polynomials.

On the other hand, there is an integral formula for Moyal star-product given by

$$(u*v)(\xi) = (\pi\hbar)^{-2n} \int u(\xi')v(\xi'') e^{\frac{2i}{\hbar}(\omega(\xi,\xi'')+\omega(\xi'',\xi')+\omega(\xi',\xi))} d\xi' d\xi'',$$

and this product $*$ gives a structure of associative algebra on the space of rapidly decreasing functions $\mathscr{S}(\mathbb{R}^{2n})$.

The formal formula converges (for $\mu = i\hbar$) in the topology of $\mathscr{S}'$ for u and v with compact Fourier transform.

With Michel Cahen and John Rawnsley, we used the method of quantization of Kähler manifolds due to Berezin, [12], as the inverse of taking symbols of operators, to construct on Hermitian symmetric spaces star-products which are convergent on a large class of functions on the manifold. I shall develop this construction in this last part of my talk.

Let me mention, before closing this introduction about convergence, the work of Rieffel [81] where he introduces the notion of strict deformation quantization. An example of strict Fréchet quantization has been recently given by Omori, Maeda, Miyazaki, and Yoshioka in [76].

Also very important are the constructions of operator representations of star-products, in particular the works of Fedosov [41] and of Bordemann, Neumaier, and Waldmann [18, 19].

4.1. Convergence of Berezin type star-products on Hermitian symmetric spaces

The method to construct a star-product involves making a correspondence between operators and functions (their Berezin symbols), transferring the operator composition to the symbols, introducing a suitable parameter into the Berezin composition of symbols, taking the asymptotic expansion in this parameter on a large algebra of functions and then showing that the coefficients of this expansion satisfy the cocycle conditions to define a star-product on the smooth functions. The idea of an asymptotic expansion appeared already in Berezin [12] and in Moreno and Ortega-Navarro [65, 66].

In [25] we show that this asymptotic expansion exists for compact M, and defines an associative multiplication on formal power series in k^{-1} with coefficients in $C^\infty(M)$ for compact coadjoint orbits. We also show that this formal power series converges on the space of symbols for M a Hermitian symmetric space of compact type.

In [55], Karabegov proves convergence for general compact coadjoint orbits (i.e., flag manifolds).

In [26] we study general Hermitian symmetric spaces of non-compact type, and use their realization as bounded domains to define an analogous algebra of symbols of polynomial differential operators.

Recently Reshetikhin and Takhtajan have announced [80] an associative formal star-product given by an asymptotic expansion on any Kähler manifold. This they do in two steps, first by building an associative product for which 1 is not a unit element and then by passing to a star-product.

4.1.1. *Berezin symbols*

We denote by (L, ∇, h) a quantization bundle for the Kähler manifold (M, ω, J) (i.e., a holomorphic line bundle L with connection ∇ admitting an invariant hermitian structure h, such that the curvature is $\mathrm{curv}(\nabla) = -2i\pi\omega$). We denote by $\mathscr{H}$ the Hilbert space of square-integrable holomorphic sections of L which we assume to be non-trivial. The coherent states are vectors $e_q \in \mathscr{H}$ such that

$$s(x) = \langle s, e_q \rangle q, \qquad \forall q \in \mathscr{L}_x, \quad x \in M, \quad s \in \mathscr{H},$$

where $\mathscr{L}$ denotes the complement of the zero-section in L. The function

$$\varepsilon(x) = |q|^2 \|e_q\|^2, \qquad q \in \mathscr{L}_x,$$

is well-defined and real analytic.

We introduce also the 2-point function

$$\psi(x,y) = \frac{|\langle e_{q'}, e_q \rangle|^2}{\|e_{q'}\|^2 \|e_q\|^2}, \qquad q \in \mathscr{L}_x, \quad q' \in \mathscr{L}_y,$$

which is a globally defined real analytic function on $M \times M$ provided ε has no zeros. It is a consequence of the Cauchy–Schwartz inequality that $\psi(x,y) \leq 1$ everywhere, with equality where the lines spanned by e_q and $e_{q'}$ coincide ($q \in \mathscr{L}_x$, $q' \in \mathscr{L}_y$).

