From Bachelier and Lundberg to insurance and weather derivatives

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I. Introduction

Among other advances, the twentieth century has witnessed a number of spectacular achievements in three fields which are today closely inter-related, namely probability theory, finance and insurance. Building on the successive visionary intuitions of the botanist Brown observing the moves of pollen particles at the surface of a liquid, of the mathematician Bachelier modeling in 1900 the stock price dynamics on the Paris Bourse and of the physicist Einstein studying in 1905 the trajectories of molecules in a gas, Wiener provided in 1915 the precise definition of a remarkable stochastic process called Brownian motion. This process, whose mathematical properties are beautiful, has been playing for more than a century a key role in the representation of random moves when the assumption of *continuity* of the trajectories appears appropriate.

In a noteworthy coincidence of dates, the actuary-mathematician Lundberg introduced in 1905 the Poisson process to model the random arrival of claims in the portfolio of contracts of an insurance company. Today, the Poisson process remains the central instrument to represent the occurrence of natural catastrophic events as well as financial market crashes or firm bankruptcies (see [4]). The "simple" Poisson process has become compound and its intensity stochastic but in all cases, the key departing point in the modeling of discontinuous trajectories belongs to Lundberg.

More than twenty five years ago, the celebrated work by Black and Scholes [1], and Merton [11] on the valuation of equity options paved the road for the explosion of derivatives markets around the world. In the following decade, these results were extended to options on futures contracts, options on currencies and so forth. All these new formulas, quite useful up to the present time, have the property of being of the same vein as the Black-Scholes-Merton formula. This paper will try to explain why options on insurance, credit or weather present in several manners new challenges.

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To start with a straightforward but crucial observation, we first need to note that in the so-called first generation of options, the quantity which defines the option payout at maturity T is the price of a traded stock or asset S. This has a number of key consequences. Firstly, the number $S(T)$, hence the terminal value $\max(0, S(T) - k)$ of the call option is fully observable and so is the stock price $S(t)$ at any date t of the lifetime of the option. This property obviously does not hold in the case of options on insurance, credit, weather for which a clear *definition* of the underlying instrument or an *index* is necessary in order to avoid litigation on a contract which was improperly underwritten. Secondly, the seller of an equity or currency option is able to continuously hedge his/her position by holding the underlying instrument in a quantity equal to the delta of the option. Hence, except for rare occurrences of "liquidity holes", hedging plain-vanilla options on stocks does not pose any real difficulties. Delta-hedging a weather derivative through a long (or short) position in a temperature index is obviously not feasible. Other answers need to be found; they will definitely not have the accuracy nor the elegant simplicity of the Black-Scholes hedging portfolio. Thirdly, the mathematical modeling of the underlying state variable dynamics, namely the claim index for insurance derivatives or the firm rating for credit options, involve stochastic processes whose proper identification and calibration require an important database; obviously, data are readily available and numerous in the case of financial options. This is not necessarily the case for other types of underlyings.

In order to clarify these different points, in particular the crucial discussion of derivatives hedging, we are going to review the Black-Scholes-Merton formula and identify the steps in the proof which may not easily extend to insurance and weather derivatives.

II. The Black-Scholes-Merton formula (1973)

Let us first review the classical graphs corresponding to the payouts and gains at maturity T of simple elementary or combinations of put and call options written on a stock S. Keeping in mind that the respective payouts of call and put options

$$C(T) = \max(0, S(T) - k) \quad \text{and} \quad P(T) = \max(0, k - S(T)),$$

where k is the strike price of the option, we obtain the familiar graphs given in Figure 1.

If an investor has paid $C(t)$ (resp. $P(t)$) to buy the call (resp. put) at date t, his algebraic gain profile at maturity, expressed in dollars of date T, is shown in Figure 2.

