Cyclic cohomology and Hopf algebra symmetry

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Abstract. Cyclic cohomology has been recently adapted to the treatment of Hopf symmetry in noncommutative geometry. The resulting theory of characteristic classes for Hopf algebras and their actions on algebras allows to expand the range of applications of cyclic cohomology. It is the goal of the present paper to illustrate these recent developments, with special emphasis on the application to transverse index theory, and point towards future directions. In particular, we highlight the remarkable accord between our framework for cyclic cohomology of Hopf algebras on one hand and both the algebraic as well as the analytic theory of quantum groups on the other, manifest in the construction of the modular square.

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Dedicated to the memory of our friend Moshé Flato

Introduction

Cyclic cohomology of noncommutative algebras is playing in noncommutative geometry a similar rôle to that of de Rham cohomology in differential topology [11]. In [14] and [15], cyclic cohomology has been adapted to Hopf algebras and their actions on algebras, which are analogous to the Lie group/algebra actions on manifolds and embody a natural notion of symmetry in noncommutative geometry. The resulting theory of characteristic classes for Hopf actions allows in turn to widen the scope of applications of cyclic cohomology to index theory. It is the goal of the present paper to review these recent developments and point towards future directions.

The contents of the paper are as follows. In §1 we recall the basic notation pertaining to the cyclic theory. The adaptation of cyclic cohomology to Hopf algebras and Hopf actions is reviewed in §2, where we also discuss the relationship with Lie group/algebra cohomology. §3 deals with the geometric Hopf algebras arising

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in transverse differential geometry and their application to transverse index theory. Finally, §4 illustrates the remarkable agreement between our framework for cyclic cohomology of Hopf algebras and both the algebraic as well as the analytic theory of quantum groups.

1. Cyclic cohomology

Cyclic cohomology has first appeared as a cohomology theory for algebras ([5], [7], [26]). In its simplest form, the cyclic cohomology $HC^*(\mathscr{A})$ of an algebra $\mathscr{A}$ (over $\mathbb{R}$ or $\mathbb{C}$ in what follows) is the cohomology of the cochain complex $\{C^*_\lambda(\mathscr{A}), b\}$, where $C^n_\lambda(\mathscr{A})$, $n \geq 0$, consists of the $(n+1)$-linear forms φ on $\mathscr{A}$ satisfying the cyclicity condition

$$\varphi(a^0, a^1, ..., a^n) = (-1)^n \varphi(a^1, a^2, ..., a^0), \quad a^0, a^1, ..., a^n \in \mathscr{A} \tag{1.1}$$

and the coboundary operator is given by

$$\begin{aligned}(b\varphi)(a^0, \ldots, a^{n+1}) = \sum_{j=0}^{n} (-1)^j \, \varphi(a^0, \ldots, a^j a^{j+1}, \ldots, a^{n+1}) \\ + (-1)^{n+1} \, \varphi(a^{n+1} a^0, a^1, \ldots, a^n).\end{aligned} \tag{1.2}$$

When the algebra $\mathscr{A}$ comes equipped with a locally convex topology for which the product is continuous, the above complex is replaced by its topological version: $C^n_\lambda(\mathscr{A})$ then consists of all continuous $(n+1)$-linear form on $\mathscr{A}$ satisfying (1.1).

Cyclic cohomology provides numerical invariants of K-theory classes as follows ([8]). Given an n-dimensional cyclic cocycle φ on $\mathscr{A}$, n even, the scalar

$$\varphi \otimes \mathrm{Tr}\,(E, E, ..., E) \tag{1.3}$$

is invariant under homotopy for idempotents

$$E^2 = E \in M_N(\mathscr{A}) = \mathscr{A} \otimes M_N(\mathbb{C}).$$

In the above formula, $\varphi \otimes \mathrm{Tr}$ is the extension of φ to $M_N(\mathscr{A})$, using the standard trace Tr on $M_N(\mathbb{C})$:

$$\varphi \otimes \mathrm{Tr}\,(a^0 \otimes \mu^0, a^1 \otimes \mu^1, \ldots, a^n \otimes \mu^n) = \varphi(a^0, a^1, \ldots, a^n)\,\mathrm{Tr}(\mu^0 \mu^1 \cdots \mu^n).$$

This defines a pairing $\langle [\varphi], [E] \rangle$ between cyclic cohomology and K-theory, which extends to the general noncommutative framework the Chern-Weil construction of characteristic classes of vector bundles.

Indeed, if $\mathscr{A} = C^\infty(M)$ for a closed manifold M and

$$\varphi(f^0, f^1, ..., f^n) = \langle \Phi, f^0 df^1 \wedge df^2 \wedge \ldots \wedge df^n \rangle, \quad f^0, f^1, ..., f^n \in \mathscr{A},$$

where Φ is an n-dimensional closed de Rham current on M, then up to normalization the invariant defined by (1.3) is equal to

$$\langle \Phi, ch^*(\mathscr{E}) \rangle,$$

here $ch^*(\mathscr{E})$ denotes the Chern character of the rank N vector bundle $\mathscr{E}$ on M whose fiber at $x \in M$ is the range of $E(x) \in M_N(\mathbb{C})$.

Note that, in the above example,

$$\varphi(f^{\sigma(0)}, f^{\sigma(1)}, ..., f^{\sigma(n)}) = \varepsilon(\sigma)\, \varphi(f^0, f^1, ..., f^n)\,,$$

for any permutation σ of the set $[n] = \{0, 1, ..., n\}$, with signature $\varepsilon(\sigma)$. However, the extension of $\varphi \otimes \mathrm{Tr}$ to $M_N(\mathscr{A})$, used in the pairing formula (1.3), retains only the *cyclic* invariance.

A simple but very useful class of examples of cyclic cocycles on a noncommutative algebra is obtained from group cohomology ([10], [3], [12]), as follows. Let Γ be an arbitrary group and let $\mathscr{A} = \mathbb{C}\Gamma$ be its group ring. Then any *normalized group cocycle* $c \in Z^n(\Gamma, \mathbb{C})$, representing an arbitrary cohomology class $[c] \in H^*(B\Gamma) = H^*(\Gamma)$, gives rise to a cyclic cocycle φ_c on the algebra $\mathscr{A}$ by means of the formula

$$\varphi_c(g_0, g_1, \dots, g_n) = \begin{cases} 0 & \text{if } g_0 \cdots g_n \neq 1 \\ c(g_1, \dots, g_n) & \text{if } g_0 \cdots g_n = 1 \end{cases} \tag{1.4}$$

extended by linearity to $\mathbb{C}\Gamma$.

In a dual fashion, one defines the cyclic homology $HC_*(\mathscr{A})$ of an algebra $\mathscr{A}$ as the homology of the chain complex $\{C_*^\lambda(\mathscr{A}), b\}$ consisting of the coinvariants under cyclic permutations of the tensor powers of $\mathscr{A}$, and with the boundary operator b obtained by transposing the coboundary formula (1.2). Then the pairing between cyclic cohomology and K-theory (1.3) factors through the natural pairing between cohomology and homology, i.e.,

$$\varphi \otimes \mathrm{Tr}(E, E, ..., E) = \langle \varphi, ch(E) \rangle\,, \tag{1.5}$$

where, again up to normalization,

$$ch_n(E) = E \otimes \cdots \otimes E \quad (n+1 \quad \text{times}) \tag{1.6}$$

represents the *Chern character* in $HC_n(\mathscr{A})$, for n even, of the K-theory class $[E] \in K_0(\mathscr{A})$.

The cyclic cohomology of an (unital) algebra $\mathscr{A}$ has an equivalent description, in terms of the bicomplex $(CC^{*,*}(\mathscr{A}), b, B)$, defined as follows. With $C^n(\mathscr{A})$ denoting the linear space of $(n+1)$-linear forms on A, set

$$\begin{aligned} &CC^{p,q}(\mathscr{A}) = C^{q-p}(\mathscr{A}), \qquad q \geq p, \\ &CC^{p,q}(\mathscr{A}) = 0, \qquad q < p\,. \end{aligned} \tag{1.7}$$

The vertical operator $b : C^n(\mathscr{A}) \to C^{n+1}(\mathscr{A})$ is defined as

$$\begin{aligned} (b\varphi)(a^0, \dots, a^{n+1}) = \sum_{j=0}^{n} (-1)^j\, \varphi(a^0, \dots, a^j a^{j+1}, \dots, a^{n+1}) \\ + (-1)^{n+1}\, \varphi(a^{n+1} a^0, a^1, \dots, a^n). \end{aligned} \tag{1.8}$$

The horizontal operator $B : C^n(\mathscr{A}) \to C^{n-1}(\mathscr{A})$ is defined by the formula

$$B = NB_0,$$

where

$$\begin{aligned} B_0\,\varphi(a^0,\dots,a^{n-1}) &= \varphi(1,a^0,\dots,a^{n-1}) - (-1)^n\,\varphi(a^0,\dots,a^{n-1},1), \\ (N\psi)(a^0,\dots,a^n) &= \sum_0^n (-1)^{nj}\,\psi(a^j,a^{j+1},\dots,a^{j-1}). \end{aligned} \tag{1.9}$$

Then $HC^*(\mathscr{A})$ is the cohomology of the *first quadrant* total complex $(TC^*(\mathscr{A}),\, b+B)$, formed as follows:

$$TC^n(\mathscr{A}) = \sum_{p=0}^{n} CC^{p,n-p}(\mathscr{A}). \tag{1.10}$$

On the other hand, the cohomology of the *full direct sum* total complex $(TC^{\Sigma}_*(\mathscr{A}),\, b+B)$, formed by taking the direct sums

$$TC^{\Sigma}_n(\mathscr{A}) = \sum_{p\in\mathbb{Z}} CC^{p,n-p}(\mathscr{A}), \tag{1.11}$$

gives the ($\mathbb{Z}/2$-graded) *periodic cyclic cohomology* groups $HC^*_{\mathrm{per}}(\mathscr{A})$.

