On G-trace and G-index in deformation quantization

Boris Fedosov
Institut für Mathematik, Universität of Potsdam
Postfach 60 15 53, 14415 Potsdam, Germany
e-mail: `fedosov@math.uni-potsdam.de`

Abstract. For a compact group G of symplectomorphisms we define a G-trace on the algebra of quantum observables by postulating its properties. We give an explicit construction of such a trace and prove a G-index theorem similar to the Atiyah-Segal-Singer equivariant index theorem for elliptic operators.

Keywords: star-product, symmetry group, fixed-point manifold, G-trace, G-index

Mathematics Subject Classifications (2000): 53D55, 37J10

Dedicated to the memory of Moshé Flato

1. Introduction

Let (M,ω) be a symplectic manifold, $\partial = \partial^s$ a symplectic connection on M, $E \to M$ a complex vector bundle with a connection ∂^E. These data give rise to a *canonical deformation quantization* [5, 6] with coefficients in the bundle $K = \mathrm{Hom}(E,E)$. The algebra $(A_h, *)$ of quantum observables is isomorphic to $C^\infty(M,K)[[h]]$ as a linear space and the associative product $*$ is a deformation of the pointwise product of sections of the coefficient bundle K.

An important quantity in quantization theory is a *trace*. This is a linear functional which maps the ideal $A_h^c \in A_h$ of compactly supported observables $a(x,h)$ to "constants". The latter are formal Laurent series in h with finite number of negative exponents (notation $\mathbb{C}[h^{-1},h]]$). Although, in general, there are no operator representations of the algebra A_h a condition

$$\mathrm{Tr}\, a * b = \mathrm{Tr}\, b * a \tag{1.1}$$

is imposed on the trace as if the observables a,b were operators in a Hilbert space. It turns out that the trace exists, is unique up to a normalization factor and has a *canonical local expression*: The algebra A_h is locally isomorphic to a standard Weyl algebra W on $\mathbb{R}^{2n}$ and for $a \in W^c$

$$\mathrm{Tr}\, a = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} \mathrm{tr}\, a(x,h) \frac{\omega^n}{n!}, \tag{1.2}$$

where tr means the trace in the coefficient bundle $K = \mathrm{Hom}(E,E)$.

G. Dito and D. Sternheimer (eds.), Conférence Moshé Flato 1999, Vol. 1, 157 – 176.

Finally, having a trace on the algebra A_h one can introduce an *index* for this algebra. In the simplest case when the manifold M is compact we define

$$\operatorname{ind} A_h = \operatorname{Tr} 1. \tag{1.3}$$

The *index theorem* for deformation quantization [6, 8] gives an explicit formula for this quantity

$$\operatorname{ind} A_h = \int_M \operatorname{ch} E \widehat{A}(M) \exp \frac{\omega}{2\pi h}, \tag{1.4}$$

where $\operatorname{ch} E$ is the Chern character form of ∂^E and $\widehat{A}(M)$ is the Atiyah-Hirzebruch form for ∂^s.

In this paper we construct a trace functional and prove an index theorem in an equivariant situation. The letter G in the title (and everywhere in this paper) means a *compact group* of symplectomorphisms of M acting also equivariantly on the bundle E. In particular, G may be a finite group. The averaging over a compact group provides us with G-invariant geometric objects: Riemannian metric, almost complex structure, G-invariant connections ∂^E and ∂^s, G-invariant canonical deformation quantization. The latter means that the pullback map $s^*a(x,h) = s^{-1}a(s(x),h)s$ is an automorphism of the algebra A_h.

For a fixed element $s \in G$ we would like to define a linear functional

$$\operatorname{Tr}_s : A_h^c \to \mathbb{C}[h]]$$

satisfying some conditions generalizing (1.1) and prompted by the representations of the algebra A_h and the group G in a Hilbert space (which do not exist in general). Assuming that they do exist so that the following equalities hold:

$$\widehat{a * b} = \widehat{a}\widehat{b}, \tag{1.5}$$

$$\widehat{s_1 s_2} = \widehat{s}_1 \widehat{s}_2, \tag{1.6}$$

$$\widehat{s}^{-1}\widehat{a}\widehat{s} = \widehat{s^* a}, \tag{1.7}$$

where the hat means the corresponding operator, we would define the trace Tr_s by

$$\operatorname{Tr}_s a = \operatorname{Tr} \widehat{s}\widehat{a}, \tag{1.8}$$

where Tr on the right means the operator trace. Formal manipulations yield

$$\begin{aligned} \operatorname{Tr}_s a * b &= \operatorname{Tr} \widehat{s}\widehat{a}\widehat{b} = \operatorname{Tr} \widehat{s}\widehat{a}\widehat{s}^{-1}\widehat{s}\widehat{b} \\ &= \operatorname{Tr} \widehat{s}\widehat{b}\widehat{(s^{-1})^* a} = \operatorname{Tr}_s b * ((s^{-1})^* a), \\ \operatorname{Tr}_{s_1 s_2} a &= \operatorname{Tr} \widehat{s}_1 \widehat{s}_2 \widehat{a} = \operatorname{Tr} \widehat{s}_2 \widehat{s}_1 \widehat{s}_2 \widehat{a} \widehat{s}_2^{-1} = \operatorname{Tr}_{s_2 s_1} (s_2^{-1})^* a. \end{aligned} \tag{1.9}$$

Dropping the intermediate terms, we postulate these identities as the properties of the functional Tr_s. Here we will mostly consider the first of them

$$\operatorname{Tr}_s a * b = \operatorname{Tr}_s b * ((s^{-1})^* a) \tag{1.10}$$

for a fixed $s \in G$. Similar motivations were used in [4] to express the Kubo-Martin-Schwinger condition in terms of deformation quantization.

As an immediate consequence of (1.10) we obtain that the distributional support of the functional Tr_s is contained in the fixed-point set of s. Indeed, let $U = \operatorname{supp} a$ and $V = \operatorname{supp}(s^{-1})^* a = s^{-1}(U)$ be compact sets with empty intersection. Taking a function ρ which separates these sets ($\rho = 1$ on U and $\rho = 0$ on V), we have

$$\mathrm{Tr}_s a = \mathrm{Tr}_s a * \rho = \mathrm{Tr}_s \rho * ((s^{-1})^* a) = 0,$$

because $\rho * (s^{-1})^* a = 0$ since $\rho = 0$ on $V = \operatorname{supp}(s^{-1})^* a$. On the other hand, if $\operatorname{supp} a$ is contained in a small neighborhood of a fixed point x_0 we find a *canonical local expression* for $\mathrm{Tr}_s a$ generalizing (1.2). This is done in Section 2. In Section 3 we study further properties of this expression, in particular (1.9). It is closely related to the notion of the $*$-exponential first introduced on an heuristic level in the pioneer work [3]. Section 4 deals with a parameter-dependent version: We calculate equivariant Chern character for a G-vector bundle with symplectic structure.