Let $A : \mathscr{H} \to \mathscr{H}$ be a bounded linear operator and let

$$\widehat{A}(x) = \frac{\langle A e_q, e_q \rangle}{\langle e_q, e_q \rangle}, \qquad q \in \mathscr{L}_x, \quad x \in M,$$

be its symbol. The function $\widehat{A}$ has an analytic continuation to an open neighbourhood of the diagonal in $M \times \overline{M}$ given by

$$\widehat{A}(x,y) = \frac{\langle A e_{q'}, e_q \rangle}{\langle e_{q'}, e_q \rangle}, \qquad q \in \mathscr{L}_x, \quad q' \in \mathscr{L}_y,$$

which is holomorphic in x and antiholomorphic in y. We denote by $\hat{E}(L)$ the space of symbols of bounded operators on $\mathscr{H}$. We can extend this definition of symbols to some unbounded operators provided everything is well-defined.

4.1.2. *Composition of operators - Parameter*

The composition of operators on $\mathscr{H}$ gives rise to a product for the corresponding symbols, which is associative and which we shall denote by $*$ following Berezin, [12]. The product $*$ of symbols is given in terms of the symbols by the integral formula

$$(\widehat{A} * \widehat{B})(x) = \int_M \widehat{A}(x,y) \widehat{B}(y,x) \psi(x,y) \varepsilon(y) \frac{\omega^n(y)}{n!}.$$

This formula is derived by use of the adjoint A^* of A so to apply it to the case where the operators are unbounded we need to be able to use the adjoint of A on coherent states. To be able to take the symbol of the composition, the result of applying B to a coherent state must be in the domain of A.

Example. The identity map has symbol $\widehat{I} = 1$ and so $\widehat{A} * 1 = 1 * \widehat{A} = \widehat{A}$ for any operator A. In particular $1 * 1 = 1$.

Let k be a positive integer. The bundle $(L^k = \otimes^k L, \nabla^k, h^k)$ is a quantization bundle for $(M, k\omega, J)$ and we denote by $\mathscr{H}^k$ the corresponding space of holomorphic sections and by $\hat{E}(L^k)$ the space of symbols of linear operators on $\mathscr{H}^k$. We let $\varepsilon^{(k)}$ be the corresponding function. We say that the **quantization is regular** if $\varepsilon^{(k)}$ is a non-zero constant for all nonnegative k and if $\psi(x,y) = 1$ implies $x = y$. The significance of these conditions has been explained in [25].

Let G be a Lie group of isometries of the Kähler manifold (M, ω, J) which lifts to a group of automorphisms of the quantization bundle (L, ∇, h). This automorphism group acts naturally on $(L^k, \nabla^{(k)}, h^k)$; if $g \in G$ and if $e_q^{(l)}$ is a coherent state of L^l, then $g.e_q^{(l)} = e_{gq}^{(l)}$ so the function $\varepsilon^{(k)}$ is invariant under G. In particular, if the quantization is homogeneous, all $\varepsilon^{(k)}$ are constants.

In the regular case, the function ψ in the integral defining the composition of symbols for powers L^k gets replaced by powers ψ^k.

When the manifold is compact, we have proven the following facts:

– When $\varepsilon^{(k)}$ is constant for all k one has the nesting property $\hat{E}(L^k) \subset \hat{E}(L^{k+1})$;

– With the same assumption $\cup_k \hat{E}(L^k)$ is dense in $\mathscr{C}^\circ(M)$.