Among the simple combinations of puts and calls that one can easily construct, let us mention one which will be particularly relevant for insurance derivatives, namely the so-called call spread. A call spread is the combination of a long position in a call with strike price k_1 and a short position in a call with strike price $k_2 > k_1$, both being written on the same underlying instrument S and having the

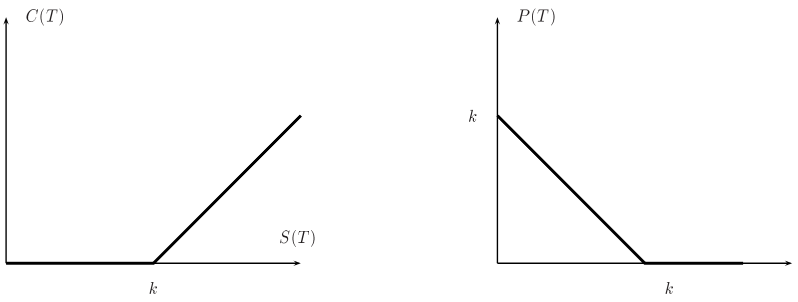

Figure 1.

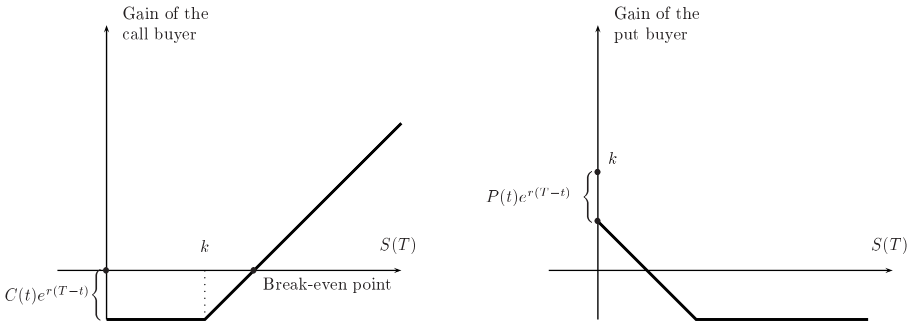

Figure 2.

same maturity T. As will be made precise below but is straight forward, the call price is higher for a lower strike price; hence the value $CS(t)$ at time t of a call spread is positive and the gain profile at maturity is represented in Figure 3.

Put-call parity

Before turning to the problem of valuing a call option, we are going to establish the relation prevailing at any time t between a call and a put written on the same underlying stock S, with the same strike price k and maturity T. It is worth noting that the result holds under fairly general assumptions, namely

A1 The market is supposed to be frictionless (no taxes, no transaction costs).

A2 The stock does not pay any dividend over the lifetime $[0,T]$ of the option (this assumption can easily be relaxed).

A3 The risk-free rate of borrowing or lending r is supposed to be constant over the period $[0,T]$.

A4 There are no arbitrage opportunities in the market, i.e., any riskless portfolio must have a return equal to the risk-free rate r.

Then, the following relationship holds at any date t:

$$S(t)+P(t)=C(t)+ke^{-r(T-t)},$$

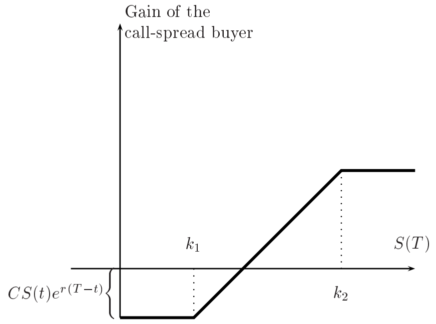

Figure 3.

where $C(t)$ is the call price, $P(t)$ the put price, $S(t)$ the stock price, and k the strike price.