Next, in Section 5 we construct the functional Tr_s globally on a smooth connected component M_0 of the fixed-point set of the map s. Our main observation consists in the fact that the algebra A_h in the vicinity of M_0 may be treated as an algebra of quantum observables *on* M_0 with a non-trivial coefficient bundle defined by the normal bundle N of M_0 in M. The map s acts on coefficients only. This allows us to define a trace tr_s on the coefficient bundle in terms of the local equivariant trace constructed in Section 2. Then the global trace Tr_s is defined by (1.2) with the replacement of the coefficient trace tr in (1.2) by the equivariant coefficient trace tr_s. Assuming that M_0 is a compact submanifold, we introduce the s-index

$$\mathrm{ind}_s A_h = \mathrm{Tr}_s 1$$

and prove an explicit index formula for this quantity in the spirit of the Atiyah-Segal-Singer equivariant index theorem. We obtain it as a simple consequence of the usual index theorem for deformation quantization with non-trivial coefficients and the properties of the trace tr_s on the coefficient bundle. This is done in Section 6.

We have not considered applications in this paper postponing them to subsequent publications. Here we only indicate two of them: An index theorem for orbifolds and a global $*$-exponential. It should be mentioned, however, that further extension of "constant coefficients" is required for applications to include rapidly oscillating or decaying exponentials $\exp(ic/h)$ where the "weights" c are complex numbers with $\mathrm{Im}\, c \geq 0$.

Acknowledgements. This work was done during the author's stay at the Université de Bourgogne. The author would like to thank the Laboratoire Gevrey de Mathématique physique and especially Professor D. Sternheimer for the kind invitation and hospitality.

2. The local *s*-trace

In this section we construct an equivariant trace assuming that all the structures in question have constant coefficients. Thus,

$$\omega = dq^i \wedge dp_i$$

(summation over $i = 1, 2, \ldots, n$) is a constant symplectic form on $\mathbb{R}^{2n}$ and the group G acts on $\mathbb{R}^{2n}$ by linear symplectic transformations. In this case the trace Tr_s actually may be defined via an operator representation, that is using (1.5)–(1.8). The price for this is that h must be a positive numerical parameter. It turns out, however, that the final expression for $\mathrm{Tr}_s\, a$ admits a stationary phase asymptotic expansion which is meaningful as a formal series in powers of h, so we may return to deformation quantization framework and define $\mathrm{Tr}_s\, a$ by this formal series. We would like to emphasize that the properties (1.9), (1.10) are fulfilled automatically on the level of asymptotic series by the very definition of $\mathrm{Tr}_s\, a$, thus, they are fulfilled on the level of formal series. This is why we need the operator representation, otherwise it would be difficult to prove these properties.

We begin with the Weyl correspondence $a \mapsto \mathrm{Op}(a)$ between functions $a(x) = a(q,p) \in C^\infty(\mathbb{R}^{2n})$ (satisfying some conditions at infinity) and pseudodifferential operators on $L^2(\mathbb{R}^n_q)$ given by

$$(\mathrm{Op}(a)u)(q) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n_p} \int_{\mathbb{R}^n_{q'}} e^{\frac{i}{h}\langle p, q-q'\rangle} a((q+q')/2, p) u(q') dq' dp. \tag{2.1}$$

The function $a(x)$ is called the Weyl symbol of the operator $\mathrm{Op}(a)$ (notation $a = \sigma_W(\mathrm{Op}(a))$. Although the operator (2.1) depends on the separation of variables $x = (q,p)$ into coordinates q and momenta p, the product $\widehat{ab}$ is independent of the separation and is given by the following formula (see, e.g., [6])

$$\sigma(\mathrm{Op}(a)\mathrm{Op}(b)) = \frac{1}{(\pi h)^{2n}} \int_{\mathbb{R}^{4n}} \exp\left(\frac{2i}{h}\omega(t,\tau)\right) a(x+t) b(x+\tau) dt d\tau. \tag{2.2}$$

Here

$$\omega(t,\tau) = \omega_{ij} t^i \tau^j, \quad (i,j = 1, 2, \ldots, 2n)$$

is the symplectic form ω evaluated on vectors t, τ, integral (2.2) is understood as an oscillatory one. Applying stationary phase method to this integral (practically it means that we expand $a(x+t)b(x+\tau)$ in Taylor series in t, τ and integrate termwise), we come to the asymptotic series

$$\sigma(\mathrm{Op}(a)\mathrm{Op}(b)) \sim a * b = \exp\left(-\frac{ih}{2}\omega^{ij}\frac{\partial}{\partial t^i}\frac{\partial}{\partial \tau^j}\right) a(x+t) b(x+\tau)\Big|_{t=\tau=0} \tag{2.3}$$

called the *Moyal $*$-product* of two functions a and b. Observe that if a or b is a polynomial, the series (2.3) is finite and coincides with the integral (2.2). The algebra of functions on $\mathbb{R}^{2n}$ with the Moyal product $*$ will be called the *Weyl algebra* (notation $W = W(\mathbb{R}^{2n})$).

Unfortunately, the action of the group $G \subset \mathrm{Sp}(2n,\mathbb{R})$ cannot be defined on $L^2(\mathbb{R}^n)$ in a single-valued manner but it is possible to define a single-valued action on the Fock space.

Let J be a positive G-invariant complex structure on $\mathbb{R}^{2n}$, so that our symplectic space may be considered as a complex space $\mathbb{C}^n$ with a G-invariant hermitian scalar product $(\cdot,\cdot)$. In an orthonormal basis we have

$$(z,w) = \sum_{i=1}^{n} \bar{z}_i w_i = z^* w, \quad \omega = \frac{1}{2i}\sum_{i=1}^{n} d\bar{z}_i \wedge dz_i = \frac{1}{2i} dz^* \wedge dz,$$

where z on the right-hand side is treated as a column and z^* is a row of complex conjugate elements. The group G acts by unitary transformations on $\mathbb{C}^n$. The Moyal $*$-product on functions $a = a(z,\bar{z})$ may be rewritten as

$$\begin{aligned} a * b = & \frac{1}{(\pi h)^{2n}} \int_{\mathbb{C}^{2n}} \exp\left(\frac{1}{h}(u^* v - v^* u)\right) a(z+u,\bar{z}+\bar{u}) b(z+v,\bar{z}+\bar{v}) du d\bar{u} dv d\bar{v} \\ & \sim \sum_{\alpha,\beta} (-1)^{|\beta|} \frac{h^{\alpha+\beta}}{\alpha!\beta!} \frac{\partial^{\alpha+\beta}}{\partial z^\alpha \partial \bar{z}^\beta} a \frac{\partial^{\alpha+\beta}}{\partial \bar{z}^\alpha \partial z^\beta} b. \end{aligned} \tag{2.4}$$

Here we assume that h is a positive small parameter. In (2.4) $du d\bar{u}$, $dv d\bar{v}$ mean the usual Lebesgue measure on $\mathbb{C}^n$.