From the nesting property, one sees that if $\hat{A}, \hat{B}$ belong to $\hat{E}(L^l)$ and if $k \geq l$ one may define

$$(\widehat{A} *_k \widehat{B})(x) = \int_M \hat{A}(x,y)\hat{B}(y,x)\psi^k(x,y)\varepsilon^{(k)}\frac{k^n\omega^n}{n!}. \tag{4.1}$$

More generally, if $\widehat{A}$ is a symbol of an operator then its analytic continuation $\widehat{A}(x,y)$ may have singularities where $\psi(x,y) = 0$ but $\widehat{A}\psi$ is always globally defined on $M \times M$. If M is not compact $\widehat{A}\psi$ may not be bounded, so we introduce the class $\mathscr{B} \subset C^\infty(M)$ of functions f which have an analytic continuation off the diagonal in $M \times \overline{M}$ so that $f(x,y)\psi(x,y)^l$ is globally defined, smooth and bounded on $K \times M$ and on $M \times K$ for each compact subset K of M for some positive power l and denote by $\mathscr{B}_l$ those for which the power l suffices. Since ψ is smooth and bounded it is clear that $\mathscr{B}$ is a subalgebra of $C^\infty(M)$. In the case M is compact we obviously have $\hat{E}(L^l) \subset \mathscr{B}_l$. If $\hat{A}, \hat{B}$ belong to $\mathscr{B}$, formula (4.1) is well defined for k large enough.

We study the behaviour of the integral (4.1) in terms of k.

4.1.3. *An asymptotic formula*
In order to localize the integral (4.1) we use a version of the Morse Lemma adapted from Combet, as in Moreno and Ortega-Navarro.

Let (M,ω,J) be a Kähler manifold with metric g. We denote by $\exp_x X$ the exponential at x of $X \in T_xM$. If g is not complete the exponential map may not be defined for all x and X, but in any case there is an open subset $V \subset TM$ where it is defined and which contains the zero-section. The differential of the exponential map at 0 is the identity so the map $\alpha : V \to M \times M$ given by $\alpha(X) = (p(X), \exp_{p(X)} X)$ where p is the projection in the tangent bundle $p\colon TM \to M$ is a diffeomorphism near the zero-section. At any point of the zero-section the differential of α is the identity.

Proposition 33 *Let (M,ω,J) be a Kähler manifold with metric g and $\alpha\colon V \to M \times M$ be the map defined above. Let (L,∇,h) be a regular quantization bundle over M and let ψ be the corresponding 2-point function on $M \times M$. Then there exists an open neighbourhood $W \subset V$ of the zero-section in TM and a smooth open embedding $\mu : W \to TM$ such that*

$$(-\log \psi \circ \alpha \circ \mu)(X) = \pi g_{p(X)}(X,X), \quad X \in W,$$

and the differential of μ at any point of the zero-section is the identity.

Denote by $\widetilde{\mathscr{B}}$ the set of functions f on $M \times M \setminus \psi^{-1}(0)$ such that $f(x,y)\,\psi(x,y)^l$ has a smooth extension to all of $M \times M$ which is bounded on $K \times M$ for each compact subset $K \subset M$ for some l and denote by $\widetilde{\mathscr{B}}_l$ those for which the power l suffices. If f, g are in $\mathscr{B}$ then $f(x,y)g(y,x)$ is in $\widetilde{\mathscr{B}}$. Note also that if $f \in \widetilde{\mathscr{B}}$ then its restriction to the diagonal $\widehat{f}(x) = f(x,x)$ is smooth.

For any f belonging to $\widetilde{\mathscr{B}}_l$, the integral

$$F_k(x) = \int_M f(x,y)\,\psi(x,y)^k k^n \frac{\omega^n(y)}{n!}, \quad \text{for k} \geq \text{l} + 1,$$

admits an asymptotic expansion

$$F_k(x) \sim \sum_{r\geq 0} k^{-r} C_r(\widehat{f})(x),$$

where C_r is a smooth differential operator of order $2r$ depending only on the geometry of M. The leading term is given by $C_0(\widehat{f})(x) = \widehat{f}$.

We are not claiming that for this very general class of functions $f(x,y)$ the integral depends smoothly on x, only that the coefficients of the asymptotic expansion do.