There is a dual description for the cyclic homology of $\mathscr{A}$, in terms of the dual bicomplex $(CC_{*,*}(\mathscr{A}),\, b,\, B)$, with $C_n(\mathscr{A}) = \mathscr{A}^{\otimes n+1}$ and the boundary operators b, B obtained by transposing the corresponding coboundaries. The periodic cyclic homology groups $HC^{\mathrm{per}}_*(\mathscr{A})$ are obtained from the *full product* total complex $(TC^*_{\Pi}(\mathscr{A}),\, b+B)$, formed by taking direct products as follows:

$$TC^n_{\Pi}(\mathscr{A}) = \Pi_p\, CC_{p,n-p}(\mathscr{A}). \tag{1.12}$$

The Chern character of an idempotent $e^2 = e \in \mathscr{A}$ is given in this picture by the periodic cycle $(ch_n(e))_{n=2,4,\dots}$, with components:

$$ch_0(e) = e, \quad ch_{2k}(e) = (-1)^k \frac{(2k)!}{k!}\left(e^{\otimes 2k+1} - \frac{1}{2}\otimes e^{\otimes 2k}\right), \quad k \ge 1. \tag{1.13}$$

The functors HC^0 and HC_0 from the category of algebras to the category of vector spaces have clear intrinsic meaning: The first assigns to an algebra $\mathscr{A}$ the vector space of traces on $\mathscr{A}$, while the second associates to $\mathscr{A}$ its abelianization $\mathscr{A}/[\mathscr{A},\mathscr{A}]$. From a conceptual viewpoint, it is important to realize the higher co/homologies HC^*, resp. HC_*, as derived functors. The obvious obstruction to such an interpretation is the non-additive nature of the category of algebras and algebra homomorphisms. This has been remedied in [9], by replacing it with the category of Λ-modules over the *cyclic category* Λ.

The cyclic category Λ is a small category, obtained by enriching with *cyclic morphisms* the familiar *simplicial category* Δ of totally ordered finite sets and increasing maps. We recall the presentation of Δ by generators and relations. It has one object $[n] = \{0 < 1 < \cdots < n\}$ for each integer $n \geq 0$, and is generated by faces $\delta_i : [n-1] \to [n]$ (the injection that misses i), and degeneracies $\sigma_j : [n+1] \to [n]$ (the surjection which identifies j with $j+1$), with the following relations:

$$\begin{aligned}
&\delta_j \delta_i = \delta_i \delta_{j-1} \qquad \text{for } i < j, \\
&\sigma_j \sigma_i = \sigma_i \sigma_{j+1} \qquad \text{for } i \leq j, \\
&\sigma_j \delta_i = \begin{cases} \delta_i \sigma_{j-1} & i < j, \\ 1_n & \text{if } i = j \text{ or } i = j+1, \\ \delta_{i-1} \sigma_j & i > j+1. \end{cases}
\end{aligned} \tag{1.14}$$

To obtain Λ one adds for each n a new morphism $\tau_n : [n] \to [n]$ such that,

$$\begin{aligned}
&\tau_n \delta_i = \delta_{i-1} \tau_{n-1}, \qquad 1 \leq i \leq n, \\
&\tau_n \sigma_i = \sigma_{i-1} \tau_{n+1}, \qquad 1 \leq i \leq n, \\
&\tau_n^{n+1} = 1_n.
\end{aligned} \tag{1.15}$$

Note that the above relations also imply:

$$\tau_n \delta_0 = \delta_n , \qquad \tau_n \sigma_0 = \sigma_n \tau_{n+1}^2 . \tag{1.16}$$

Alternatively, Λ can be defined by means of its "cyclic covering", the category $E\Lambda$. The latter has one object $(\mathbb{Z}, n)$ for each $n \geq 0$ and the morphisms $f : (\mathbb{Z}, n) \to (\mathbb{Z}, m)$ are given by non decreasing maps $f : \mathbb{Z} \to \mathbb{Z}$, such that $f(x+n) = f(x) + m, \quad \forall x \in \mathbb{Z}$. One has $\Lambda = E\Lambda / \mathbb{Z}$, with respect to the obvious action of $\mathbb{Z}$ by translation.

To any algebra $\mathscr{A}$ one associates a module $\mathscr{A}^\natural$ over the category Λ by assigning to each integer $n \geq 0$ the vector space $C^n(\mathscr{A})$ of $(n+1)$-linear forms $\varphi(a^0, \ldots, a^n)$ on $\mathscr{A}$, and to the generating morphisms the operators $\delta_i : C^{n-1} \to C^n$, $\sigma_i : C^{n+1} \to C^n$ defined as follows:

$$\begin{aligned}
&(\delta_i \varphi)(a^0, \ldots, a^n) = \varphi(a^0, \ldots, a^i a^{i+1}, \ldots, a^n), \quad i = 0, 1, \ldots, n-1, \\
&(\delta_n \varphi)(a^0, \ldots, a^n) = \varphi(a^n a^0, a^1, \ldots, a^{n-1}); \\
&(\sigma_0 \varphi)(a^0, \ldots, a^n) = \varphi(a^0, 1, a^1, \ldots, a^n), \\
&(\sigma_j \varphi)(a^0, \ldots, a^n) = \varphi(a^0, \ldots, a^j, 1, a^{j+1}, \ldots, a^n), \quad j = 1, \ldots, n-1, \\
&(\sigma_n \varphi)(a^0, \ldots, a^n) = \varphi(a^0, \ldots, a^n, 1); \\
&(\tau_n \varphi)(a^0, \ldots, a^n) = \varphi(a^n, a^0, \ldots, a^{n-1}).
\end{aligned} \tag{1.17}$$

These operations satisfy the relations (1.14) and (1.15), which shows that $\mathscr{A}^\natural$ is indeed a Λ-module.

One thus obtains the desired interpretation of the cyclic co/homology groups of a k-algebra $\mathscr{A}$ over a ground ring k in terms of derived functors over the cyclic category ([9]):

$$HC^n(\mathscr{A}) \simeq Ext^n_\Lambda(k^\natural, \mathscr{A}^\natural) \quad \text{and} \quad HC_n(\mathscr{A}) \simeq Tor^\Lambda_n(\mathscr{A}^\natural, k^\natural).$$

Moreover, all of the fundamental properties of the cyclic co/homology of algebras, such as the long exact sequence relating it to Hochschild co/homology ([8], [23]), are shared by the functors $Ext^*_\Lambda/Tor^\Lambda_*$-functors and, in this generality, can be attributed to the coincidence between the classifying space $B\Lambda$ of the small category Λ and the classifying space $BS^1 \simeq P_\infty(\mathbb{C})$ of the circle group.

Let us finally mention that, from the very definition of $Ext^*_\Lambda(k^\natural, F)$ and the existence of a canonical projective biresolution for $k^\natural$ ([9]), it follows that the cyclic cohomology groups $HC^*(F)$ of a Λ-module F, as well as the periodic ones $HC^*_{\text{per}}(F)$, can be computed by means of a bicomplex analogous to (1.7). A similar statement holds for the cyclic homology groups.

2. Cyclic theory for Hopf algebras

The familiar antiequivalence between suitable categories of *spaces* and matching categories of *associative algebras*, effected by the passage to coordinates, is of great significance in both the purely algebraic context (affine schemes versus commutative algebras) as well as the topological one (locally compact spaces versus commutative C^*-algebras). By extension, it has been adopted as a fundamental principle of noncommutative geometry. When applied to the realm of *symmetry*, it leads to promoting the notion of a *group*, whose coordinates form a commutative Hopf algebra, to that of a general *Hopf algebra*. The cyclic categorical formulation recalled above allows to adapt cyclic co/homology in a natural way to the treatment of symmetry in noncommutative geometry. This has been done in [14], [15] and will be reviewed below.

We consider a Hopf algebra $\mathscr{H}$ over $k = \mathbb{R}$ or $\mathbb{C}$, with unit $\eta : k \to \mathscr{H}$, counit $\varepsilon : \mathscr{H} \to k$ and antipode $S : \mathscr{H} \to \mathscr{H}$. We use the standard definitions ([25]) together with the usual convention for denoting the coproduct:

$$\Delta(h) = \sum h_{(1)} \otimes h_{(2)}, \qquad h \in \mathscr{H}. \tag{2.1}$$

Although we work in the algebraic context, we shall include a datum intended to play the rôle of the modular function of a locally compact group. For reasons of consistency with the Hopf algebra context, this datum has a self-dual nature: It comprises both a character $\delta \in \mathscr{H}^*$:

$$\delta(ab) = \delta(a)\delta(b), \quad \forall\, a, b \in \mathscr{H}, \tag{2.2}$$

and a group-like element $\sigma \in \mathscr{H}$:

$$\Delta(\sigma) = \sigma \otimes \sigma, \qquad \varepsilon(\sigma) = 1, \tag{2.3}$$

related by the condition

$$\delta(\sigma) = 1. \tag{2.4}$$

Such a pair (δ, σ) will be called a *modular pair*. The character δ gives rise to a δ-*twisted antipode* $\widetilde{S} = S_\delta : \mathscr{H} \to \mathscr{H}$, defined by

$$\widetilde{S}(h) = \sum_{(h)} \delta(h_{(1)})\, S(h_{(2)}), \quad h \in \mathscr{H}. \tag{2.5}$$

Like the untwisted antipode, $\widetilde{S}$ is an algebra antihomomorphism:

$$\begin{aligned} \widetilde{S}(h^1 h^2) &= \widetilde{S}(h^2)\widetilde{S}(h^1), \quad \forall h^1, h^2 \in \mathscr{H}, \\ \widetilde{S}(1) &= 1, \end{aligned} \tag{2.6}$$

a coalgebra twisted antimorphism:

$$\Delta\widetilde{S}(h) = \sum_{(h)} S(h_{(2)}) \otimes \widetilde{S}(h_{(1)}), \quad \forall h \in \mathscr{H}, \tag{2.7}$$

and it also satisfies the identities:

$$\varepsilon \circ \widetilde{S} = \delta, \qquad \delta \circ \widetilde{S} = \varepsilon. \tag{2.8}$$