Consider an operator $\mathrm{Op}(p)$ on $L^2(\mathbb{R}^n_q)$ whose Weyl symbol has the following form in complex coordinates

$$p(z,\bar{z}) = 2^n \exp\left(-\frac{1}{h}|z|^2\right). \tag{2.5}$$

Using the integral formula (2.4), it is easy to verify that $\mathrm{Op}(p)$ is a projector in $L^2(\mathbb{R}^n_q)$. Moreover, since

$$\mathrm{Tr}\,\mathrm{Op}(p) = \frac{1}{(2\pi h)^n} \int_{\mathbb{C}^n} p(z,\bar{z}) dz d\bar{z} = 1,$$

its range is a one-dimensional subspace. We call it a *vacuum projector*.

The Fock space F corresponding to the Weyl algebra W and the vacuum projector (2.5) consists of all symbols having the form $a * p$ where $a \in W$. In other words, F is the left ideal in W generated by p. It is easy to see that $z_i * p = 0$, so F is generated by vectors

$$e^\alpha = \frac{(\bar{z})^\alpha * p}{\sqrt{\alpha!(2h)^{|\alpha|}}}. \tag{2.6}$$

At $\alpha = 0$ we obtain $e^0 = p$, this vector will be called the *vacuum vector*. Define a scalar product taking for two vectors $u = a * p$ and $v = b * p$

$$(u, v) = \mathrm{Tr}\,\mathrm{Op}(u)^* \mathrm{Op}(v) = \frac{1}{(2\pi h)^n} \int_{\mathbb{C}^n} p * \bar{a} * b * p \, dz d\bar{z}. \tag{2.7}$$

We assume that F is completed with respect to this scalar product, so that F is the Hilbert space with an orthonormal basis (2.6). The action of the Weyl algebra on F is defined by the $*$-multiplication: For $u = b * p \in F$ and $a \in W$ we set

$$\widehat{a}u = a * b * p,$$

so that W is represented by operators on the space F. We denote by $\widehat{a}$ the operator on the Fock space reserving notation $\mathrm{Op}(a)$ for the operator on $L^2(\mathbb{R}^n_q)$ given by (2.1). The representation on the Fock space has an advantage comparing with the original representation (2.1) in $L^2(\mathbb{R}^n)$, it admits a well-defined action of the unitary group $\mathrm{U}(n)$. Indeed, the unitary group acts on W:

$$s: \; a(z, \bar{z}) \mapsto a(s^{-1}z, \overline{s^{-1}z}) \tag{2.8}$$

by automorphisms. Considering $u = a * p \in F$ as an element of W, we have an action (2.8) of $s \in \mathrm{U}(n)$ on u. Since p is invariant for this action and $*$-product is preserved, the ideal F is preserved by this action too. Thus, we have a representation of $\mathrm{U}(n)$ on F which we denote by $s \mapsto \widehat{s}$ where

$$\widehat{s}u = u(s^{-1}z, \overline{s^{-1}z}).$$

In particular,

$$\widehat{s}e^\alpha = \frac{(\overline{s^{-1}z})^\alpha * p}{\sqrt{\alpha!(2h)^{|\alpha|}}}. \tag{2.9}$$

On the algebra W considered as the algebra of operators on F we have an adjoint representation

$$s: \widehat{a} \mapsto \widehat{s}\widehat{a}\widehat{s}^{-1}. \tag{2.10}$$

One can see immediately that the action (2.10) coincides with the previous action (2.8).

We are ready now to construct a G-trace on W for $G = \mathrm{U}(n)$. As was explained in the introduction, we set

$$\mathrm{Tr}_s a = \mathrm{Tr}\,\widehat{s}\widehat{a} \tag{2.11}$$

for $s \in \mathrm{U}(n)$, $a \in W$ where $\widehat{a}$ means the corresponding operator on F. This definition seems rather complicated. Quite remarkably, it admits a simple explicit formula in the spirit of deformation quantization.

Theorem 2.1 *Assume that 1 is not an eigenvalue of the matrix $s \in U(n)$. Then*

$$\mathrm{Tr}_s a = \det{}^{-1}(1 - s^{-1}) \exp\left(h \frac{\partial}{\partial z} \frac{1 + s^{-1}}{1 - s^{-1}} \frac{\partial}{\partial z^*} \right) a \Big|_{z = \bar{z} = 0} \tag{2.12}$$

where $\partial/\partial z$ means a row $(\partial/\partial z_1, \partial/\partial z_2, \dots, \partial/\partial z_n)$ and $\partial/\partial z^*$ is a column of complex conjugate elements.

We will need well-known properties of the Cayley transformation

$$\sigma = C(s) = (1-s)(1+s)^{-1}$$

and its inverse

$$s = C(\sigma) = (1-\sigma)(1+\sigma)^{-1}$$

summarized in the following lemma (see also [6, Appendix A]).

Lemma 2.2 *Let s be a contraction in $\mathbb{C}^n$, that is*

$$|sz|^2 < |z|^2$$

for any $z \in \mathbb{C}^n$. The Cayley transform $\sigma = C(s)$ exists and belongs to the "Siegel right half-plane", that is the matrix $\sigma + \sigma^*$ is positive definite. Vice versa, if σ belongs to the "right half-plane" then $s = C(\sigma)$ exists and is a contraction in $\mathbb{C}^n$.

Proof. Since s is a contraction, the number -1 cannot be its eigenvalue, so the Cayley transform exists. Consider the quadratic form

$$2\mathrm{Re}(\sigma z, z) = (\sigma z, z) + (z, \sigma z). \tag{2.13}$$

Taking $z = (1+s)u$ (it is possible since $1+s$ is non-degenerate), we obtain

$$\begin{aligned} 2\mathrm{Re}(\sigma z, z) &= ((1-s)u, (1+s)u) + ((1+s)u, (1-s)u) \\ &= 2|u|^2 - 2|su|^2 > 0. \end{aligned} \tag{2.14}$$

Conversely, if the form (2.13) is positive for any $z \in \mathbb{C}^n$, then -1 cannot be an eigenvalue of σ, so that $s = C(\sigma)$ exists. Further, taking $z = (1+\sigma)u$, we obtain

$$\begin{aligned} (z,z) - (\sigma z, \sigma z) &= ((1+\sigma)u, (1+\sigma)u) - (1-\sigma)u, (1-\sigma)u) \\ &= 2(\sigma u, u) + 2(u, \sigma u) > 0, \end{aligned}$$

proving the lemma. ∎

Remark 2.3 As a limit case, we see that the Cayley transform gives a one-to-one correspondence between anti-hermitian matrices σ and unitary matrices s, provided that -1 is not an eigenvalue of σ or s.