In the regular case $\varepsilon^{(k)}$ has an asymptotic expansion $\sum_{r\geq 0} \varepsilon_r/k^r$ as k tends to infinity with $\varepsilon_0 = 1$. Indeed,

$$1 = 1 *_k 1 = \varepsilon^{(k)} \int_M \psi(x,y)^k k^n \frac{\omega^n(y)}{n!},$$

has an asymptotic expansion in k^{-1} with leading term 1 by the previous proposition and we can then invert the asymptotic expansion to obtain one for $\varepsilon^{(k)}$.

Theorem 34 *Let (M,ω,J) be a Kähler manifold and (L,∇,h) be a regular quantization bundle over M. Let $\widehat{A},\widehat{B}$ be in $\mathscr{B}$. Then*

$$(\widehat{A}*_k\widehat{B})(x)=\int_M\widehat{A}(x,y)\widehat{B}(y,x)\psi^k(x,y)\varepsilon^{(k)}k^n\frac{\omega^n(y)}{n!}(y),$$

defined for k sufficiently large, admits an asymptotic expansion in k^{-1} as $k\to\infty$

$$(\widehat{A}*_k\widehat{B})(x)\sim\sum_{r\geq 0}k^{-r}C_r(\widehat{A},\widehat{B})(x),$$

and the cochains C_r are smooth bidifferential operators, invariant under the automorphisms of the quantization and determined by the geometry alone. Furthermore

$$C_0(\widehat{A},\widehat{B})=\widehat{A}\widehat{B},$$

and

$$C_1(\widehat{A},\widehat{B})-C_1(\widehat{B},\widehat{A})=\frac{i}{\pi}\{\widehat{A},\widehat{B}\}.$$

4.1.4. *A convergent star-product for flag manifolds*
We first would like to show that the asymptotic expansion obtained above defines an associative formal star-product. For this we assume that (M,ω,J) is a flag manifold. Reshetikhin and Takhtajan [80] have announced an analogous result for general Kähler manifolds. Observe that if G is a Lie group of isometries of the Kähler manifold (M,ω,J) which lifts to a group of automorphisms of the quantization bundle (L,∇,h) it acts naturally on $(L^k,\nabla^{(k)},h^k)$:

$$g^*(\widehat{A}*_k\widehat{B})(x)=(g^*\hat{A}*g^*\hat{B})(x),$$

for any $\hat{A},\hat{B}$ in $\hat{E}(L^l)$ and any $k\geq l$. Observe also that the bidifferential operators C_r depend on the geometry alone thus are invariant under G.

Lemma 35 *Let (M,ω,J) be a flag manifold with $M=G/K$ where G is a compact simply-connected Lie group and K the centralizer of a torus. Assume the geometric quantization conditions are satisfied and let (L,∇,h) be a quantization bundle over M. Let $\mathscr{C}_L=\cup_k\hat{E}(L^k)$ be the union of the symbol spaces. Then $\mathscr{C}_L$ coincides with the space E of vectors in $C^\infty(M)$ whose G-orbit is contained in a finite-dimensional subspace.*

In the case of a flag manifold as above, the group G lifts to a group of automorphisms of the quantization bundle $(L^k,\nabla^{(k)},h^k)$ hence the map $\hat{E}(L^l)\otimes\hat{E}(L^l)\to C^\infty(M)$ given by $\hat{A}\otimes\hat{B}\mapsto\widehat{A}*_k\widehat{B}$ intertwines the action of G and the bidifferential operators C_r are invariant under G. Thus, if $\hat{A}$, $\hat{B}$ belong to $\hat{E}(L^l)$, there exists an integer $a(l)$ such that $\widehat{A} *_k \widehat{B}$ belongs to $\hat{E}(L^{a(l)})$ for all $k \geq l$, and such that $C_r(\hat{A},\hat{B})$ belongs to $\hat{E}(L^{a(l)})$ for every integer r.