We start by associating to $\mathscr{H}$, viewed only as a *coalgebra*, the standard cosimplicial module known as the *cobar resolution* ([1], [4]), twisted by the insertion of the group-like element $\sigma \in \mathscr{H}$. Specifically, we set $C^n(\mathscr{H}) = \mathscr{H}^{\otimes n}$, $\forall n \geq 1$, and $C^0(\mathscr{H}) = k$, then define the *face* operators $\delta_i : C^{n-1}(\mathscr{H}) \to C^n(\mathscr{H})$, $0 \leq i \leq n$, as follows, if $n > 1$:

$$\begin{aligned} \delta_0(h^1 \otimes \ldots \otimes h^{n-1}) =& 1 \otimes h^1 \otimes \ldots \otimes h^{n-1}, \\ \delta_j(h^1 \otimes \ldots \otimes h^{n-1}) =& h^1 \otimes \ldots \otimes \Delta h^j \otimes \ldots \otimes h^{n-1} \\ &= \sum_{(h_j)} h^1 \otimes \ldots \otimes h^j_{(1)} \otimes h^j_{(2)} \otimes \ldots \otimes h^{n-1}, \quad 1 \leq j \leq n-1, \\ \delta_n(h^1 \otimes \ldots \otimes h^{n-1}) =& h^1 \otimes \ldots \otimes h^{n-1} \otimes \sigma, \end{aligned} \tag{2.9}$$

while, if $n = 1$:

$$\delta_0(1) = 1, \qquad \delta_1(1) = \sigma.$$

Next, the *degeneracy* operators $\sigma_i : C^{n+1}(\mathscr{H}) \to C^n(\mathscr{H})$, $0 \leq i \leq n$, are defined by:

$$\begin{aligned} \sigma_i(h^1 \otimes \ldots \otimes h^{n+1}) =& h^1 \otimes \ldots \otimes \varepsilon(h^{i+1}) \otimes \ldots \otimes h^{n+1} \\ =& \varepsilon(h^{i+1})\, h^1 \otimes \ldots \otimes h^i \otimes h^{i+2} \otimes \ldots \otimes h^{n+1}, \end{aligned} \tag{2.10}$$

and, for $n = 0$:

$$\sigma_0(h) = \varepsilon(h), \quad h \in \mathscr{H}.$$

The remaining features of the given data, namely the *product* and the *antipode* of $\mathscr{H}$ together with the character $\delta \in \mathscr{H}^*$, are used to define the candidate for the *cyclic operator*, $\tau_n : C^n(\mathscr{H}) \to C^n(\mathscr{H})$, as follows:

$$\begin{aligned}\tau_n(h^1 \otimes \ldots \otimes h^n) =& (\Delta^{n-1}\widetilde{S}(h^1)) \cdot h^2 \otimes \ldots \otimes h^n \otimes \sigma \\ =& \sum_{(h^1)} S(h^1_{(n)})h^2 \otimes \ldots \otimes S(h^1_{(2)})h^n \otimes \widetilde{S}(h^1_{(1)})\sigma.\end{aligned} \tag{2.11}$$

Note that

$$\tau_1^2(h) = \tau_1(\widetilde{S}(h)\sigma) = \sigma^{-1}\widetilde{S}^2(h)\sigma,$$

therefore the following is a necessary condition for cyclicity:

$$(\sigma^{-1} \circ \widetilde{S})^2 = I. \tag{2.12}$$

The remarkable fact is that this condition is also sufficient for the implementation of the sought-for Λ-module.

A modular pair (δ, σ) satisfying (2.12) is called a *modular pair in involution*.

Theorem 1. **([14, 15])** *Let $\mathscr{H}$ be a Hopf algebra endowed with a modular pair (δ, σ) in involution. Then $\mathscr{H}^\natural_{(\delta,\sigma)} = \{C^n(\mathscr{H})\}_{n\geq 0}$ equipped with the operators given by (2.9) – (2.11) is a module over the cyclic category Λ.*

The cyclic cohomology groups corresponding to the Λ-module $\mathscr{H}^\natural_{(\delta,\sigma)}$, denoted by $HC^*_{(\delta,\sigma)}(\mathscr{H})$, can be computed from the bicomplex $(CC^{*,*}(\mathscr{H}), b, B)$, analogous to (1.7), defined as follows:

$$\begin{aligned}&CC^{p,q}(\mathscr{H}) = C^{q-p}(\mathscr{H}), \quad q \geq p, \\ &CC^{p,q}(\mathscr{H}) = 0, \quad q < p,\end{aligned} \tag{2.13}$$

the operator

$$b : C^{n-1}(\mathscr{H}) \to C^n(\mathscr{H}), \qquad b = \sum_{i=0}^{n} (-1)^i \delta_i, \tag{2.14}$$

is explicitly given, if $n \geq 1$, by

$$\begin{aligned}b(h^1 \otimes \ldots \otimes h^{n-1}) =& 1 \otimes h^1 \otimes \ldots \otimes h^{n-1} \\ &+ \sum_{j=1}^{n-1} (-1)^j \sum_{(h_j)} h^1 \otimes \ldots \otimes h^j_{(1)} \otimes h^j_{(2)} \otimes \ldots \otimes h^{n-1} \\ &+ (-1)^n h^1 \otimes \ldots \otimes h^{n-1} \otimes \sigma,\end{aligned}$$

and, if $n=0$

$$b(c) = c\cdot(1-\sigma), \quad c\in k,$$

the operator $B : C^{n+1}(\mathscr{H}) \to C^n(\mathscr{H})$ is defined by the formula

$$B = N_n \circ \widetilde{\sigma}_{-1} \circ (1+(-1)^n \tau_{n+1}), \quad n\geq 0, \tag{2.15}$$

where $\widetilde{\sigma}_{-1} : C^{n+1}(\mathscr{H}) \to C^n(\mathscr{H})$, is the *extra degeneracy* operator

$$\begin{aligned}\widetilde{\sigma}_{-1}(h^1\otimes\ldots\otimes h^{n+1}) =&(\Delta^{n-1}\widetilde{S}(h^1))\cdot h^2\otimes\ldots\otimes h^{n+1}\\ =&\sum_{(h^1)} S(h^1_{(n)})h^2\otimes\ldots\otimes S(h^1_{(2)})h^n\otimes\widetilde{S}(h^1_{(1)})h^{n+1},\end{aligned} \tag{2.16}$$

$$\widetilde{\sigma}_1(h) = \delta(h), \qquad h\in\mathscr{H},$$

and

$$N_n = 1+(-1)^n\tau_n+\ldots+(-1)^{n^2}\tau_n{}^n. \tag{2.17}$$

Explicitly:

$$\begin{aligned}&N_n(h^1\otimes\ldots\otimes h^n) \\ &\quad=\sum_{j=0}^n(-1)^{nj}\Delta^{n-1}\widetilde{S}(h^j)\cdot h^{j+1}\otimes\ldots\otimes h^n\otimes\sigma\otimes\widetilde{S}^2(h^0)\sigma\otimes\ldots\otimes\widetilde{S}^2(h^{j-1})\sigma\\ &\quad=\sum_{j=0}^n(-1)^{nj}\Delta^{n-1}\widetilde{S}(h^j)\cdot h^{j+1}\otimes\ldots\otimes h^n\otimes\sigma\otimes\sigma h^0\otimes\ldots\otimes\sigma h^{j-1}\\ &\quad=\sum_{j=0}^n(-1)^{nj}\sum_{(h_j)}S(h^j_{(n)})\otimes\ldots\otimes S(h^j_{(2)})\otimes\widetilde{S}(h^j_{(1)})\cdot\\ &\qquad\quad\cdot h^{j+1}\otimes\ldots\otimes h^n\otimes\sigma\otimes\sigma h^0\otimes\ldots\otimes\sigma h^{j-1}.\end{aligned} \tag{2.18}$$

In particular, for $n=0$:

$$B(h) = \delta(h)+\varepsilon(h), \quad \forall\, h\in\mathscr{H}. \tag{2.19}$$

The expression of the B-operator can be simplified by passing to the quasi-isomorphic *normalized bicomplex* $(C\bar{C}^{*,*}_{(\delta,\sigma)}(\mathscr{H}), b, \bar{B})$, defined as follows

$$\begin{aligned}C\bar{C}^{p,q}_{(\delta,\sigma)}(\mathscr{H}) =&\bar{C}^{q-p}(\mathscr{H}), \quad q\geq p,\\ C\bar{C}^{p,q}_{(\delta,\sigma)}(\mathscr{H}) =&0, \qquad q<p,\end{aligned} \tag{2.20}$$

where

$$\bar{C}^n(\mathscr{H}) = (\mathrm{Ker}\,\varepsilon)^{\otimes n}, \quad \forall n\geq 1, \qquad \bar{C}^0(\mathscr{H}) = k,$$

while the formula for the b-operator remains unchanged, the new horizontal operator becomes

$$\bar{B} = N_n\circ\widetilde{\sigma}_{-1}, \quad n\geq 0, \tag{2.21}$$

in particular, for $n = 0$, one has

$$\bar{B}(h) = \delta(h), \quad \forall\, h \in \mathscr{H}. \tag{2.22}$$

An alternate description of the cyclic cohomology groups $HC^*_{(\delta,1)}(\mathscr{H})$, in terms of the Cuntz-Quillen formalism, is given by M. Crainic in [16].

We should also mention that the corresponding *cyclic homology* groups

$$HC_n^{(\delta,\sigma)}(\mathscr{H}) = Tor_n^{\Lambda}(\mathscr{H}^{\natural}_{(\delta,\sigma)}, k^{\natural}) \tag{2.23}$$

can be computed from the bicomplex $(CC_{*,*}(\mathscr{H}), b, B)$, obtained by dualizing (2.13) in the obvious fashion:

$$\begin{aligned} CC_{p,q}(\mathscr{H}) =& Hom(C^{q-p}(\mathscr{H}), k), \quad q \geq p, \\ CC_{p,q}(\mathscr{H}) =& 0, \quad q < p, \end{aligned} \tag{2.24}$$

with the boundary operators b and B the transposed of the corresponding coboundaries.

When applied to the usual notion of symmetry in differential geometry, the "Hopf algebraic" version of cyclic cohomology discussed above recovers both the Lie algebra co/homology and the differentiable cohomology of Lie groups, as illustrated by the following results.

Proposition 2. **([14])** *Let* $\mathfrak{g}$ *be a Lie algebra and let* $\delta : \mathfrak{g} \to \mathbb{C}$ *be a character of* $\mathfrak{g}$. *With* $\mathscr{U}(\mathfrak{g})$ *denoting the enveloping algebra of* $\mathfrak{g}_{\mathbb{C}}$, *viewed as a Hopf algebra with modular pair* $(\delta, 1)$, *one has*

$$HC^*_{\mathrm{per}(\delta,1)}(\mathscr{U}(\mathfrak{g})) \simeq \sum_{i \equiv *(2)}^{\oplus} H_i(\mathfrak{g}, \mathbb{C}_\delta),$$

where $\mathbb{C}_\delta$ *is the* 1*-dimensional* $\mathfrak{g}$*-module associated to the character* δ.

Remark 3 In a dual fashion, one can prove that

$$HC_*^{\mathrm{per}(\delta,1)}(\mathscr{U}(\mathfrak{g})) \simeq \sum_{i \equiv *(2)}^{\oplus} H^i(\mathfrak{g}, \mathbb{C}_\delta). \tag{2.25}$$

Proposition 4. **([14])** *Let* $\mathscr{H}(G)$ *be the Hopf algebra of polynomial functions on a simply connected affine algebraic nilpotent group G, with Lie algebra* $\mathfrak{g}$. *Then its periodic cyclic cohomology with respect to the trivial modular pair* $(\varepsilon, 1)$ *coincides with the Lie algebra cohomology:*

$$HC^*_{\mathrm{per}}(\mathscr{H}(G)) \simeq \sum_{i \equiv *(2)}^{\oplus} H^i(\mathfrak{g}, \mathbb{C}).$$

Remark 5 Since by Van Est's Theorem, the cohomology of the nilpotent Lie algebra $\mathfrak{g}$ is isomorphic to the differentiable group cohomology $H^*_{\rm d}(G)$, the above isomorphism can be reformulated as

$$HC^*_{\rm per}(\mathscr{H}(G))) \simeq \sum_{i\equiv *(2)}^{\oplus} H^i_{\rm d}(G,\mathbb{C}).$$

Under the latter form it continues to hold for any affine algebraic Lie group G, with the same proof as in [14] .