Proof of Theorem 2.1. Let us calculate the trace (2.11) as the sum of the diagonal matrix entries in the basis (2.6)

$$\begin{aligned} \mathrm{Tr}\,\widehat{s}\widehat{a} &= \sum_{|\alpha|=0}^{\infty} (e^\alpha, \widehat{s}\widehat{a} e^\alpha) = \sum_{|\alpha|=0}^{\infty} (\widehat{s}^{-1} e^\alpha, \widehat{a} e^\alpha) \\ &= \frac{1}{(2\pi h)^n} \sum_{|\alpha|=0}^{\infty} \int_{\mathbb{C}^n} \frac{p * (s^{-1}z)^\alpha * a * \bar{z}^\alpha * p}{\alpha!(2h)^{|\alpha|}} dz d\bar{z}. \end{aligned}$$

We have made use of the formula (2.7) for the scalar product. The integral may be interpreted as a trace of an operator on $L^2(\mathbb{R}^n_q)$ via Weyl correspondence (2.1). In particular, the cyclic permutations of $*$-factors are possible. Since $p * p = p$, we obtain

$$\mathrm{Tr}\widehat{s}\widehat{a} = \frac{1}{(2\pi h)^n} \sum_{|\alpha|=0}^{\infty} \int_{\mathbb{C}^n} a * \frac{\bar{z}^\alpha * p * (s^{-1}z)^\alpha}{\alpha!(2h)^{|\alpha|}} dz d\bar{z}.$$

An important feature of the Weyl correspondence is that

$$\mathrm{Tr}\,\mathrm{Op}(a)\mathrm{Op}(b) = \frac{1}{(2\pi h)^n} \int_{\mathbb{C}^n} ab\, dz d\bar{z}$$

with the usual product ab instead of $*$-product (see, e.g. [6]). This gives us the possibility to reduce the number of $*$-products, obtaining

$$\mathrm{Tr}\widehat{s}\widehat{a} = \frac{1}{(2\pi h)^n} \sum_{|\alpha|=0}^{\infty} \int_{\mathbb{C}^n} a \frac{\bar{z}^\alpha * p * (s^{-1}z)^\alpha}{\alpha!(2h)^{|\alpha|}} dz d\bar{z}.$$

Observe now that for a polynomial $f(z)$ we have by (2.4)

$$p * f(z) = \sum \frac{1}{\alpha!} \frac{\partial^\alpha f(z)}{\partial z^\alpha} z^\alpha = f(2z)p,$$

so that only one $*$-product is left

$$\mathrm{Tr}\widehat{s}\widehat{a} = \frac{1}{(2\pi h)^n} \sum_{|\alpha|=0}^{\infty} \int_{\mathbb{C}^n} a \frac{\bar{z}^\alpha * (p(s^{-1}z)^\alpha)}{\alpha! h^{|\alpha|}} dz d\bar{z}.$$

For this remaining $*$-product we need the integral formula (2.4). The function $a(z,\bar{z})$ is supposed to have compact support. To feel free with infinite summation, we will assume that the factor $\exp(-\varepsilon|u|^2 - \varepsilon|v|^2)$ is inserted into the integrand of (2.4) and the limit for $\varepsilon \to 0$ is taken after integration (this is the usual trick for oscillating integrals). We also will assume that s^{-1} is a contraction, so that $(1 \pm s^{-1})^{-1}$ exists.

Thus,

$$\frac{\bar{z}^\alpha * (p(s^{-1}z)^\alpha)}{\alpha! h^{|\alpha|}} = \frac{1}{(\pi h)^{2n}} \int_{\mathbb{C}^n} \left(\frac{1}{h}\right)^{|\alpha|} \frac{1}{\alpha!} \exp\left(\frac{1}{h}(u^* v - v^* u - |z+v|^2)\right) (z^* + u^*)^\alpha (s^{-1}(z+v))^\alpha du d\bar{u} dv d\bar{v}.$$

The series may be summed giving an exponential function

$$\exp\left(\frac{1}{h}(z^* + u^*) s^{-1}(z+v)\right),$$

and we finally obtain an oscillating integral

$$\frac{1}{(\pi h)^{3n}}\int_{\mathbb{C}^{3n}} a(z,\bar z)\exp\big(\frac{1}{h}(u^*v-v^*u-(z^*+v^*)(z+v) \\ +(z^*+u^*)s^{-1}(z+v))\big)dud\bar u dvd\bar v dzd\bar z,$$

which may be calculated asymptotically for $h\to 0$ by the stationary phase method. Moreover, the integrals over v, $\bar v$, u, $\bar u$ are Gaussian, so the integration over v, $\bar v$, u, and $\bar u$ may be performed explicitly. The integration over v, $\bar v$ yields

$$\frac{1}{(\pi h)^{2n}}\int_{\mathbb{C}^{2n}} a(z,\bar z)\exp\left(-\frac{1}{h}(u^*(1+s^{-1})u-z^*(1-s^{-1})u+u^*z)\right)dud\bar u dzd\bar z.$$

The next integration over u, $\bar u$ results in

$$\frac{\det^{-1}(1+s^{-1})}{(\pi h)^n}\int_{\mathbb{C}^n} a(z,\bar z)\exp\left(-\frac{1}{h}z^*\frac{1-s^{-1}}{1+s^{-1}}z\right)dzd\bar z. \tag{2.15}$$

At this step it is essential that the quadratic form $\mathrm{Re}\,u^*(1+s^{-1})^{-1}(1-s^{-1})u$ is positive definite since s^{-1} is a contraction. The exponent in (2.15) also has a negative real part. The stationary phase method applied to (2.15) means that we integrate termwise the formal Taylor expansion for $a(z,\bar z)$ at $z=\bar z=0$. The result, obviously, may be written in the form (2.12). ■

Remark 2.4 Passing to the limit $s\to 1$ in (2.15), we obtain the usual formula for the trace

$$\mathrm{Tr}\,\widehat{a}=\frac{1}{(2\pi h)^n}\int_{\mathbb{C}^n} a(z,\bar z)dzd\bar z. \tag{2.16}$$

We now may forget about the origin of the formula (2.12) and take it as a definition of $\mathrm{Tr}_s a$ for the algebra W in the case when s has no eigenvalue 1. This expression is meaningful even in the framework of deformation quantization. Moreover, if s has an eigenvalue 1, we split $\mathbb{C}^n$ into orthogonal direct sum of two subspaces $\mathbb{C}_1\oplus\mathbb{C}_1^\perp$ of dimensions k and $n-k$. On $\mathbb{C}_1$, s is equal to the identity matrix and on $\mathbb{C}_1^\perp$ the eigenvalues of s are not equal to 1. Thus, denoting $z=(z_1,z_\perp)$, $s=1\oplus s_\perp$, we would have

$$\mathrm{Tr}_s a = \\ \frac{1}{(2\pi h)^k}\int_{\mathbb{C}_1} dz_1 d\bar z_1 \det{}^{-1}(1-s_\perp^{-1})\exp\left(h\frac{\partial}{\partial z_\perp}\frac{1+s_\perp^{-1}}{1-s_\perp^{-1}}\frac{\partial}{\partial z_\perp^*}\right)a\Big|_{z_\perp=\bar z_\perp=0}. \tag{2.17}$$

We will also consider a regularization of the trace (2.17) taking $a(z,\bar z)$ in the form

$$a(z,\bar z)=\lim_{\varepsilon\to 0}a_\varepsilon(z,\bar z)=\lim_{\varepsilon\to 0}a(z,\bar z)e^{-\frac{\varepsilon}{h}|z|^2}. \tag{2.18}$$

Substituting into (2.15), we get

$$\operatorname{Tr}_s a_\varepsilon = \frac{\det^{-1}(1+s^{-1})}{(\pi h)^n} \int_{\mathbb{C}^n} a(z,\bar z) \exp\left(-\frac{1}{h}\left(z^*\left(\varepsilon + \frac{1-s^{-1}}{1+s^{-1}}\right) z\right)\right) dz d\bar z,$$

whence by the stationary phase method

$$\operatorname{Tr}_s a_\varepsilon = \tag{2.19}$$
$$\det{}^{-1}(\varepsilon(1+s^{-1})+1-s^{-1}) \exp\left(h\frac{\partial}{\partial z}\frac{1+s^{-1}}{\varepsilon(1+s^{-1})+1-s^{-1}}\frac{\partial}{\partial z^*}\right) a\Big|_{z=\bar z=0}.$$

For $\varepsilon > 0$ this expression is meaningful as a formal series in h for any unitary matrix s. Moreover, if s has no eigenvalue eual to 1 then we have

$$\operatorname{Tr}_s a = \lim_{\varepsilon\to+0} \operatorname{Tr}_s a_\varepsilon$$

with the termwise limit of the formal series.