The proof is straightforward and consists, as it is usual in finance, in considering an appropriate portfolio. Here, we build a position at date t which comprises a stock, a put and a short position in the call. The position is held and liquidated at date T.

	t	T	
		$S(T) < k$	$S(T) \geq k$
Buy a stock	$-S(t)$	$S(T)$	$S(T)$
Buy a put	$-P(t)$	$k - S(T)$	0
Sell a call	$C(t)$	0	$k - S(T)$
		k	k

We see that in all *states of the world*, the terminal value of the portfolio is equal to k, a fixed quantity *known* at date t. Hence, the portfolio is *riskless* since risk in finance (and insurance) is equivalent to randomness. The no arbitrage assumption then implies that the initial dollar value at date t of this portfolio is $+ke^{-r(T-t)}$, which needs to be invested in order to build the position. This proves the put-call parity relation. It is extremely important to observe that:

- This result does not depend on a particular choice for the spot price evolution between t and T. In particular the stock price process $S(t)$ may have continuous or discontinuous trajectories.

- A key point, however, is the possibility of buying the stock at date t and holding it up to time T.

When we turn to credit, insurance or weather derivatives, it will become difficult to buy a temperature or insurance index or a credit rating and keep it in a portfolio or in a position up to maturity. One has to keep the proofs of the Black-Scholes-Merton results in mind in order to identify the situations where their extension becomes hazardous.

To conclude on this put-call parity, let us make a final observation on a simple but fundamental use of the put option. Suppose that one buys at date t a stock $S(t)$ and a put written on the stock, with maturity T and strike price k. At date T, the value of this portfolio P is

$$\begin{aligned} V(T) &= S(T) + \max(0, k - S(T)) \\ &= \max(k, S(T)). \end{aligned}$$

This expression shows that the investor will benefit from a rise of the stock market since the portfolio value $V(T)$ at date T will be equal to $S(T)$; however, in the case of a collapse of the stock market, $V(T)$ will not vanish to zero but will be equal to k. The portfolio P is called an *insured portfolio* and k is called the floor of the portfolio insurance strategy. This use of put options plays a central role in the management of portfolios, both for small and institutional investors. It also illuminates the function of *insurance* structurally built in all financial options.

The Black-Scholes-Merton formula (1973)
We can observe first that if we only kept assumptions used to establish the put-call parity, we would not be able to find the price at date t (equally fair to the buyer and the seller) of the call option. One key assumption is missing, namely the one which describes the evolution of the stock price between t and T.

The Black-Scholes-Merton model
We make again the assumptions **A1** to **A4** and add the assumption

A5 $$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \qquad (1)$$

where $(W_t)_{t \geq 0}$ denotes the Brownian motion and the positive constant σ denotes the so-called volatility of the stock price.

We can observe that, since the stock does not pay any dividend, the return over the interval $(t, t + dt)$ comes from the algebraic gain in capital represented by $S(t + dt) - S(t)$, hence is the left-hand side of Equation (1). Since, the right-hand side (an affine function of dWt) follows a normal distribution, it appears that the normality of returns is a key condition of applicability of the Black and Scholes formula and of its possible extension to other underlying instruments.

Moreover, if S_t is replaced by an index I_t of aggregate catastrophe claims when pricing an insurance derivative, the continuity of the trajectories implied by Equation (1) may not be desirable since it will poorly reflect the jumps in the amount insured losses following the occurrence of an earthquake or a hurricane. In the case of weather derivatives, S_t is replaced by a temperature index (or an arithmetic average of temperatures) and it is not clear that the relative change $\frac{dT_t}{I_t}$ over the time interval $(t, t + dt)$ is best represented by a normal distribution. Hence, in both cases, the mathematical modeling of the underlying source of uncertainly needs to rethought and an adequate option pricing formula established.

Besides the nature of the stochastic process itself, another key ingredient in the proof of the Black-Scholes-Merton is the *risk-neutrality* argument applied to a portfolio consisting of financial instruments, the underlying stock and the option. When the option is written on insurance, credit or weather, the portfolio to which one may apply this argument is more difficult to identify, if it exists at all.

Coming to the mathematics of the Black-Scholes-Merton formula, the well-know proof given by the authors is the following: The call price $C(t)$ at date t may be viewed as a function of the two state-variables t and S_t, namely $C(t) = f(t, S_t)$.