Hopf algebras often arise implemented as endomorphisms of associative algebras. A *Hopf action* of a Hopf algebra $\mathscr{H}$ on an algebra $\mathscr{A}$ is given by a linear map, $\mathscr{H}\otimes\mathscr{A}\to\mathscr{A}, \quad h\otimes a\to h(a)$ satisfying the *action property*

$$h_1(h_2a) = (h_1h_2)(a), \qquad \forall h_i\in\mathscr{H}, a\in\mathscr{A}, \tag{2.26}$$

and the *Hopf-Leibniz rule*

$$h(ab) = \sum h_{(1)}(a)\, h_{(2)}(b), \qquad \forall a,b\in\mathscr{A}, h\in\mathscr{H}. \tag{2.27}$$

In practice, $\mathscr{H}$ first appears as a subalgebra of endomorphisms of an algebra $\mathscr{A}$ fulfilling (2.27), and it is precisely the Hopf-Leibniz rule which dictates the coproduct of $\mathscr{H}$. In turn, the modular pair of $\mathscr{H}$ arises in connection with the existence of a twisted trace on $\mathscr{A}$.

Given a Hopf action $\mathscr{H}\otimes\mathscr{A}\to\mathscr{A}$ together with a modular pair (δ,σ), a linear form $\tau:\mathscr{A}\to\mathbb{C}$ is called a σ-*trace* under the action of $\mathscr{H}$ if

$$\tau(ab) = \tau(b\sigma(a)), \qquad \forall a,b\in\mathscr{A}. \tag{2.28}$$

The σ-trace τ is called δ-*invariant* under the action of $\mathscr{H}$ if

$$\tau(h(a)b) = \tau(a\widetilde{S}(h)(b)), \qquad \forall a,b\in\mathscr{A},\ h\in\mathscr{H}. \tag{2.29}$$

If $\mathscr{A}$ is unital, the "integration by parts" formula (2.29) is equivalent to the δ-invariance condition

$$\tau(h(a)) = \delta(h)\,\tau(a), \qquad \forall a\in\mathscr{A},\ h\in\mathscr{H}.$$

With the above assumptions, the very definition of the cyclic co/homology of $\mathscr{H}$, with respect to a modular pair in involution (δ,σ), is uniquely dictated such that the following *Hopf action principle* holds:

Theorem 6. **([14, 15])** *Let $\tau:\mathscr{A}\to\mathbb{C}$ be a δ-invariant σ-trace under the Hopf action of $\mathscr{H}$ on $\mathscr{A}$. Then the assignment*

$$\gamma(h^1\otimes\ldots\otimes h^n)(a^0,\ldots,a^n) = \tau(a^0h^1(a^1)\ldots h^n(a^n)), \tag{2.30}$$

$\forall a^0,\ldots,a^n \in \mathscr{A},\ h^1,\ldots,h^n \in \mathscr{H}$, defines a map of Λ-modules $\gamma^\natural : \mathscr{H}^\natural_{(\delta,\sigma)} \to \mathscr{A}^\natural$, which in turn induces characteristic homomorphisms in cyclic co/homology:

$$\gamma^*_\tau : HC^*_{(\delta,\sigma)}(\mathscr{H}) \to HC^*(\mathscr{A}); \tag{2.31}$$

$$\gamma^\tau_* : HC_*(\mathscr{A}) \to HC^{(\delta,\sigma)}_*(\mathscr{H}). \tag{2.32}$$

As a quick illustration, let us assume that $\mathfrak{g}$ is a Lie algebra of derivations of an algebra $\mathscr{A}$ and τ is a δ-invariant trace on $\mathscr{A}$. Then (2.32), composed with the Chern character in cyclic homology (1.13) on one hand and with the isomorphism (2.25) on the other, recovers the additive map

$$ch^*_\tau : K_*(\mathscr{A}) \to H^*(\mathfrak{g},\mathbb{C}_\delta), \tag{2.33}$$

previously introduced in [6], in terms of $\mathfrak{g}$-invariant curvature forms associated to an arbitrary $\mathfrak{g}$-connection. Before applying the isomorphism (2.25), the periodic cyclic class $\gamma_*(ch_*(e)) \in HC^{\mathrm{per}\,(\delta,1)}_*(\mathscr{U}(\mathfrak{g}))$, for $e^2 = e \in \mathscr{A}$, is given by the cycle with the following components:

$$\begin{aligned}
&\gamma_*(ch_0(e)) = \tau(e),\\
&\gamma_*(ch_{2k}(e))\,(h^1,\ldots,h^{2k})\\
&\quad = (-1)^k \frac{(2k)!}{k!}\left(\tau(eh^1(e)\cdots h^{2k}(e)) - \frac{1}{2}\tau(h^1(e)\cdots h^{2k}(e))\right),\\
&\qquad\qquad \forall k \geq 1,\ \forall h^1,\ldots,h^{2k} \in \mathscr{U}(\mathfrak{g}),
\end{aligned} \tag{2.34}$$

The Lie algebra cocycle representing the class $ch^*_\tau(e) \in H^*(\mathfrak{g},\mathbb{C}_\delta)$ in terms of the Grassmannian connection is obtained by restricting $\gamma^\tau_*(ch_*(e))$ to $\wedge^\bullet\mathfrak{g}$ via antisymmetrization.

3. Transverse index theory on general foliations

The developments discussed in the preceding section have been largely motivated by a challenging computational problem concerning the index of transversely hypoelliptic differential operators on foliations [13]. In turn, they were instrumental in settling it [14]. The goal of this section is to highlight the main steps involved.

The transverse geometry of a foliation $(V,\mathscr{F})$, i.e., the geometry of the "space" of leaves $V/\mathscr{F}$, provides a prototypical example of noncommutative space, which already exhibits many of the distinctive features of the general theory. In noncommutative geometry, a geometric space is given by a *spectral triple* $(\mathscr{A},\mathscr{H},D)$, where $\mathscr{A}$ is an involutive algebra of operators in a Hilbert space $\mathscr{H}$, representing the "local coordinates" of the space, and D is an unbounded self-adjoint operator on $\mathscr{H}$. The operator $D^{-1} = ds$ corresponds to the infinitesimal line element in Riemannian geometry and, in addition to its metric significance, it also carries nontrivial homological meaning, representing a *K-homology* class of $\mathscr{A}$. The construction of such a spectral triple associated to a general foliation ([13]) comprises several steps and incorporates important ideas from [10] and [20].

To begin with, we recall that a codimension n foliation $\mathscr{F}$ of an N-dimensional manifold V can be given by means of a *defining cocycle* (U_i, f_i, g_{ij}), where $\{U_i\}$ is an open cover of V, $f_i : U_i \to T_i$ are submersions with connected fibers onto n-dimensional manifolds $\{T_i\}$ and

$$g_{ij} : f_j(U_i \cap U_j) \to f_i(U_i \cap U_j),$$

are diffeomorphisms such that

$$\forall\ (i,j), \quad f_i = g_{ij} \circ f_j \text{ on } U_i \cap U_j.$$

The disjoint union $M = \cup_i T_i \times \{i\}$ can be regarded as a complete transversal for the foliation, while the collection of local diffeomorphisms $\{g_{ij}\}$ of M generates the transverse *holonomy pseudogroup* Γ. We should note that the notion of *pseudogroup* used here is slightly different from the standard one, since we do not enforce the customary hereditary condition; in particular, any group of diffeomorphisms is such a pseudogroup.

We shall assume $\mathscr{F}$ *transversely oriented*, which amounts to stipulating that M is oriented and that Γ consists of orientation preserving local diffeomorphisms. From M we shall pass by a Diff^+-functorial construction ([10]) to a quotient bundle, $\pi : P \to M$, of the frame bundle, whose sections are the Riemannian metrics on M. Specifically, $P = F/SO(n)$, where F is the $GL^+(n,\mathbb{R})$-principal bundle of oriented frames on M. The total space P admits a canonical *para-Riemannian structure* as follows. The vertical subbundle $\mathscr{V} \subset TP$, $\mathscr{V} = \operatorname{Ker} \pi_*$, carries natural Euclidean structures on each of its fibers, determined solely by the choice of a $GL^+(n,\mathbb{R})$-invariant Riemannian metric on the symmetric space $GL^+(n,\mathbb{R})/SO(n)$. On the other hand, the quotient bundle $\mathscr{N} = (TP)/\mathscr{V}$ comes equipped with a tautologically defined Riemannian structure: Every $p \in P$ is an Euclidean structure on $T_{\pi(p)}(M)$ which is identified to $\mathscr{N}_p$ via π_*.

The naturality of the above construction with respect to Diff^+ ensures that the action of the holonomy pseudogroup Γ lifts to both bundles F and P. One can thus form for each the associated *smooth étale groupoid* $F \rtimes \Gamma$, resp. $P \rtimes \Gamma$. An element of $F \rtimes \Gamma$ or of $P \rtimes \Gamma$ is given by a pair

$$(x, \varphi)\,, \qquad x \in \operatorname{Range} \varphi\,,$$

while the composition law is

$$(x,\varphi) \circ (y,\psi) = (x, \varphi \circ \psi) \quad \text{if} \quad y \in \operatorname{Dom} \varphi \quad \text{and} \quad \varphi(y) = x\,.$$

We let

$$\mathscr{A}_F = C_c^\infty(F \rtimes \Gamma)\,, \quad \text{resp.} \quad \mathscr{A}_P = C_c^\infty(P \rtimes \Gamma)$$

denote the corresponding convolution algebras. The elements of $\mathscr{A} = \mathscr{A}_P$ will serve as “functions of local coordinates” for the noncommutative space $V/\mathscr{F}$. A generic element of $\mathscr{A}$ can be represented as a linear combination of monomials

$$a = f\,U_\psi^*\,, \quad f \in C_c^\infty(\operatorname{Dom} \psi)\,,$$

where the star indicates a contravariant notation. The multiplication rule is

$$f_1\, U^*_{\psi_1} \cdot f_2\, U^*_{\psi_2} = f_1 \cdot (f_2 \circ \widetilde{\psi}_1)\, U^*_{\psi_2 \psi_1}\,,$$

where by hypothesis the support of $f_1 (f_2 \circ \widetilde{\psi}_1)$ is a compact subset of

$$\mathrm{Dom}\, \psi_1 \cap \psi_1^{-1}\, \mathrm{Dom}\, \psi_2 \subset \mathrm{Dom}\, \psi_2\, \psi_1\,.$$

The algebras $\mathscr{A} = \mathscr{A}_P$ and $\mathscr{A}_F$ admit canonical $*$-representations on the Hilbert spaces

$$L^2(P) = L^2(P, vol_P)\,, \quad \text{resp.} \quad L^2(F) = L^2(F, vol_F)\,,$$

where vol_P, resp. vol_F, denotes the canonical Diff^+-invariant volume form on P, resp. on F. Explicitly, for $\mathscr{A} = \mathscr{A}_P$,

$$((f\, U^*_\psi)\, \xi)(p) = f(p)\, \xi(\psi(p)) \quad \forall\, p \in P\,,\ \xi \in L^2(P)\,,\ f\, U^*_\psi \in \mathscr{A}\,, \tag{3.1}$$

and similarly for $\mathscr{A}_F$. We shall denote by $A = \bar{\mathscr{A}}$ the norm closure of $\mathscr{A}$ in this representation.