Proposition 2.5 *The properties* (1.9) *and* (1.10) *hold.*

Proof. Consider the operators $\widehat{s}$, $\widehat{a}$, $\widehat{b}$ in the Fock space corresponding to the symbols $a(x)$, $b(x) \in W$ (with compact supports) and to the unitary matrix s. Then

$$\operatorname{Tr}\widehat{s}\widehat{a}\widehat{b} = \operatorname{Tr}\widehat{s}\widehat{b}\widehat{s}\widehat{a}\widehat{s}^{-1}.$$

But $\widehat{s}\widehat{a}\widehat{s}^{-1} = \widehat{a(sx)}$, so we come to an identity

$$\operatorname{Tr}\widehat{s}\widehat{a}\widehat{b} = \operatorname{Tr}\widehat{s}\widehat{b}\widehat{a(sx)}.$$

Now, on both sides of this identity we pass to their deformation quantization counterparts given by (2.17). They must coincide since they give the asymptotic expansions of the identical operator expressions. This proves (1.10).

The proof of (1.9) is similar. ■

Finally, we show that $\operatorname{Tr}_s a$ is independent of the choice of a positive complex structure needed for its definition.

Proposition 2.6 *Let S be a symplectic matrix whose eigenvalues are not equal to* 1 *and let J be a positive complex structure on $\mathbb{R}^{2n}$ commuting with S. Let s be a unitary matrix corresponding to S in a given complex structure. Then* $\operatorname{Tr}_s a$ *given by* (2.12) *is independent of J.*

Proof. Taking

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

we have a correspondence between complex $n \times n$ matrices $s = s_1 + is_2$ with s_1, s_2 real, and real $2n \times 2n$ matrices commuting with J

$$S = \begin{pmatrix} s_1 & s_2 \\ -s_2 & s_1 \end{pmatrix}.$$

Being rational, the Cayley transformation preserves this correspondence. It means that

$$\frac{\partial}{\partial z} \frac{1 - s^{-1}}{1 + s^{-1}} \frac{\partial}{\partial z^*} = \frac{\partial}{\partial x} \frac{1 - S^{-1}}{1 + S^{-1}} \frac{\partial}{\partial x'}$$

where $\partial/\partial x$ is a row $(\partial/\partial x^1, \partial/\partial x^2, \dots, \partial/\partial x^{2n})$ and $\partial/\partial x'$ is a column obtained by transposition. Thus, the second factor in (2.12) admits an expression in terms of the symplectic matrix S only, so it is independent of J. Let us show that the first factor

$$\det{}^{-1}(1 - s^{-1}) \tag{2.20}$$

is also independent of J. To this end we show that the variation of (2.20) with respect to J is identically 0. This implies the desired independence since any two positive complex structures are homotopic. So, consider variations δJ of complex structures commuting with S. From the identity $J^2 = 1$ it follows that $J\delta J + \delta J J = 0$, thus δJ commutes with S and anticommutes with J. Let us reduce $J + \delta J$ to its canonical form J. The reducing matrix $C = 1 + \delta C$ satisfies the equation

$$J\delta C - \delta C J + \delta J = 0.$$

Its solutions have the form

$$\delta C = \frac{1}{2} J \delta J + X,$$

where

$$X = \begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix}$$

is an arbitrary matrix commuting with J. This yields

$$C^{-1} S C = [S, \delta C] = [S, X],$$

since S commutes with J and δJ. Thus, the variation $\delta s = \delta s_1 + i \delta s_2$ of the corresponding complex matrix s is

$$\delta s = [s, x],$$

where $x = x_1 + i x_2$. Now,

$$\delta \det(1 - s^{-1}) = \det(1 - s^{-1}) \operatorname{tr} (1 - s^{-1})^{-1} s^{-1} \delta s^{-1}.$$

But for a rational function $f(s)$

$$\operatorname{tr} f(s) \delta s = \operatorname{tr} f(s) [s, x] = 0,$$

proving the proposition. ∎

Remark 2.7 In the general situation, when S has an eigenvalue equal to 1 the functional Tr_s given by (2.17) is also independent of the positive complex structure commuting with the symplectic matrix S. Indeed, the k-dimensional complex eigensubspace $\mathbb{C}_1$ of the unitary matrix s corresponds to a $2k$-dimensional real eigensubspace of the symplectic matrix S. The orthogonal complement $\mathbb{C}_1^{\perp}$ with respect to the hermitian scalar product on $\mathbb{C}^n$ corresponds to the orthogonal complement of the real eigensubspace of S with respect to the symplectic form. So, everything in (2.17) may by defined in terms of symplectic geometry on $\mathbb{R}^{2n}$.

Remark 2.8 An important quantity for index theory is the trace of the identity element $1 \in W$. From (2.12) we have

$$\mathrm{Tr}_s 1 = \det{}^{-1}(1 - s^{-1}) \tag{2.21}$$

if s has no eigenvalues 1. This result has a simple explanation. The right-hand side of (2.21) may be viewed as the character of the representation of the unitary group $\mathrm{U}(n)$ on an infinite direct sum of symmetric tensor products $\mathrm{Sym}(\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n)$. But the bosonic Fock space clearly coincides with this direct sum and the action of the unitary group is the same, so the characters must coincide. What is a little bit surprising is that the quantum effects are inessential.

3. The local $*$-exponential

In this section we consider a link between the G-trace and the $*$-exponential introduced in the pioneer work [3] as a hypothetical solution of the Schrödinger evolution equation

$$\frac{dU}{dt} = \frac{i}{h} H * U, \quad U|_{t=0} = 1.$$

Unfortunately, in general, such a solution makes no sense in the framework of deformation quantization.

Proposition 3.1 *Let $s(t) \in U(n)$ be a smooth family. Then for any $a \in W$ with compact support and any $\varepsilon > 0$*

$$2h \frac{d}{dt} \mathrm{Tr}_{s(t)} a_\varepsilon = \mathrm{Tr}_{s(t)} H * a_\varepsilon \tag{3.1}$$

where

$$H(t) = z^* s^{-1}(t) \dot{s}(t) z - p * (z^* s^{-1} \dot{s} z) * p = z^* s^{-1}(t) \dot{s}(t) * z \tag{3.2}$$

is a normal-ordered quadratic Hamiltonian and a_ε is defined as in (2.18).