By application of Itô's lemma

$$dC_t = \frac{\partial C_t}{\partial t} + \frac{\partial C_t}{\partial S_t}\partial S_t + \frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}(\partial S_t)^2,$$

where, from Equation (1),

$$dS_t = \mu S_t dt + \sigma S_t dW_t. \tag{1'}$$

Hence

$$dC_t = \Big[\frac{\partial C_t}{\partial t} + \mu S_t \frac{\partial C_t}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C_t}{\partial S_t^2}\Big] dt + \sigma S_t \frac{\partial C_t}{\partial S_t} dW_t. \tag{2}$$

In order to eliminate the risk expressed only by the presence of the Brownian motion (W_t), it suffices to consider the portfolio P built at time t with one call option and n stocks, where n is a real number held constant over the time interval $(t, t+dt)$.

The portfolio value at date t is

$$V_P(t) = C_t + nS_t.$$

Hence

$$\begin{aligned} dV_P(t) &= dC_t + ndS_t \\ &= dt\Big[\frac{\partial C_t}{\partial t} + \mu S_t \frac{\partial C_t}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C_t}{\partial S_t^2} + nS_t\mu\Big] + dW_t\Big[\sigma S_t \frac{\partial C_t}{\partial S_t} + \mu n S_t\Big]. \end{aligned} \tag{3}$$

By choosing $n = -\frac{\partial C_t}{\partial S_t}$, the coefficient of dW_t is nullified and the portfolio P becomes *riskless* over the interval $(t, t+dt)$.

Hence $V_P(t+dt)$, is non random at date t. By the *no-arbitrage* assumption **A4**, the return on this riskless portfolio has to be equal to the risk-free rate r and we obtain a second expression for $dV_P(t)$

$$dV_P(t) = rV_P(t)dt = r\Big[C_t - \frac{\partial C_t}{\partial S_t}S_t\Big]. \tag{4}$$

The comparison of Equations (3) and (4) leads to the fundamental *partial differential equation*

$$\frac{\partial C_t}{\partial t} + rS_t \frac{\partial C_t}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C_t}{\partial S_t^2} - rC_t = 0 \tag{5}$$

with the boundary condition $C(T) = \max(0, S(T) - k)$.

Up to a change of variables, Equation (5) may be reduced to the *heat equation* solved by Einstein. This leads to the call price at date t

$$C(t) = S(t)N(d_1) - ke^{r(T-t)}N(d_2), \tag{6}$$

where

- N denotes the cumulative function of a normal variable with mean 0 and variance 1
- $d_1 = \dfrac{\ln\left(\frac{S_t}{ke^{-r(T-t)}}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$
- $d_2 = d_1 - \sigma\sqrt{T-t}$

Before turning to the computation of the call price sensitivities to the different variables (the Greeks), some important observations are in order. Equation (4) establishes that the portfolio consisting of a long call and a number of stocks equal to $-\frac{\partial C_t}{\partial S_t}$ ($\frac{\partial C_t}{\partial S_t}$ will be defined below as the delta of the option) behaves over the interval $(t, t+dt)$ like a money market account; equivalently, the call is replicated on the interval $(t, t+dt)$ by Δ shares and some cash amount (in fact negative), the two quantities being functions of time (for instance, $\frac{\partial C_t}{\partial S_t}$ obviously varies with time), hence needing to be adjusted over time. This is *the dynamic replication* of the call the seller must perform over the interval $[0, T]$ in order to come up with the payout of the option $\max(0, S(T) - k)$ in all states of the world at time T.

This continuous replication of the option with the underlying instrument (and the cash, which is held by any institution) is obviously problematic for electricity derivatives, for insurance derivatives and for weather derivatives. In each case, an appropriate solution in terms of a hedging portfolio — if it exists — needs to be exhibited.

Sensitivities of the call price to the different variables

1. Delta of the call

The first quantity whose impact on the option price needs to be monitored is obviously the underlying source of risk S_t; hence the importance of the so-called delta of the call

$$\Delta = \frac{\partial C_t}{\partial S_t} = N(d_1),$$

as computed from Equation (6).