Evidently, the algebra $\mathscr{A}$ depends on the choice of the defining cocycle (U_i, f_i, g_{ij}). However, if (U_i', f_i', g_{ij}') is another cocycle defining the same foliation $\mathscr{F}$, with corresponding algebra $\mathscr{A}'$ (resp. A'), then $\mathscr{A}$ and $\mathscr{A}'$ are *Morita equivalent*, while the C^*-algebras A and A' are *strongly Morita equivalent*. We recall that Morita equivalence preserves the cyclic co/homology and the K-theory/K-homology. Also, in the commutative case it simply reduces to isomorphism; e.g., absent any nontrivial pseudogroup of diffeomorphisms, two manifolds N and N' are diffeomorphic iff the algebras $C_c^\infty(N)$ and $C_c^\infty(N')$ are Morita equivalent.

To complete the description of the spectral triple associated to $V/\mathscr{F}$, we need to define the operator D. In practice, it is more convenient to work with another representative of the same K-homology class, namely the *hypoelliptic signature operator* $Q = D|D|$. The latter is a second order differential operator, acting on the Hilbert space

$$\mathscr{H} = \mathscr{H}_P := L^2(\wedge^\bullet \mathscr{V}^* \otimes \wedge^\bullet \mathscr{N}^*,\, vol_P), \tag{3.2}$$

it is defined as a graded sum

$$Q = (d_V^*\, d_V - d_V\, d_V^*) \oplus (d_H + d_H^*)\,, \tag{3.3}$$

where d_V denotes the vertical exterior derivative and d_H stands for the horizontal exterior differentiation with respect to a fixed connection on the frame bundle. When $n \equiv 1$ or $2 \pmod 4$, for the vertical component to make sense, one has to replace P with $P \times S^1$ so that the dimension of the vertical fiber stays even.

Proposition 7. **([13])** *For any $a \in \mathscr{A}$, $[D, a]$ is bounded. For any $f \in C_c^\infty(P)$ and $\lambda \notin \mathbb{R}$, $f(D - \lambda)^{-1}$ is p-summable, $\forall\, p > m = \frac{n(n+1)}{2} + 2n$.*

One now confronts a well-posed index problem. The operator D determines an *index pairing* map $\mathrm{Index}_D : K_*(\mathscr{A}) \to \mathbb{Z}$ as follows:

(0) In the *graded* (or *even*) case,

$$\mathrm{Index}_D([e]) = \mathrm{Index}\,(eD^+e)\,, \quad e^2 = e \in \mathscr{A}\,;$$

(1) In the *ungraded* (or *odd*) case,

$$\mathrm{Index}_D([u]) = \mathrm{Index}\,(P^+uP^+)\,, \quad u \in GL_1(\mathscr{A})\,,$$

where $P^+ = \frac{1+F}{2}$, with $F = \mathrm{Sign}\,(D)$.

One of the main functions of cyclic co/homology, its *raison d'être* in some sense, is to compute the index pairing via the equality

$$\mathrm{Index}_D(\kappa) = \langle ch_*(D), \mathrm{ch}^*(\kappa)\rangle, \qquad \forall\, \kappa \in K_*(\mathscr{A}). \tag{3.4}$$

The cyclic cohomology class $ch_*(D) \in HC^*_{\mathrm{per}}(\mathscr{A})$, i.e., its *Chern character in K-homology*, is defined in the *ungraded case* by means of the cyclic cocycle

$$\tau_F(a^0,\ldots,a^n) = \mathrm{Trace}\,(a^0[F,a^1]\ldots[F,a^n])\,, \qquad a^j \in \mathscr{A}\,, \tag{3.5}$$

where n is any odd integer exceeding the dimension of the spectral triple $(\mathscr{A},\mathscr{H},D)$; in the *graded case*, Trace is replaced with the graded trace Trace_s and n is even. Being defined by means of the *operator trace*, the cocycle (3.5) is inherently difficult to compute. The problem is therefore to *provide an explicit formula for the Chern character in K-homology*.

We should note at this point that, for smooth groupoids such as those associated to foliations, the answer to (3.4) is indeed known for all *K-theory classes in the range of the assembly map* from the corresponding geometric K-group to the analytic one (cf. [11]).

As mentioned before, the functors K-theory/K-homology and cyclic co/homology are Morita invariant. Moreover, the corresponding Chern characters are *Morita equivariant*, in such a way that both sides of (3.4) are preserved by the canonical isomorphisms associated with a Morita equivalence datum. Thus, one may as well take advantage of the Morita invariant nature of the problem and choose from the start a defining cocycle (U_i, f_i, g_{ij}) for $\mathscr{F}$ with all local transversals T_i *flat affine manifolds*. This renders M itself as a flat affine manifold, although it *does not* allow one to assume that the affine structure is preserved by Γ. One can however take the horizontal component in (3.3) with respect to a *flat connection* ∇. It is then readily seen that the operator Q belongs to the class of operators of the form

$$R = \pi_a(R_{\mathscr{U}}), \quad \text{with} \quad R_{\mathscr{U}} \in (\mathscr{U}(G_a(n)) \otimes \mathrm{End}(E))^{SO(n)}, \tag{3.6}$$

where $G_a(n) = \mathbb{R}^n \rtimes GL(n,\mathbb{R})$ is the affine group, π_a denotes its right regular representation and E is a unitary $SO(n)$-module.

A differential operator R of the form (3.6) will be called *affine*. If in addition the principal symbol of R, with respect to (3.7), is invertible then R will be called an *hypoelliptic affine operator*.

By an easy adaptation of a classical theorem of Nelson and Stinespring, one can show that *any hypoelliptic affine operator R which is formally self-adjoint is in fact essentially self-adjoint*, with core any dense $G_a(n)$-invariant subspace of C^∞-vectors for π_a.

The hypoelliptic calculus adapted to the para-Riemannian structure of the manifold P and to the treatment of the above operators is a particular case of the pseudodifferential calculus on Heisenberg manifolds ([2]). One simply modifies the notion of homogeneity of symbols $\sigma(p,\xi)$ by using the homotheties

$$\lambda \cdot \xi = (\lambda \xi_v, \lambda^2 \xi_n), \quad \forall \lambda \in \mathbb{R}_+^*, \tag{3.7}$$

where ξ_v, ξ_n are the vertical, resp. normal, components of the covector ξ. The above formula depends on local coordinates (x_v, x_n) adapted to the vertical foliation, but the corresponding pseudodifferential calculus is independent of such choices. The principal symbol of a hypoelliptic operator of order k is a function on the fibers of $\mathscr{V}^* \oplus \mathscr{N}^*$, homogeneous of degree k in the sense of (3.7). The distributional kernel $k(x,y)$ of a pseudodifferential operator T in this hypoelliptic calculus has the following behavior near the diagonal:

$$k(x,y) \sim \sum a_j(x, x-y) - a(x) \log |x-y|' + O(1), \tag{3.8}$$

where a_j is homogeneous of degree $-j$ in $x-y$ in the sense of (3.7), and the metric $|x-y|'$ is locally of the form

$$|x-y|' = ((x_v - y_v)^4 + (x_n - y_n)^2)^{1/4}. \tag{3.9}$$

The 1-density $a(x)$ is independent on the choice of metric $|\cdot|'$ and can be obtained from the symbol of order $-m$ of T, where

$$m = \frac{n(n+1)}{2} + 2n$$

is the *Hausdorff dimension* of the metric space $(P, |\cdot|')$. Like in the ordinary pseudodifferential calculus, this allows to define a residue of Wodzicki-Guillemin-Manin type, extending the Dixmier trace to operators of all degrees, by the equality

$$⨍ T = \frac{1}{m} \int_P a(x). \tag{3.10}$$

One uses the hypoelliptic calculus to prove ([13]) that the spectral triple $(\mathscr{A}, \mathscr{H}, D)$, or more generally that obtained by replacing Q with any hypoelliptic affine operator R (in which case $D|D| = R$), fulfills the hypotheses of the operator theoretic *local index theorem* of [13]. Its application allows to express the corresponding Chern character

$$ch_*(R) = ch_*(D) \in HC^*_{\mathrm{per}}(\mathscr{A}_P)$$

in terms of the *locally computable* residue (3.10). In the odd case, it is given by the cocycle $\Phi_R = \{\varphi_n\}_{n=1,3,\ldots}$ in the (b,B)-bicomplex of $\mathscr{A}$ defined as follows:

$$\varphi_n(a^0,\ldots,a^n) = \sum_k c_{n,k} \,⨍\, a^0[R,a^1]^{(k_1)}\cdots[R,a^n]^{(k_n)}\,|R|^{-n-2|k|}, \quad a^j \in \mathscr{A}, \quad (3.11)$$

where we used the abbreviations

$$T^{(k)} = \nabla^k(T) \qquad \text{and} \qquad \nabla(T) = D^2T - TD^2,$$

k is a multi-index, $|k| = k_1 + \ldots + k_n$, and

$$c_{n,k} = (-1)^{|k|}\sqrt{2i}(k_1!\ldots k_n!)^{-1}\left((k_1+1)\ldots(k_1+\ldots+k_n+n)\right)^{-1}\Gamma(|k|+\frac{n}{2});$$

there are finitely many nonzero terms in the above sum and only finitely many components of Φ_R are nonzero. In the even case, the corresponding cocycle $\Phi_R = \{\varphi_n\}_{n=0,2,\ldots}$ is defined in a similar fashion, except for φ_0 (see [13]).