Proof. Consider the operator counterpart of the left-hand side of (3.1)

$$2h \frac{d}{dt} \mathrm{Tr}\, \widehat{s}(t)\, \widehat{a}_\varepsilon = 2h \mathrm{Tr}\, \widehat{s}\, \widehat{s^{-1}\dot{s}}\, \widehat{a}_\varepsilon.$$

Let us show that the operator $2h\widehat{s}^{-1}\dot{\widehat{s}}$ is equal to $\widehat{H}$ with $H \in W$ given by (3.2). Indeed,

$$\widehat{s}(t)(\bar{z}^\alpha * p) = \overline{s^{-1}z}^\alpha * p.$$

Thus, in the algebra W we have

$$\frac{d}{dt}(s^{-1}(t)z)^\alpha = -\frac{\partial z^\alpha}{\partial z_k}(s^{-1}\dot{s}z)_k\Big|_{z\mapsto s^{-1}z} \qquad = \frac{1}{2h}[z^* s^{-1}\dot{s}z, z^\alpha]\Big|_{z\mapsto s^{-1}z},$$

where the commutator is understood with respect to the $*$-product (2.4). Denoting $H_0 = z^* s^{-1}\dot{s}z \in W$, we see that

$$2h\frac{d}{dt}\widehat{s}(t)(\bar{z}^\alpha * p) = \widehat{s}(t)((H_0 * \bar{z}^\alpha - \bar{z}^\alpha * H_0) * p).$$

We may add any constant C to H_0 without changing the result. If we take $C = -p * H_0 * p$, then $(H_0 + C) * p = 0$, so we obtain

$$2h\frac{d}{dt}\widehat{s}(t)(\bar{z}^\alpha * p) = \widehat{s}(t)(\widehat{H_0 + C})(\bar{z}^\alpha * p),$$

which means that

$$2h\frac{d}{dt}\widehat{s}(t) = \widehat{s}(t)\,\widehat{H},$$

where $H = H_0 + C$ is the normal-ordered Hamiltonian. Multiplying both sides by $\widehat{a}_\varepsilon$ and taking traces, we get

$$2h\frac{d}{dt}\mathrm{Tr}\,\widehat{s}(t)\,\widehat{a}_\varepsilon = \mathrm{Tr}\,\widehat{s}(t)\widehat{H}\,\widehat{a}_\varepsilon.$$

It remains to pass to deformation quantization counterparts of both sides to obtain (3.1). ■

4. Equivariant Chern character

In this section we continue to study G-trace on the Fock space but in a parameter-dependent situation. Let M be an hermitian vector bundle with an hermitian connection ∂ and let a compact group G act on E by unitary transformations of fibers. Thus, choosing local orthonormal frames of the bundle E, we represent an element $s \in G$ as a unitary matrix $s(x)$ acting on fibers E_x and depending on a parameter $x \in M$. We suppose that the matrix $s(x)$ has no eigenvalue equal to 1 for any x. For any fiber E_x we may construct a Fock space F_x as was described in Section 2 and obtain an associated Fock bundle F over M. Moreover, we have a trace $\mathrm{Tr}_{s(x)}$ on fibers. The problem is to calculate the Chern character of the Fock bundle with respect to this trace.

First of all let us describe an associated connection $\widehat{\partial}$ on the Fock bundle. In a local orthonormal frame of E the connection ∂ is given by a matrix-valued 1-form $\Gamma = \Gamma_{ij}$, $i, j = 1, 2, \ldots, n$, which is an anti-hermitian matrix ($\Gamma = -\Gamma^*$). Consider a local 1-form

$$\widehat{\Gamma} = \frac{1}{2h}\Gamma_{ij}\bar{z}_i * z_j = \frac{1}{2h}z^*\Gamma * z \tag{4.1}$$

whose values are normally ordered quadratic Hamiltonians in F. We set for the basis (2.6)

$$\widehat{\partial}(\bar{z}^\alpha * p) = [\widehat{\Gamma}, \bar{z}^\alpha * p] = \widehat{\Gamma} * \bar{z}^\alpha * p, \tag{4.2}$$

the latter identity follows since $p * \Gamma = 0$ because of the normal ordering. In particular, the vacuum section p is covariantly constant, that is $\widehat{\partial} p = 0$.

The result of this section consists in the following theorem.

Theorem 4.1 *Let $\widehat{R}$ be the curvature of $\widehat{\partial}$. Then*

$$\mathrm{Tr}_s \exp\left(\frac{1}{2\pi i}\widehat{R}\right) = \det{}^{-1}\left(1 - s^{-1}(x)\exp\left(-\frac{R}{2\pi i}\right)\right). \tag{4.3}$$

Proof. First let us calculate the curvature. We have

$$\widehat{\partial}^2 e^\alpha = \widehat{\partial}(\widehat{\Gamma} e^\alpha) = (d\widehat{\Gamma} + \widehat{\Gamma} * \widehat{\Gamma})e^\alpha$$

so that

$$\begin{aligned}\widehat{R} &= d\widehat{\Gamma} + \widehat{\Gamma} * \widehat{\Gamma} = d\widehat{\Gamma} + \frac{1}{2}[\widehat{\Gamma}, \widehat{\Gamma}] \\ &= \frac{1}{2h}d\Gamma_{ij}\bar{z}_i * z_j + \frac{1}{8h^2}\Gamma_{ij} \wedge \Gamma_{kl}[\bar{z}_i * z_j, \bar{z}_k * z_l].\end{aligned}$$

But, using (2.4), we have

$$\begin{aligned}[\bar{z}_i * z_j, \bar{z}_k * z_l] = [\bar{z}_i z_j, \bar{z}_k z_l] &= 2h\delta_{jk}\bar{z}_i z_l - 2h\delta_{il}\bar{z}_k z_j \\ &= 2h(\delta_{jk}\bar{z}_i * z_l - \delta_{il}\bar{z} * z_j),\end{aligned}$$

so, that

$$\widehat{R} = \frac{1}{2h}z^*(d\Gamma + \Gamma \wedge \Gamma) * z = \frac{1}{2h}z^* R * z, \tag{4.4}$$

where R is the curvature of the connection ∂.

Now, since M is a finite-dimensional manifold, the element $\widehat{R}$ is nilpotent, so the $*$-exponential $\exp(\widehat{R}/2\pi i)$ is defined correctly by its Taylor series or via differential equation, of course, both definitions coincide. Thus, we may apply Proposition 3.1 to calculate the $*$-exponential and its trace. According to this proposition the $*$-exponential acts on the Fock space as a unitary matrix $\sigma = \exp(-R/2\pi i)$. In other words,

$$\exp\left(\frac{\widehat{R}}{2\pi i}\right) = \widehat{\exp\left(-\frac{R}{2\pi i}\right)} = \widehat{\sigma}.$$

Then

$$\operatorname{Tr}_s \widehat{\sigma} = \operatorname{Tr} \widehat{s} \widehat{\sigma} = \operatorname{Tr} \widehat{s\sigma}\, 1 = \operatorname{Tr}_{s\sigma} 1 = \det{}^{-1}(1-(s\sigma)^{-1}),$$

proving the theorem. ■

5. Quantization of the fixed-point manifold

Let $(M,\ \omega)$ be a symplectic manifold with a symplectic connection ∂^s, $E \to M$ a complex vector bundle with a connection ∂^E, s a symplectic diffeomorphism belonging to a compact group G of symplectomorphisms preserving the connection ∂^s. We assume also that G has an equivariant lifting to E preserving the connection ∂^E, that is E is a G-bundle.