Hence, this sensitivity is positive, which is not surprising: The higher the stock price, the higher the option price. Moreover, Δ is smaller than 1, which shows that one uses less than one stock at any date t to replicate the call. Lastly, observing that over a time interval δt, one can write

$$\delta C_t \approx \Delta \cdot \delta S_t$$
$$\frac{\delta C_t}{C_t} \approx \frac{\delta S_t}{S_t} \frac{S_t}{C_t} \cdot \Delta.$$

From Equation (6), we have $S_t \cdot \Delta > C_t$, which leads to

$$\left| \frac{\delta C_t}{C_t} \right| > \left| \frac{\delta S_t}{S_t} \right|.$$

The return generated over $(t, t + dt)$ by investing in the call has the same sign as an investment in the stock but is always greater in absolute value, hence the *leverage effect* obtained by investing the same amount of money in the call rather than in the underlying stock when one has a view (upward or downward) on the stock moves. Lastly, we can observe that the delta of the put can be easily derived from the delta of the call which has the same strike price and maturity. Returning to the put-call parity relationship $S(t) + P(t) = C(t) + ke^{-r(T-t)}$, it suffices to take the partial derivative with respect to S_t on both sides to obtain $\Delta_{\text{put}} = \Delta_{\text{call}} - 1$.

It shows that the put is at any date t a decreasing function of the stock price (this is obvious at time T from the expression $\max(0, k - S(T))$ of the put price) and that the dynamic replication of the put will involve a short position in the underlying stock.

2. Gamma of the call
Given the importance of the delta of the call, a quantity of interest is the sensitivity of the delta itself to the stock price, namely

$$\Gamma = \frac{\partial \Delta_t}{\partial S_t} = \frac{\partial^2 C}{\partial S_t^2} = N'(d_1) \frac{\partial d_1}{\partial S_t} = \frac{1}{S_t} \frac{N'(d_1)}{\sigma \sqrt{T-t}}$$

which is obviously positive.

Hence at any date t, the call price is a convex function of the underlying stock price.

3. Sensitivity to time

$$\Theta = \frac{-dS_t}{dt} = -\left[re^{-r(T-t)} kN(d_2) + \frac{\sigma e^{-r(T-t)}}{2\sqrt{T-t}} kN'(d_1) \right] < 0.$$

As time goes by, the period during which the stock price may move to very high values before maturity shrinks (the downward moves below k remaining constantly translated into a zero payout of the call at date T) and the *time value* of the call price diminishes.

4. Sensitivity to the volatility: Vega of the option

$$\frac{\partial C}{\partial \sigma} = S\sqrt{T-t}N'(d_1) > 0.$$

A number of comments are in order at this point, particularly important from a practical standpoint. Firstly, let us mention that even though the volatility parameter σ is supposed to be constant in the Black-Scholes model, practitioners have been computed for more than twenty years the so-called *implied volatility* $\sigma_{\text{imp}}(t)$ embedded in the option market price $C(t)$ at date t (in a unique manner since $\frac{\partial C}{\partial \sigma} > 0$). In contrast with the model, $\sigma_{\text{imp}}(t)$ as reconstructed daily from prices of options written on a given stock, varies with time. Among the number of questions on this issue, the first one — which has been widely documented in the finance literature without any decisive conclusion — is the following: If one writes an option at time t for the maturity T, what is the best quantity to incorporate for σ in the Black-Scholes formula in order to obtain a quasi-perfect replication of the option by delta hedging? The main quantities which have been and are tested are the historical volatility, computed on a time series of past prices of the underlying stock, and the implied volatility derived from current option prices. As far as the latter is concerned, another difficulty arises from the fact that it also only varies with the strike price k of the option, generating the famous volatility smile $\sigma_{\text{impl}}(k)$ prevailing at date t.