The expression (3.11) is definitely explicitly computable, but its actual computation is exceedingly difficult to perform. Already in the case of codimension 1 foliations, where we did carry through its calculation by hand, it involves computing thousands of terms. On the other hand, in the absence of a guiding principle, computer calculations are unlikely to produce an illuminating answer. However, a simple inspection of (3.11) reveals some helpful general features. For the clarity of the exposition, we shall restrict our comments to the case $R = Q$, which is our main case of interest anyway.

First of all, since the passage from $(\mathscr{A}_F, \mathscr{H}_F)$ to $(\mathscr{A}_P, \mathscr{H}_P)$ involves the rather harmless operation of taking K-invariants with respect to the compact group $K = SO(n)$, we may work directly at the level of the frame bundle, equivariantly with respect to K. Secondly, since we are interested in the flat case, we may as well assume for starters that $M = \mathbb{R}^n$, with the trivial connection. This being the case, we shall identify F with the affine group $G_a(n)$. We may also replace Γ by the full group $\mathrm{Diff}^+(\mathbb{R}^n)$ and thus work for a while with the algebra

$$\mathscr{A}(n) = C_c^\infty(F(\mathbb{R}^n) \rtimes \mathrm{Diff}^+(\mathbb{R}^n)).$$

We recall that Q is built from the vertical vector fields $\{Y_i^j;\, i,j = 1,\ldots,n\}$ which form the canonical basis of $\mathfrak{gl}(n,\mathbb{R})$ and the horizontal vector fields $\{X_k,\, k = 1,\ldots,n\}$ coming from the canonical basis of $\mathbb{R}^n$. Therefore, the expression under the residue-integral in (3.11) involve iterated commutators of these vector fields with multiplication operators of the form $a = f\,U_\psi^*$, $f \in C_c^\infty(F,)$, $\psi \in \Gamma$. Now the canonical action of $GL^+(n,\mathbb{R})$ on F commutes with the action of Γ and hence extends canonically to the crossed product $\mathscr{A}_F$. At the Lie algebra level, this implies that the operators on $\mathscr{A}_F$ defined by

$$Y_i^j(f\,U_\psi^*) = (Y_i^j f)\,U_\psi^* \qquad (3.12)$$

are derivations:

$$Y_i^j(ab) = Y_i^j(a)\,b + a\,Y_i^j(b)\,. \tag{3.13}$$

The horizontal vector fields X_k on F can also be made to act on the crossed product algebra, according to the rule

$$X_k(f\,U_\psi^*) = X_k(f)\,U_\psi^*\,. \tag{3.14}$$

However, since the trivial connection is not preserved by the action of Γ, the operators X_k are no longer derivations of $\mathscr{A}_F$; they satisfy instead

$$X_i(ab) = X_i(a)\,b + a\,X_i(b) + \sum \delta_{ji}^k(a)\,Y_k^j(b)\,, \qquad a,b \in \mathscr{A}\,. \tag{3.15}$$

The linear operations δ_{ij}^k are of the form

$$\delta_{ij}^k(f\,U_\psi^*) = \gamma_{ij}^k\,f\,U_\psi^*\,, \tag{3.16}$$

with $\gamma_{ij}^k \in C^\infty(F \rtimes \Gamma)$ characterized by the identity

$$\psi^*\,\omega_j^i - \omega_j^i = \sum_k \gamma_{jk}^i\,\theta^k\,, \tag{3.17}$$

where ω is the standard *flat connection form* and θ is the *fundamental form* on $F = F(\mathbb{R}^n)$. From (3.17) it easily follows that each δ_{ij}^k is a derivation:

$$\delta_{ij}^k(ab) = \delta_{ij}^k(a)\,b + a\,\delta_{ij}^k(b)\,. \tag{3.18}$$

The commutation of the Y_i^j with δ_{ab}^c preserves the linear span of the δ_{ab}^c. However, the successive commutators with X_k produces new operators:

$$\delta_{ab,i_1\ldots i_r}^c = [X_{i_r}, \ldots [X_{i_1}, \delta_{ab}^c] \ldots]\,, \qquad r \geq 1\,,$$

symmetric in the indices $i_1, \ldots, i_r$. They are all of the form $T\,(f\,U_\psi^*) = h\,f\,U_\psi^*$, with $h \in C^\infty(F \rtimes \Gamma)$; in particular, they pairwise commute.

A first observation is that the linear space

$$\mathfrak{h}(n) = \sum \mathbb{C}\cdot Y_i^j \oplus \sum \mathbb{C}\cdot X_k \oplus \sum \mathbb{C}\cdot \delta_{ab,i_1\ldots i_r}^c$$

forms a Lie algebra and furthermore, if we let

$$\mathscr{H}(n) = \mathscr{U}(\mathfrak{h}(n))$$

denote the corresponding enveloping algebra, then $\mathscr{H}(n)$ acts on $\mathscr{A}$ satisfying a Leibniz rule of the form

$$h(ab) = \sum h_{(0)}(a)\,h_{(1)}(b)\,, \qquad \forall\, a,b \in \mathscr{A}\,. \tag{3.19}$$

A second observation is that the product rules (3.13), (3.15) and (3.18) can be converted into *coproduct* rules

$$\begin{aligned} \Delta Y_i^j &= Y_i^j \otimes 1 + 1 \otimes Y_i^j, \\ \Delta X_i &= X_i \otimes 1 + 1 \otimes X_i + \sum_k \delta_{ji}^k \otimes Y_k^j, \\ \Delta \delta_{ij}^k &= \delta_{ij}^k \otimes 1 + 1 \otimes \delta_{ij}^k. \end{aligned} \tag{3.20}$$

Together with the requirement:

$$\Delta[Z_1, Z_2] = [\Delta Z_1, \Delta Z_2], \qquad \forall Z_1, Z_2 \in \mathfrak{h}(n),$$

which is satisfied on generators because of the flatness of the connection, they uniquely determine a multiplicative coproduct

$$\Delta : \mathscr{H}(n) \to \mathscr{H}(n) \otimes \mathscr{H}(n). \tag{3.21}$$

One can check that this coproduct is coassociative and also that it is compatible with the Leibniz rule (3.19), in the sense that

$$\Delta h = \sum h_{(0)} \otimes h_{(1)} \quad \text{iff} \quad \Delta(h_1 h_2) = \Delta h_1 \cdot \Delta h_2, \qquad \forall\, h_j \in \mathscr{H}(n),\ \forall\, a, b \in \mathscr{A}. \tag{3.22}$$

Simple computations show that there is a unique antiautomorphism S of $\mathscr{H}(n)$ such that

$$S(Y_i^j) = -Y_i^j, \qquad S(\delta_{ab}^c) = -\delta_{ab}^c, \qquad S(X_a) = -X_a + \delta_{ab}^c Y_c^b.$$

Moreover, S serves as antipode for the bialgebra $\mathscr{H}(n)$; we should note though that $S^2 \neq I$.

To summarize, one has:

Proposition 8. **([14])** *The enveloping algebra $\mathscr{H}(n)$ of the Lie algebra generated by the canonical action of $\mathbb{R}^n \rtimes \mathfrak{gl}(n,\mathbb{R})$ on $\mathscr{A}(n)$ has a unique coproduct which turns it into a Hopf algebra such that its tautological action on $\mathscr{A}(n)$ is a Hopf action.*

Remark 9 The Hopf algebra $\mathscr{H}(n)$ acts canonically on the crossed product algebra $\mathscr{A}_{F(M)} = C_c^\infty(F(M) \rtimes \Gamma)$, for any flat affine manifold M with a pseudogroup Γ of orientation preserving diffeomorphisms. Using Morita equivalence, one can always reduce the case of a general manifold M to the flat case. The obstruction one encounters in trying to transfer the action of $\mathscr{H}(n)$, via the Morita equivalence data, from the flattened version to a non-flat M is exactly the curvature of the manifold M. *The analysis of this obstruction in the general context of actions of Hopf algebras on algebras should provide the correct generalization of the notion of Riemannian curvature in the framework of noncommutative geometry.*

There is a more revealing definition ([14]) of the Hopf algebra $\mathcal{H}(n)$, in terms of a bicrossproduct construction (cf. e.g. [21]) whose origin, in the case of finite groups, can be traced to the work of G. I. Kac. In our case, it leads to the interpretation of $\mathcal{H}(n)$ as a bicrossproduct of two Hopf algebras, $\mathcal{U}_a(n)$ and $\mathcal{S}_u(n)$, canonically associated to the decomposition of the diffeomorphism group as a set-theoretic product

$$\mathrm{Diff}(\mathbb{R}^n) = G_a(n)\cdot G_u(n),$$

where $G_u(n)$ is the group of diffeomorphisms of the form $\psi(x) = x + o(x)$. $\mathcal{U}_a(n)$ is just the universal enveloping $\mathcal{U}(\mathfrak{g}_a(n))$ of the group $G_a(n)$ of affine motions of $\mathbb{R}^n$, with its natural Hopf structure. $\mathcal{S}_u(n)$ is the Hopf algebra of polynomial functions on the pro-nilpotent group of formal diffeomorphisms associated to $G_u(n)$.

The modular character of the affine group $\delta = \mathrm{Trace} : \mathfrak{g}_a(n) \to \mathbb{R}$ extends to a character $\delta \in \mathcal{H}(n)^*$. It can be readily verified that the corresponding twisted antipode satisfies the involution condition $\widetilde{S}^2 = I$. It follows that the pair $(\delta, 1)$ fulfills (2.12) and hence forms a modular pair in involution.

The preceding bicrossproduct interpretation allows to relate the cyclic cohomology of $\mathcal{H}(n)$ with respect to $(\delta, 1)$ to the Gelfand-Fuchs cohomology [18] of the infinite-dimensional Lie algebra $\mathfrak{a}_n$ of formal vector fields on $\mathbb{R}^n$.

Theorem 10. ([14]) *There is a canonical Van Est-type map of complexes which induces an isomorphism*

$$\sum_{i\equiv *(2)} H^i(\mathfrak{a}_n) \simeq HC^*_{\mathrm{per}(\delta,1)}(\mathcal{H}(n)). \tag{3.23}$$

We now return to the spectral triple $(\mathcal{A}_F, \mathcal{H}_F, D)$ associated to (M, Γ) with M flat. Then we have the canonical Hopf action $\mathcal{H}(n)\otimes\mathcal{A}_F \to \mathcal{A}_F$. In addition, the crossed product $\mathcal{A}_F$ inherits a canonical trace $\tau_F : \mathcal{A}_F \to \mathbb{C}$, dual to the volume form vol_F,

$$\tau_F(f U^*_\psi) = 0 \quad \text{if} \quad \psi \neq 1 \quad \text{and} \quad \tau_F(f) = \int_F f\, vol_F\,. \tag{3.24}$$

Using the Γ-invariance of vol_F, it is easy to check that the trace τ_F is δ-invariant under the action of $\mathcal{H}(n)$, i.e., the property (2.29) holds. We therefore obtain a characteristic map

$$\gamma^*_F : HC^*_{\mathrm{per}(\delta,1)}(\mathcal{H}(n)) \to HC^*_{\mathrm{per}}(\mathcal{A}_F),$$

which together with (3.23) gives rise to a new characteristic homomorphism:

$$\chi_F : H^*(\mathfrak{a}_n) \to HC^*_{\mathrm{per}}(\mathcal{A}_F). \tag{3.25}$$

Passing to $SO(n)$-invariants, one obtains an induced characteristic map from the relative Lie algebra cohomology:

$$\chi_P : \sum_{i+p\equiv *(2)} H^i(\mathfrak{a}_n, SO(n)) \to HC^*_{\mathrm{per}}(\mathcal{A}_P), \tag{3.26}$$

which instead of τ_F involves the analogous trace τ_P of $\mathscr{A}_P$ and where $p = \dim P$.