Consider now the asymptotic development:

$$\widehat{A} *_k \widehat{B} = \sum_{r=0}^{N} k^{-r} C_r(\hat{A},\hat{B}) + R_N(\hat{A},\hat{B},k),$$

where

$$\lim_{k\to\infty} k^N R_N(\hat{A},\hat{B},k) = 0.$$

The above tells us that $R_N(\hat{A},\hat{B},k)$ belongs to $\hat{E}(L^{a(l)})$ where $a(l)$ is independent of k. So, we can write

$$\begin{aligned}(\widehat{A} *_k \widehat{B}) *_k \hat{C} &= \sum_{r=0}^{N} k^{-r} C_r(\hat{A},\hat{B}) *_k \hat{C} + R_N(\hat{A},\hat{B}) *_k \hat{C} \\ &= \sum_{r,s=0}^{N} k^{-r-s} C_s(C_r(\hat{A},\hat{B}),\hat{C}) \\ &\quad + \sum_{r=0}^{N} k^{-r} R_N(C_r(\hat{A},\hat{B}),\hat{C},k) + R_N(\hat{A},\hat{B},k) *_k \hat{C}.\end{aligned} \tag{4.2}$$

The last two terms multiplied by k^N tend to zero when k tends to infinity.

Theorem 36 *The asymptotic expansion $\sum_{r\geq 0} k^{-r} C_r(u,v)$ yields a formal associative deformation of the usual product of functions in $\mathscr{C}_L$. It is a formal star-product which extends to all of $C^\infty(M)$, using uniform convergence.*

We prove that $\widehat{A} *_k \widehat{B}$ is a rational function of k with no pole at infinity, when the flag manifold is a hermitian symmetric space, by using structure theory of these spaces.

Theorem 37 *Let M be a compact hermitian symmetric space and let (L,∇,h) be a quantization bundle over M. Let $L^k = \otimes^k L$ and let $\mathscr{H}^k$ be the space of holomorphic sections of L^k. Let $\hat{E}(L^k)$ be the space of symbols of operators on $\mathscr{H}^k$. If $\hat{A}$, $\hat{B}$ belong to $\hat{E}(L^l)$ and $k \geq l$, the product $\hat{A} *_k \hat{B}$ depends rationally on k and has no pole at infinity, hence the asymptotic expansion of $\widehat{A} *_k \widehat{B}$ is convergent.*

This result is generalized by Karabegov [55]:

Theorem 38 *For any generalized flag manifold the $*_k$ product of two symbols is a rational function of k without pole at infinity.*

4.1.5. *Star-product on bounded symmetric domains*

Bounded symmetric domains. Let $\mathscr{D}$ denote a bounded symmetric domain. We shall use the Harish-Chandra embedding to realise $\mathscr{D}$ as a bounded subset of its Lie algebra of automorphisms. More precisely, if G_0 is the connected component of the group of holomorphic isometries then $\mathscr{D}$ is the homogeneous space G_0/K_0 where G_0 is a non-compact semi-simple Lie group and K_0 is a maximal compact subgroup. Let $\mathfrak{g}$ be the Lie algebra of G_0, $\mathfrak{k}$ the subalgebra corresponding with K_0, $\mathfrak{g}$ and $\mathfrak{k}$ the complexifications, and G, K the corresponding complex Lie groups containing G_0 and K_0. The complex structure on $\mathscr{D}$ is determined by K_0-invariant abelian subalgebras $\mathfrak{m}_+$ and $\mathfrak{m}_-$ with

$$\mathfrak{g} = \mathfrak{m}_+ + \mathfrak{k} + \mathfrak{m}_-, \qquad [\mathfrak{m}_+, \mathfrak{m}_-] \subset \mathfrak{k}, \qquad \overline{\mathfrak{m}_+} = \mathfrak{m}_-,$$