III. Extensions of the Black-Scholes-Merton formula

A number of deviations from the original Black-Scholes-Merton model were documented both by academics and practitioners over the last twenty five years. We noted in the previous paragraph that the implied volatility rapidly became a time-varying parameter $\sigma_{\text{imp}}(t)$, depending also not only on the strike price of the option under analysis but also on its maturity T (for instance, options with a short time to maturity have a much higher volatility than the one predicted by the Black-Scholes formula). This dependence in t and T defines the so-called term structure of volatilities at date t, $\sigma_{\text{impl}}(t,T)$, whose dependence in k as well translates into the volatility surface $\sigma_{\text{impl}}(k,T)$ at date t.

The celebrated formula has been extended in several manners which can be roughly classified in three categories:

- Replacement of the classical geometric Brownian motion to represent the underlying price process by a more complicated stochastic process.
- Replacement of the classical call option payoff $\max(0, S(T)-k)$ by an "exotic" payoff involving not only $S(T)$ but the trajectory of the process $S(t)$ over the lifetime $[0,T]$ of the option.
- Replacement of the stock price as the underlying source of risk in the option by interest rates, currencies, and very recently by credit, insurance, weather,

electricity. As this article will try to show all these extensions apply to the case of derivatives on insurance, electricity, and weather.

A) Alternative representations of the underlying source of risk dynamics

We observed in the previous section that Equation (1), representing the return of the stock price over the interval $(t, t+dt)$ as the quantity $\mu dt + \sigma dW_t$, hence a normal distribution, was a key assumption in the Black-Scholes model. On the other hand, a huge literature in finance has been dedicated to the analysis of the normality of asset returns since this property play a central role in the construction of the Markowitz frontier of efficient portfolios and Capital Asset Pricing Model. The property of normality also has an impact on the nature of the *tails* of the distribution, which represent extreme events and are conveyed by the $1-(\alpha)$-quantile ($\alpha = 0.95$ for instance) called Value at Risk at the level of confidence α and playing a major role as a risk indicator. Deviations of returns from normality have been consistently exhibited in the finance literature and this is all the more true as the time period is shorter [5] — like a day or a week, time length which has become for practical purposes the horizon in portfolio management. Besides the fat tails or kurtosis, the distribution of observed returns is more skewed than the normal distribution. One way of addressing the issue of skewness and kurtosis is to introduce jumps in the representation of the stock price dynamics.

1. Jump diffusion processes for the underlying asset price

Back in 1976, Merton had proposed to extend the model proposed in [1] and [11] for the stock price dynamics by adding a jump component, namely

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + U dN_t, \tag{7}$$

where:
- (N_t) is a Poisson process with intensity λ;
- U is a real-valued random variable;
- The other quantities have the same meaning as before.

This assumption has a number of consequences:

a) From a mathematical viewpoint, the solution of (7) is obviously more complicated than the solution of (1) (in particular, it is not explicit any more but involves the so-called Doléans-Dade exponential).

b) From an economic viewpoint, the number of sources of randomness driving the stock price has been drastically increased, not only because of the presence of the Poisson process (N_t) but also because the magnitude of the jumps U is a random quantity. Hence, it is not any more possible to duplicate the call option with the underlying stock and the money-market account and in fact, an *infinite number* of hedging instruments are necessary and do not exist in practice.

In order to eliminate this market incompleteness, Merton had made the simplifying assumption that the jump risk could be diversified away, hence the economic difficulty — the most important one — was eliminated. And some mathematical considerations allowed to exhibit the option price in this new setting. This assumption of a null *market price of jump risk* is clearly unacceptable for *catastrophic insurance risk*. It is not valid either in the case of electricity derivatives where after the spike of June 1998 in the Illinois region — the spot price of electricity went from 25 dollars to 7500 dollars — options with a strike price of 1000 dollars got their market prices multiplied by 100, clearly indicating the existence of risk aversion and risk premium associated with the jump.