We pick a positive almost complex structure J on M which is preserved by G, this means that $\omega(Ju,v)$ is a G-invariant Riemannian metric on M. Such a structure always exists. A standard way to construct it consists in taking first any Riemannian metric g. Averaging it over a compact group G, we make it G-invariant. Then goes a standard procedure which for a pair $g,\ \omega$ gives an automorphism J of the tangent bundle such that $g(u,v) = \omega(Ju,v)$ (see, e.g. [6, Chapter 2]). The tangent bundle TM with the almost complex structure J becomes a complex vector bundle $T^{\mathbb{C}}M$ with a G-invariant hermitian scalar product (u,v) such that

$$\omega(u,v) = \operatorname{Im}(u,v). \tag{5.1}$$

Let, further, M_0 be a connected component of the fixed-point set of s. We assume that M_0 is a smooth compact submanifold and let N be its normal bundle (with respect to the metric g). At any point $x \in M_0$, s acts on T_xM as an orthogonal matrix. The vectors $u \in T_xM$ tangent to M_0 are the eigenvectors of s with the eigenvalue 1. We assume also that s is *non-degenerate*, that is there are no more eigenvectors with the eigenvalue 1. So, we have an s-invariant decomposition $T_xM = T_xM_0 \oplus N_x$ and s acts on N without fixed vectors (except the origin). Since the complex structure J is preserved s acts on $T_x^{\mathbb{C}}M$ as a unitary matrix and the decomposition

$$T_x^{\mathbb{C}}M = T_x^{\mathbb{C}}M_0 \oplus N_x^{\mathbb{C}}$$

also takes place. In particular, this decomposition is orthogonal with respect to the symplectic form ω on T_xM, too. Thus, the restriction of ω to T_xM_0 and N_x gives the symplectic structure on both of these subspaces. We come to the following proposition.

Proposition 5.1 *The restriction $\omega_0 = \omega|_{M_0}$ is a symplectic form on M_0. The normal bundle N has an hermitian structure $(\cdot,\cdot)$, s acts on N as a unitary automorphism and, moreover $1 \notin \operatorname{spec} s|_N$.*

In the sequel we treat N as a complex bundle simplifying the notation $N^{\mathbb{C}}$ to N. Pick an s-invariant hermitian connection on N, let $\Gamma^N = (\Gamma^N_{ij})$ be a connection 1-form ($i, j = 1, 2, \dots, k = \dim_{\mathbb{C}} N$) in a local orthonormal frame, this is an anti-hermitian matrix (cf. Section 4). The connection gives rise to a closed 2-form $\widetilde{\omega}$ on the total space N. Denoting by (x, z) local coordinates on N where

$$z = (z_1, z_2, \dots, z_k)^t$$

is a column of coordinates of the vector $z \in N_x$, set

$$\widetilde{\omega} = \frac{1}{2i} d(z^* \partial^N z) + \omega_0 = \frac{1}{2i}(dz^* dz + d(z^* \Gamma^N z)) + \omega_0. \tag{5.2}$$

This form is non-degenerate for z small enough, so we obtain a symplectic form $\widetilde{\omega}$ on a neighborhood N_ε of the zero section. The exponential mapping (with respect to the metric g) maps this neighborhood onto a tubular neighborhood of M_0 in M. The symplectic form ω on M is pulled back to N_ε and the forms ω and $\widetilde{\omega}$ coincide being restricted to a tangent space $T_x M$ at any point $x \in M_0$. The Weinstein arguments (see e.g. [6, Section 2.3]) show that there exists an s-equivariant diffeomorphism of N_ε to the tubular neighborhood of M_0 in M transforming ω to $\widetilde{\omega}$. So, in the sequel we will assume that this neighborhood is identified with N_ε with the symplectic form (5.2) which from now on we will denote simply by ω. This gives us a *normal form* of M in the vicinity of the fixed-point set component M_0.

Consider now a G-invariant canonical deformation quantization on M. The word "canonical" means that the algebra A_h of quantum observables on M is realized as the algebra of flat sections of the Weyl algebra bundle $W(TM)$ over M with respect to an Abelian connection D whose Weyl curvature is $-i\omega/h$ (see [6, Chapter 5] for more details). Restriction to the tubular neighborhood gives a canonical deformation quantization on N_ε with the symplectic form (5.2). But the structure of the vector bundle allows us to construct another realization of the algebra A_h (see [6, Section 6.6]). This construction goes in two steps. The first step is simple: We quantize each fiber N_x of N. The result is the Weyl algebra $W_x(\mathbb{C}^k)$ on the complex space N_x described in Section 2. These algebras form a bundle $W(N)$ over M_0 with fibers $W_x(\mathbb{C}^k)$. We may think of it as the bundle of "operators on the Fock bundle" as in Section 4. The trouble is, however, that such a viewpoint requires h to be a number. So, only the bundle $W(N)$ is meaningful for deformation quantization. Nevertheless, it is very helpful to keep in mind such a "realization" of the bundle $W(N)$. The connection $\widehat{\partial}^N$ gives rise to an associated connection $\widehat{\partial}$ on $W(N)$: If $a = a(x, z, \bar{z})$ is a section of $W(N)$, then

$$\widehat{\partial}^N a = d_x a + \frac{1}{2h}[z^* \Gamma * z, a] \tag{5.3}$$

(cf. Section 4). Observe, that the normal ordering in (5.3) may be replaced by the symmetric one, that is we may take $z^*\Gamma z$ instead of $z^*\Gamma * z$. Indeed, by (2.4)

$$\bar{z}_i * z_j = \bar{z}_i z_j - h\delta_{ij},$$

so that

$$z^*\gamma * z = z^*\Gamma^N z - \operatorname{tr}\Gamma^N$$

and the scalar $\operatorname{tr}\Gamma$ disappears under the commutator.

Remark 5.2 We see once again that it is useful to keep in mind the hypothetical Fock bundle $F(N)$. The normal ordering gives a "connection" on the Fock bundle with the connection form

$$\widehat{\Gamma}^N = \frac{1}{2h} z^*\Gamma^N * z$$

and the change of the ordering changes the connection itself (cf. Section 4).