where $^-$ denotes conjugation over the real form $\mathfrak{g}$ of $\mathfrak{g}$. The exponential map sends $\mathfrak{m}_\pm$ diffeomorphically onto subgroups $M_\pm$ of G such that M_+KM_- is an open set in G containing G_0 and the multiplication map

$$M_+ \times K \times M_- \to M_+KM_-$$

is a diffeomorphism. KM_- is a parabolic subgroup of G and the quotient G/KM_- a generalized flag manifold. The G_0-orbit of the identity coset can be identified with G_0/K_0 and lies inside $M_+KM_-/KM_- \cong M_+$. Composing this identification with the inverse of the exponential map gives the desired Harish-Chandra embedding of $\mathscr{D}$ as a bounded open subset of $\mathfrak{m}_+$. We shall assume from now on that $\mathscr{D} \subset \mathfrak{m}_+$ via this embedding. In this realization it is clear that the action of K_0 on $\mathscr{D}$ coincides with the adjoint action of K_0 on $\mathfrak{m}_+$.

Following Satake, we define maps

$$k \colon \mathscr{D} \times \mathscr{D} \to K, \qquad m_\pm \colon \mathscr{D} \times \mathscr{D} \to \mathfrak{m}_\pm,$$

by

$$\exp -\overline{Z'} \exp Z = \exp m_+(Z,Z')\; k(Z,Z')^{-1} \exp m_-(Z,Z'),$$

for $Z, Z' \in \mathscr{D}$. They satisfy

$$\overline{k(Z,Z')} = k(Z',Z)^{-1}, \qquad \overline{m_+(Z,Z')} = -m_-(Z',Z).$$

The holomorphic quantization. For any unitary character χ of K_0 there is a Hermitian holomorphic line bundle L over $\mathscr{D}$ whose curvature is the Kähler form of an invariant Hermitian metric on $\mathscr{D}$. If χ also denotes the holomorphic extension to K then the Hermitian metric has Kähler potential $\log \chi(k(Z,Z))$ and L has a zero-free holomorphic section s_0 with

$$|s_0(Z)|^2 = \chi(k(Z,Z))^{-1}.$$

For χ sufficiently positive s_0 is square-integrable and the representation U of G_0 on the space $\mathscr{H}$ of square-integrable sections of L is one of Harish-Chandra's holomorphic discrete series. s_0 is a highest weight vector for the extremal K-type so is a smooth vector for the representation. We form the coherent states e_q and see that $e_{s_0(0)}$ transforms the same way as s_0 and so they must be equal up to a multiple. This means that the coherent states are also smooth vectors of the representation. Further, since the quantization is homogeneous, ε will be constant. Thus

$$\langle e_{s_0(Z')}, e_{s_0(Z)} \rangle = \varepsilon \chi(k(Z,Z')).$$

Lemma 39 *Up to a constant (determined by the normalization of Haar measure on G_0) ε is the formal degree d_U of the discrete series representation U, hence by Harish-Chandra's formula, it is a polynomial function of the differential $d\chi$ of the character χ.*

The two-point function ψ is given by

$$\psi(Z,Z') = \frac{|\chi(k(Z,Z'))|^2}{\chi(k(Z,Z))\chi(k(Z',Z'))};$$

it takes the value 1 only on the diagonal.

Polynomial differential operators and symbols. We let $\mathscr{A}$ denote the algebra of holomorphic differential operators on functions on $\mathscr{D}$ with polynomial coefficients. We filter $\mathscr{A}$ by both the orders of the differentiation and the degrees of the coefficients: $\mathscr{A}_{p,q}$ denotes the subspace of operators of order at most p with coefficients of degree at most q. Obviously, the composition of operators gives a map

$$\mathscr{A}_{p,q} \times \mathscr{A}_{p',q'} \to \mathscr{A}_{p+p',q+q'}.$$

The global trivialization by s_0 of the holomorphic line bundle L corresponding with the character χ allows us to transport the above operators to act on sections of L by sending $D \in \mathscr{A}$ to D^χ where

$$D^\chi(fs_0) = (Df)s_0.$$

Let $\mathscr{A}(\chi)$ denote the resulting algebra of operators on sections of L and $\mathscr{A}_{p,q}(\chi)$ the corresponding subspaces.