Let us assume that the action of Γ on M has no degenerate fixed point. Recall the local formula (3.11) for $ch_*(R)$. Using the built-in affine invariance of a hypoelliptic affine operator R, one can show that any cochain on $\mathscr{A}_P$ of the form,

$$\varphi(a^0,\ldots,a^n) = \int\!\!\!\!\!- \, a^0[R,a^1]^{(k_1)}\cdots[R,a^n]^{(k_n)}\,|R|^{-(n+|2k|)}, \qquad \forall\, a^j \in \mathscr{A}_P,$$

can be written as a finite linear combination

$$\varphi(a^0,\ldots,a^n) = \sum_\alpha \tau_P(a^0\, h_1^\alpha(a^1)\cdots h_n^\alpha(a^n)), \quad \text{with} \quad h_i^\alpha \in \mathscr{H}(n),$$

and therefore belongs to the range of the characteristic map χ_P. The structure of the cohomology ring $H^*(\mathfrak{a}_n, SO(n))$ is well-known ([19]). It is computed by the cohomology of the finite-dimensional complex

$$\{E(h_1,h_3,\ldots,h_m)\otimes P(c_1,\ldots,c_n)\ ,\ d\},$$

where $E(h_1,h_3,\ldots,h_m)$ is the exterior algebra in the generators h_i of dimension $2i-1$, with m the largest odd integer less than n and i odd, while $P(c_1,\ldots,c_n)$ is the polynomial algebra in the generators c_i of degree $2i$ truncated by the ideal of elements of weight $> 2n$. The coboundary d is defined by

$$dh_i = c_i,\ \ i \text{ odd}, \quad dc_i = 0 \text{ for all } i.$$

In particular, the Pontryagin classes $p_i = c_{2i}$ are non-trivial for all $2i \le n$.

The final outcome of the preceding discussion is the following index theorem for transversely hypoelliptic operators on foliations:

Theorem 11. **([14])** *For any hypoelliptic affine operator R on P, there exists a characteristic class $\mathscr{L}(R) \in \sum_{i+n\equiv *(2)} H^i(\mathfrak{a}_n, SO(n))$ such that*

$$ch_*(R) = \chi_P(\mathscr{L}(R)) \,\in\, HC^*_{\rm per}(\mathscr{A}_P).$$

Remark 12 We conclude with the remark that similar considerations, leading to analogous results, can be implemented for more specialized cases of transverse geometries, such as complex analytic and symplectic (or Hamiltonian). For example, in the *symplectic case*, which is the less obvious of the two, the transverse data consists of a symplectic manifold (M^{2n},ω) together with a pseudogroup Γ_{sp} of local symplectomorphisms. The corresponding frame bundle is the principal $Sp(n,\mathbb{R})$-bundle F_{sp} of symplectic frames. Its quotient mod K, where $K = Sp(n,\mathbb{R})\cap O_{2n} \simeq U(n)$, is the bundle P_{sp} whose fiber at $x \in M$ consists of the almost complex structures on T_xM compatible with ω_x. It carries an intrinsic *para-Kählerian structure*, obtained as follows. The typical fiber of P_{sp} can be identified with the noncompact Hermitian symmetric space $Sp(n,\mathbb{R})/U(n)$ and, as such, it inherits a canonical Kähler structure. This gives rise to a natural Kähler structure on the vertical subbundle $\mathscr{V}$ of the tangent bundle TP_{sp}. On the other hand, the normal bundle $\mathscr{N} = TP_{sp}/\mathscr{V} \simeq TM$ possesses a tautological Kähler structure. Indeed, a point in P_{sp} is by definition an almost complex structure and thus, together with ω, determines a "moving" Kähler structure. The entire construction is functorial with respect to local symplectomorphisms. One can therefore define the symplectic analogue Q_{sp} of the hypoelliptic signature operator as a graded direct sum

$$Q_{sp} = (\bar{\partial}_V^* \bar{\partial}_V - \bar{\partial}_V \bar{\partial}_V^*) \oplus (\bar{\partial}_H + \bar{\partial}_H^*)\,,$$

where the horizontal $\bar{\partial}$-operator $\bar{\partial}_H$ is associated to a symplectic connection. *The corresponding index theorem asserts that, with (M^{2n}, ω) flat, acted upon by an arbitrary pseudogroup of local symplectomorphisms Γ_{sp}, and with*

$$\chi_{sp} : H^*(\mathfrak{a}_n^{sp}, U(n)) \to HC^*_{\mathrm{per}}(\mathscr{A}_{P_{sp}}), \tag{3.27}$$

denoting the characteristic map corresponding to the Lie algebra $\mathfrak{a}_n^{sp}$ of formal Hamiltonian vector fields on $\mathbb{R}^{2n}$, there exists a characteristic class $\mathscr{L}_{sp} \in \sum_{i \equiv *(2)} H^i(\mathfrak{a}_n^{sp}, U(n))$ such that

$$ch_*(Q_{sp}) = \chi_{sp}(\mathscr{L}_{sp}) \in HC^*_{\mathrm{per}}(\mathscr{A}_{P_{sp}}).$$

As in the para-Riemannian case, the proof relies on the local Chern character formula (3.11) and on the cyclic cohomology of the Hopf algebra $\mathscr{H}_{sp}(n)$, associated to the group of symplectomorphism of $\mathbb{R}^{2n}$ in the same manner as $\mathscr{H}(n)$ was constructed from $\mathrm{Diff}(\mathbb{R}^n)$.

4. Quantum groups and the modular square

The definition of cyclic co/homology of Hopf algebras hinges on the existence of modular pairs in involution. The necessity of this condition may appear as artificial. In fact, quite the opposite is true and the examples given below serve to illustrate that most Hopf algebras arising in "nature", including quantum groups and their duals, do come equipped with intrinsic modular pairs.

1. We begin with the class of *quasitriangular Hopf algebras*, introduced by Drinfeld [17], in connection with the quantum inverse scattering method for constructing quantum integrable systems. Such a Hopf algebra comes endowed with an universal *$\mathscr{R}$-matrix*, inducing solutions of the Yang-Baxter equation on each of their modules. (For a lucid introduction into the subject, see [21]).

A *quasitriangular Hopf algebra* is a Hopf algebra $\mathscr{H}$ which admits an invertible element $R = \sum_i s_i \otimes t_i \in \mathscr{H} \otimes \mathscr{H}$, such that

$$\begin{aligned} \Delta^{\mathrm{op}}(x) &= R\Delta(x)R^{-1}, \quad \forall x \in \mathscr{H}\,, \\ (\Delta \otimes I)(R) &= R_{13}R_{23}\,, \\ (I \otimes \Delta)(R) &= R_{13}R_{12}, \end{aligned} \tag{4.1}$$

where we have used the customary "leg numbering" notation, e.g.,

$$R_{23} = \sum_i 1 \otimes s_i \otimes t_i .$$

The square of the antipode is then an inner automorphism $S^2(x) = uxu^{-1}$, with

$$u = \sum S(t_i) s_i, \qquad u^{-1} = \sum S^{-1}(t_i) S(s_i).$$

Furthermore, $uS(u) = S(u)u$ is central in $\mathscr{H}$ and one has

$$\varepsilon(u) = 1, \quad \Delta u = (R_{21}R)^{-1}(u \otimes u) = (u \otimes u)(R_{21}R)^{-1} .$$

A quasitriangular Hopf algebra $\mathscr{H}$ is called a *ribbon algebra* [24], if there exists a *central element* $\theta \in \mathscr{H}$ such that

$$\Delta(\theta) = (R_{21} R)^{-1} (\theta \otimes \theta) , \quad \varepsilon(\theta) = 1 , \quad S(\theta) = \theta . \tag{4.2}$$

Any quasitriangular Hopf $\mathscr{H}$ algebra has a "double cover" cover $\widetilde{\mathscr{H}}$ satisfying the ribbon condition (4.2). More precisely (cf.[24]):

$$\widetilde{\mathscr{H}} = \mathscr{H}[\theta]/(\theta^2 - uS(u))$$

has a unique Hopf algebra structure such that, under the natural inclusion, $\mathscr{H}$ is a Hopf subalgebra.

If $\mathscr{H}$ is a ribbon algebra, by setting $\sigma = \theta^{-1} u$, one gets a group-like element

$$\Delta\sigma = \sigma \otimes \sigma, \quad \varepsilon(\sigma) = 1, \quad S(\sigma) = \sigma^{-1},$$

such that, for any $x \in \widetilde{\mathscr{H}}$,

$$\begin{aligned}(\sigma^{-1} \circ S)^2(x) &= \sigma^{-1} S(\sigma^{-1} S(x)) = \sigma^{-1} S^2(x) \sigma \\ &= \sigma^{-1} u x u^{-1} \sigma = \theta x \theta^{-1} = x.\end{aligned}$$

Thus, (ε, σ) is a modular pair in involution for $\mathscr{H}$.

By dualizing the above definitions one obtains the notion of *coquasitriangular*, resp. *coribbon algebra*. Among the most prominent examples of coribbon algebras are the function algebras of the classical quantum groups ($GL_q(N)$, $SL_q(N)$, $SO_q(N)$, $O_q(N)$, and $Sp_q(N)$). The analogue of the above *ribbon group-like element* σ for a coribbon algebra $\mathscr{H}$, is the *ribbon character* $\delta \in \mathscr{H}^*$. The corresponding twisted antipode satisfies the condition $\widetilde{S}^2 = 1$, which renders $(\delta, 1)$ as a canonical modular pair in involution for $\mathscr{H}$.

We thus have:

Proposition 13. **([15])** *Coribbon algebras and compact quantum groups are each intrinsically endowed with a modular pair in involution* $(\delta, 1)$. *Dually, ribbon algebras and duals of compact quantum groups are each intrinsically endowed with a modular pair in involution* $(1, \sigma)$.