The curvature of the connection (5.3) is by definition

$$\widehat{R}^N = d_x\widehat{\Gamma}^N + \widehat{\Gamma}^N * \widehat{\Gamma}^N = \frac{1}{2h} z^* R^N * z = \frac{1}{2h} z^* R^N z - \frac{1}{2}\operatorname{tr} R^N.$$

The second step is to quantize M_0 taking $W(N)$ as a coefficient bundle. More precisely, we have a bundle E and the original coefficient bundle $K = \operatorname{Hom}(E,E)$. Now, having constructed a bundle

$$W(N) = \text{“}\operatorname{Hom}(F(N),F(N))\text{”},$$

let us take

$$L = K \otimes W(N) = \text{“}K \otimes \operatorname{Hom}(F(E),F(E))\text{”}$$

as a new coefficient bundle with the connection

$$\partial^L = \partial^E \otimes 1 + 1 \otimes \widehat{\partial}^N \tag{5.4}$$

whose curvature is

$$R^L = R^E \otimes 1 + 1 \otimes \widehat{R}^N.$$

We apply a quantization scheme with a non-trivial coefficient bundle [6, Section 6.5]. According to this scheme we need to consider the Weyl algebra bundle $W(TM_0) \otimes L$ on the tangent bundle to M_0 and a connection on it

$$D = d + [\gamma, \cdot]$$

with

$$\gamma = \frac{i}{h}\omega^0_{ij} y^i dx^j + \frac{i}{2h}\Gamma^s_{ij} y^i y^j + \Gamma^L + \frac{i}{h} r.$$

Here ω_{ij}^0 are coefficients of the symplectic form ω_0 in local Darboux coordinates on M_0, Γ^s is a connection form of the symplectic connection on M_0, Γ^L a connection form of ∂^L, $y = (y_1, y_2, \ldots, y^{2m})$, $2m = \dim M_0$, is a tangent vector to M_0 and r is an unknown 1-form subject to the condition

$$d\gamma + \gamma^2 = -\frac{i}{h}\omega_0 + \frac{1}{2}\operatorname{tr} R^N. \tag{5.5}$$

On the right-hand side of (5.5) stands the desired scalar curvature form (called the *Weyl curvature*). The additional lower-order term $1/2 \operatorname{tr} R^N$ is included to compensate the change of the ordering (from normal to symmetric ones) needed in future. In [6, Chapter 6] it is proved that such an r exists. Moreover, the quantum algebra A_h on the total space N_ε with the Weyl curvature $-i/h\,\omega$ given by (5.2) and the quantum algebra on M_0 with coefficients in L and the Weyl curvature (5.5) are isomorphic.

Thus, we have realized A_h as an algebra of flat sections on M_0 with coefficient bundle $L = K \otimes W(N)$. It is very important that s acts on coefficients only but not on M_0 or the Weyl algebra bundle $W(TM_0)$.

6. *G*-index theorem

To define the global s-trace for a given fixed-point set component M_0 we follow general prescriptions of [6, Chapters 5,6]. The algebra A_h constructed in Section 5 consists of flat sections of the Weyl algebra bundle $W(TM_0)$ over M_0 with coefficients in the bundle $L = K \otimes W(N)$. Restricting this algebra to a small coordinate neighborhood $O \subset M_0$, we first trivialize the coefficient bundle taking local frames of E and N over O. Then, similarly to [6, Section 6.5], we construct a trivialization isomorphism

$$I : A_h \to W(\mathbb{R}^{2m}) \otimes L$$

to a standard Weyl algebra with coefficients in the trivial bundle L. The group G acts on coefficients only, thus, the local s-trace (Section 2) makes sense (we denote here this trace by tr_s). Finally, define the global s-trace by

$$\operatorname{Tr}_s a = (2\pi h)^{-m} \int_{\mathbb{R}^{2m}} \operatorname{tr}_s Ia \frac{\omega_0^m}{m!}. \tag{6.1}$$

The general case is reduced to the case of a small support by using partition of unity. As usual, this construction is independent of the choice of trivialization isomorphisms.

The properties of the global trace Tr_s follow directly from the corresponding properties of the local coefficient trace tr_s.

Having constructed the desired s-trace, we calculate now $\operatorname{Tr}_s 1$. The following theorem which we call the *equivariant index theorem for deformation quantization* gives an explicit formula for this quantity.

Theorem 6.1 *Let M_0 be a compact submanifold, TM_0 its real tangent bundle, N its complex normal bundle. Let*

$$\kappa^t = \frac{R^t}{2\pi i}, \quad \kappa^N = \frac{R^N}{2\pi i}, \quad \kappa^E = \frac{R^E}{2\pi i},$$

be curvature forms for TM_0, N, and E, respectively. Then

$$\mathrm{Tr}_s 1 = \int_{M_0} \frac{\mathrm{tr}\, s^E \exp \kappa^E}{\det(1-(s^N)^{-1}\exp(-\kappa^N))} \det{}^{-1/2}\left(\frac{\sinh \kappa^t/2}{\kappa^t/2}\right) \exp\left(\frac{\omega_0}{2\pi h} - \frac{1}{2}\mathrm{tr}\, \kappa^N\right) \tag{6.2}$$

where s^E and s^N mean automorphisms of E and N defined by s.

Proof. We apply the usual index theorem for deformation quantization [6] with a non-trivial coefficient bundle [7]. In our case the symplectic manifold is M_0 and the coefficient bundle is $L = K \otimes W(N)$ with the coefficient trace equal to tr_s. We take the Weyl curvature to be

$$\Omega_W = -\frac{i}{h}\omega_0 + \frac{1}{2}\mathrm{tr}\, R^N$$

and the connection in $K \otimes W(N)$ defined by (5.4) with the curvature

$$R^L = R^E \otimes 1 + 1 \otimes \frac{1}{2h} z^* R^N * z. \tag{6.3}$$

The only problem is to calculate the Chern character form for (6.3), that is $\mathrm{tr}_s \exp \kappa^L$. But

$$\begin{aligned} \mathrm{tr}_s \exp \kappa^L &= \mathrm{tr}_s \left(\exp \kappa^E \otimes \exp\left(\frac{1}{2h} z^* R^N * z\right)\right) \\ &= \mathrm{tr}_s \exp \kappa^E \, \mathrm{tr}_s \exp\left(\frac{1}{2h} z^* R^N * z\right). \end{aligned} \tag{6.4}$$

On the bundle E, s acts as a homomorphism, so $\mathrm{tr}_s \exp \kappa^E = \mathrm{tr}\, s^E \exp \kappa^E$. The second factor in (6.4) was calculated in Section 4. The result is

$$\det{}^{-1}(1-(s^N)^{-1}\exp(-\kappa^N)).$$

Substituting these expressions into the index formula

$$\mathrm{Tr}_s 1 = \int_{M_0} \mathrm{ch}\, L \exp\left(\frac{-\Omega_W}{2\pi i}\right) \widehat{A}(M_0),$$

we come to (6.2). ■

Let us examine the expression (6.2) in two extreme cases. The first one is when s is the identity. Then $M_0 = M$, N is a zero bundle, so $\kappa^N = 0$, and we come to the usual index formula (1.4). In the other case when M_0 is a fixed point x_0 we obtain a familiar fixed-point contribution

$$\frac{\operatorname{tr} s^E(x_0)}{\det(1-(s^N)^{-1}(x_0))}.$$

It should be mentioned that this expression is defined up to a phase factor which depends on how the group G acts on the vacuum. Our normalization in (6.2) means that the action is trivial.

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