In this non-compact situation elements of $\mathscr{A}(\chi)$ do not define bounded operators on the Hilbert space $\mathscr{H}$, but the fact that the coherent states are smooth vectors of the holomorphic discrete series representation and that polynomials are bounded on $\mathscr{D}$ means that each operator in $\mathscr{A}(\chi)$ maps the coherent states into $\mathscr{H}$ so that it makes sense to speak of the symbols of these operators.

Lemma 40 *The analytically continued symbol $\widehat{D^\chi}(Z,Z')$ of an operator D^χ in $\mathscr{A}_{p,q}(\chi)$ is a polynomial in Z and $m_-(Z,Z')$ of bidegree p,q.*

The space of symbols of the operators in $\mathscr{A}_{k,l}(\chi)$ is the space of polynomials in Z and $m_-(Z,Z')$ of bidegree k,l so, in particular, is independent of χ.

Denote by $\mathscr{E}_{p,q}$ the space of polynomials in Z and $m_-(Z,Z')$ of bidegree p,q and by $\mathscr{E}$ the union of these spaces. $\mathscr{E}$ is an algebra under pointwise multiplication, the algebra of symbols . If we take a symbol in $\mathscr{E}_{p,q}$ then it is the symbol of an operator in $\mathscr{A}_{p,q}(\chi)$. Taking two such operators and composing them corresponds with the composition of two polynomial operators in $\mathscr{A}_{p,q}$ and so can be expressed in terms of a basis for $\mathscr{A}_{2p,2q}$ as a rational function of $d\chi$. In other words the Berezin product $f*g$ of two symbols f, g in $\mathscr{E}_{p,q}$ is a symbol in $\mathscr{E}_{2p,2q}$ depending rationally on $d\chi$.
The star-product. We construct a formal deformation of the algebra $C^\infty(\mathscr{D})$ by first constructing it on the subalgebra $\mathscr{E}$.

We consider the powers L^k of the line bundle L which correspond with the powers χ^k of χ. These powers have differentials $kd\chi$, so the Berezin product $f*_k g$ of two symbols f, g in $\mathscr{E}_{p,q}$ is a rational function of k by the results of the previous section.

The symbols in $\mathscr{E}_{p,q}$ are in $\mathscr{B}_0$ (it is enough to show that $\operatorname{ad} m_-(Z,Z')$ is bounded on $X\times\mathscr{D}$ and $\mathscr{D}\times X$ for any compact subset X of $\mathscr{D}$); thus the asymptotic expansion exists.

Since the Berezin product is associative for each k, the same argument as in the compact situation shows that its asymptotic expansion in k^{-1} is an associative formal deformation on $\mathscr{E}$ with bidifferential operators as coefficients. To see that it extends to all of $C^\infty(\mathscr{D})$ we show that $\mathscr{E}$ contains enough functions to determine these operators, hence the asymptotic expansion of $f*_k g$ has bidifferential operator coefficients which satisfy the cocycle conditions to define a formal product on $C^\infty(\mathscr{D})$ which is associative.

Theorem 41 *Let $\mathscr{D}$ be a bounded symmetric domain and $\mathscr{E}$ the algebra of symbols of polynomial differential operators on a homogeneous holomorphic line bundle L over $\mathscr{D}$ which gives a realization of a holomorphic discrete series representation of G_0 (i.e., $\mathscr{E}$ is the algebra of functions on $\mathscr{D}$ which are polynomials in Z and $m_-(Z,Z)$), then for f and g in $\mathscr{E}$ the Berezin product $f*_k g$ has an asymptotic expansion in powers of k^{-1} which converges to a rational function of k. The coefficients of the asymptotic expansion are bidifferential operators which define an invariant and covariant star-product on $C^\infty(\mathscr{D})$.*

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