For a *compact quantum group* in the sense of Woronowicz, the stated property follows from Theorem 5.6 of [28], describing the modular properties of the analogue of Haar measure.

2. Evidently, one can produce modular pairs in involution with both δ and σ nontrivial by forming tensor products of dual classes of Hopf algebras as in the preceding statement. The fully non-unimodular situation arises naturally however, in the case of *locally compact quantum groups*, because of the existence, by fiat or otherwise, of left and right Haar weights. We refer to [22] for the most recent and concise formalization of this notion, which is in remarkable agreement with our framework for cyclic co/homology of Hopf algebras and of Hopf actions. This accord is manifest in the following *construction of a modular square* associated to a Hopf algebra $\mathscr{H}$ modeling a locally compact group. Since the inherent analytic intricacies are beyond the scope of the present exposition, we shall keep the illustration at a formal level (comp. [27] for an algebraic setting).

By analogy with the definition of a C^*-algebraic quantum group in [22], we assume the existence and uniqueness (up to a scalar) of a *left invariant weight* φ, satisfying a *KMS-like condition*. The invariance means that

$$(I\otimes\varphi)((I\otimes x)\Delta(y)) = S((I\otimes\varphi)\Delta(x)(I\otimes y)), \qquad \forall x,y\in\mathscr{H}, \tag{4.3}$$

while the KMS condition stipulates the existence of a *modular group of automorphisms* σ_t of $\mathscr{H}$, such that

$$\varphi\circ\sigma_t = \varphi, \qquad \varphi(xy) = \varphi(\sigma_i(y)x), \quad \forall x,y\in\mathscr{H}. \tag{4.4}$$

Taking $\psi = \varphi\circ S^{-1}$, one obtains a a *right invariant weight*,

$$(\psi\otimes I)(\Delta(x)(y\otimes 1)) = S((\psi\otimes I)(x\otimes 1)\Delta(y)), \tag{4.5}$$

which is also unique up to a scalar and has modular group $\sigma'_t = S\circ\sigma_t\circ S^{-1}$.

It will be convenient to express the above properties in terms of the natural left and right actions of the dual Hopf algebra $\hat{\mathscr{H}} = \mathscr{H}^*$ on $\mathscr{H}$. Given $\omega\in\hat{\mathscr{H}}$ and $x\in\mathscr{H}$, we denote

$$\begin{aligned} \omega\cdot x &= \sum \omega(x_{(1)})\,x_{(2)} = (\omega\otimes I)\Delta(x)\,,\\ x\cdot\omega &= \sum x_{(1)}\,\omega(x_{(2)}) = (I\otimes\omega)\Delta(x)\,. \end{aligned} \tag{4.6}$$

With respect to the natural product of $\hat{\mathscr{H}}$,

$$(\omega_1 * \omega_2)(x) = <\omega_1\otimes\omega_2, \Delta(x)> = \sum \omega_1(x_{(1)})\,\omega_2(x_{(2)})\,, \qquad \forall x\in\mathscr{H},$$

the left action in (4.6) is the transpose of the left regular representation of $\hat{\mathscr{H}}$, hence defines a representation of the opposite algebra $\hat{\mathscr{H}}^{\mathrm{op}}$ on $\mathscr{H}$, while the right action in (4.6), being the transpose of the right regular representation of $\hat{\mathscr{H}}$, gives a representation of the algebra $\hat{\mathscr{H}}$ on $\mathscr{H}$. On the other hand, it is easy to check that both actions satisfy the rule (2.27) and therefore (4.6) *defines a Hopf action of the tensor product Hopf algebra* $\widetilde{\mathscr{H}} := \hat{\mathscr{H}}^{\rm op} \otimes \hat{\mathscr{H}}$ *on* $\mathscr{H}$,

$$\widetilde{\mathscr{H}} \otimes \mathscr{H} \to \mathscr{H}\,, \quad (\omega_1 \otimes \omega_2, x) \to \omega_1 \cdot x \cdot \omega_2\,. \tag{4.7}$$

The invariance conditions (4.3) and (4.5) can now be rewritten as follows:

$$\begin{aligned}\varphi((\omega \cdot x)\, y) &= \varphi(x\,(S^{-1}(\omega) \cdot y)),\\ \psi((x \cdot \omega)\, y) &= \psi(x\,(y \cdot S(\omega))),\end{aligned} \tag{4.8}$$

where S^{-1} occurs in the first identity as the antipode of $\hat{\mathscr{H}}^{\rm op}$.

The left invariance property of ψ gives the analogue of the modular function, namely a *group-like element* $\delta \in \mathscr{H}$ such that

$$(I \otimes \psi)\Delta(x) = \psi(x)\delta\,, \qquad \forall x \in \mathscr{H}\,,$$

or equivalently

$$\psi(\omega \cdot x) = \omega(\delta)\,\psi(x)\,, \qquad \forall x \in \mathscr{H}\,. \tag{4.9}$$

The *modular element* δ also relates the left and right Haar weights:

$$\varphi(x) = \psi(\delta^{\frac{1}{2}}\, x\, \delta^{\frac{1}{2}}), \qquad \forall x \in \mathscr{H}\,. \tag{4.10}$$

In particular, the full invariance property of φ under the action of $\widetilde{\mathscr{H}}$ is given by

$$\varphi((\omega_1 \cdot x \cdot \omega_2)\, y) = \varphi(x\,(S^{-1}(\omega_1) \cdot y \cdot S_{\delta^{-1}}(\omega_2))), \quad \forall \omega_1, \omega_2 \in \widetilde{\mathscr{H}},\ x, y \in \mathscr{H}\,, \tag{4.11}$$

where $S_{\delta^{-1}}$ denotes the twisted antipode (2.5) corresponding to δ^{-1}.

Let us now form the *midweight* τ,

$$\tau(x) = \varphi(\delta^{-\frac{1}{4}}\, x\, \delta^{-\frac{1}{4}}), \qquad \forall x \in \mathscr{H}\,. \tag{4.12}$$

One checks that its behavior under the action of $\widetilde{\mathscr{H}}$ is as follows:

$$\tau((\omega_1 \cdot x \cdot \omega_2)\, y) = \tau(x\,(S^{-1}_{\delta^{1/2}}(\omega_1) \cdot y \cdot S_{\delta^{-1/2}}(\omega_2)))\,, \tag{4.13}$$

for any $\omega_1, \omega_2 \in \widetilde{\mathscr{H}}$ and $x, y \in \mathscr{H}$. In other words, τ is $\widetilde{\delta}$*-invariant under the action (4.7), with respect to the character*

$$\widetilde{\delta} = \delta^{\frac{1}{2}} \otimes \delta^{-\frac{1}{2}} \in \widetilde{\mathscr{H}}^* = \mathscr{H}^{\rm op} \otimes \mathscr{H}\,. \tag{4.14}$$

On the other hand, it follows as in [22], but with a slightly different notation, that there exists a group-like element $\sigma \in \hat{\mathscr{H}}$, such that the modular groups of φ, ψ can be expressed as follows:

$$\begin{aligned}\sigma_t(x) &= \delta^{it/2}(\sigma^{it/2} \cdot x \cdot \sigma^{it/2})\,\delta^{-it/2}\,,\\ \sigma'_t(x) &= \delta^{-it/2}(\sigma^{it/2} \cdot x \cdot \sigma^{it/2})\,\delta^{it/2}\,, \quad \forall x \in \mathscr{H}\,.\end{aligned} \tag{4.15}$$

In terms of the modular group of τ, to be denoted by σ_t^τ , (4.15) is equivalent to

$$\sigma_t^\tau(x) = \sigma^{it/2} \cdot x \cdot \sigma^{it/2}, \quad \forall x \in \mathscr{H}. \tag{4.16}$$

This shows that τ is a $\widetilde{\sigma}$*-trace for the action (4.7) , with group-like element*

$$\widetilde{\sigma} = \sigma^{\frac{1}{2}} \otimes \sigma^{\frac{1}{2}} \in \widetilde{\mathscr{H}}^* = \mathscr{H}^{\mathrm{op}} \otimes \mathscr{H}. \tag{4.17}$$

It remains to compute the square of the corresponding doubly twisted antipode of $\widetilde{\mathscr{H}}$,

$$\widetilde{\sigma}^{-1} \circ S_{\widetilde{d}} = \sigma^{-1/2} \circ S^{-1}_{\delta^{1/2}} \otimes \sigma^{-1/2} \circ S_{\delta^{-1/2}} : \hat{\mathscr{H}}^{\mathrm{op}} \otimes \hat{\mathscr{H}} \to \hat{\mathscr{H}}^{\mathrm{op}} \otimes \hat{\mathscr{H}}. \tag{4.18}$$

It suffices to compute the square of $\sigma^{-1/2} \circ S_{\delta^{-1/2}} : \hat{\mathscr{H}} \to \hat{\mathscr{H}}$, or equivalently, the square of its transpose, for which a straightforward calculation gives:

$$(\sigma^{-1/2} \circ S_{\delta^{-1/2}})^2 = < \sigma^{-1/2}, \delta^{-1/2} > I_{\hat{\mathscr{H}}}.$$

Since the passage to the opposite algebra gives the reciprocal scalar, it follows that

$$(\widetilde{\sigma}^{-1} \circ S_{\widetilde{\delta}})^2 = I_{\widetilde{\mathscr{H}}}. \tag{4.19}$$

We summarize the conclusions of the preceding discussion in the following result:

Theorem 14 (i) *The Hopf algebra* $\widetilde{\mathscr{H}} = \hat{\mathscr{H}}^{\mathrm{op}} \otimes \hat{\mathscr{H}}$ *possesses a canonical modular pair in involution* $(\widetilde{\delta} = \delta^{\frac{1}{2}} \otimes \delta^{-\frac{1}{2}}, \widetilde{\sigma} = \sigma^{\frac{1}{2}} \otimes \sigma^{\frac{1}{2}})$.

(ii) *The Haar midweight* τ*, given by (4.12), is a* $\widetilde{\delta}$*-invariant* $\widetilde{\sigma}$*-trace for the canonical action of* $\widetilde{\mathscr{H}}$ *on* $\mathscr{H}$.

The first statement characterizes the construction we referred to as *the modular square* associated to a Hopf algebra $\mathscr{H}$ that models a locally compact quantum group. Together with the Haar midweight τ of the second statement, it determines in cyclic cohomology a *modular characteristic homomorphism*:

$$\gamma_\tau^* : HC^*_{(\widetilde{\delta},\widetilde{\sigma})}(\widetilde{\mathscr{H}}) \to HC^*(\mathscr{H}). \tag{4.20}$$

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