2. Time-varying volatility for the underlying asset price
Another way of explaining the volatility smile mentioned in Section II as well as the fact that the implied volatility $\sigma_{\text{impl}}(t)$ derived from a call option written on a given stock varies with time is to explicitly introduce in Equation (1) a volatility which varies with time.

a) In a first approach, $\sigma(t)$ may be expressed as a deterministic function of time

$$\frac{dS_t}{S_t} = \mu dt + \sigma(t) dW_t.$$

It is then easy to show that all Black-Scholes results hold, including the hedging strategy; it suffices to replace everywhere, including in the valuation formula (6), σ with Σ, where Σ is defined by

$$\Sigma^2(T-t) = \int_t^T \sigma^2(u) dw.$$

b) Going one step further, $\sigma(t)$ becomes stochastic but has the property of being a function of $S(t)$. In this case, we still have only one source of randomness, namely the Brownian motion involved in the dynamics of $(S(t))$. Hence, the option price (as well as the hedging portfolio) is unique. It can be obtained as a closed-form expression if $\sigma(S_t)$ is a simple function; otherwise, $C(t)$ may be computed using Monte-Carlo simulations.

c) In the most general *stochastic volatility model*, $\sigma(t)$ is a stochastic process; hence, assuming diffusion processes for the stock and the volatility, the dynamics of $(S(t))$ are defined by the two stochastic differential equations

$$\begin{cases} dS_t = \mu_1(t, S_t)dt + \sigma(t, S_t)dW_t^1 \\ dy(t) = \mu_2(t, y_t)dt + \eta(t, y(t))dW_t^2 \end{cases}$$

where $y(t) = [\sigma(t)]^2$, $dW_t^1 \cdot dW_t^2 = \rho(t)dt$.

The coefficient $\rho(t)$ allows for dependence of the Brownian motions $W_1(t)$ and $W_2(t)$; the functions μ_1 and μ_2 may eventually translate a property of mean reversion for the stock price or volatility. In this case, not only do the closed-form solutions for option prices become problematic but the unicity of the call value and of the hedging portfolio is lost since we still have only two hedging instruments and an extra-source of risk represented by the Brownian motion (W_t^2). A possible answer to this market incompleteness is to consider that the at-the-money call C^m (i.e., with a strike price equal to the value of the stock today) is a *basic security* whose price is agreed upon by all market players.

If then becomes possible to hedge all other calls and puts using the stock, the money-market account and the call C^m. The existence of this very liquid option completing the market is quite legitimate for options on stocks, on stock indexes or on currencies; it is certainly problematic for insurance or weather derivatives, where the generality of a stochastic volatility model is paid for by the loss of a unique price for the options.

B) Exotic pay-offs

As of the beginning of the 80's, so called exotic options were introduced to answer the hedging needs of financial and industrial firms as well as the search for new opportunities on the part of investors. We will describe the most popular exotic options, namely the Asian and the barrier options

a) Asian or average-rate options

Instead of the Black-Scholes pay-off $\max(0, ST - k)$, the buyer of an Asian call option receives at maturity T $\max(0, A_T - k)$, where A_T represents the arithmetic average of the stock price computed over the lifetime $[0, T]$ or over a subinterval of $[0, T]$.

Asian options are very much used for thinly traded assets such as gold since they obviously prevent a possible manipulation of prices on the maturity date of standard options by institutions holding large positions in the underlying asset. They represent a large percentage of the options traded on oil or oil spreads since oil indexes are themselves defined as averages of spot prices over a time period. In the FX markets, they allow corporate treasurers to hedge a series of cash-flows by buying a single average-rate option, as an alternative to a portfolio of standard options associated with each payment date, hence reducing the cost of FX risk management.

Other examples of Asian options will be presented in this article and explain why their valuation is such an important issue. Unfortunately, it is not a straightforward problem, the difficulties stemming from the fact that an average of prices S_t satisfying Equation (1) does not satisfy an equation of the same type or, in other terms, is not a geometric Brownian motion. The main methods used today to value Asian options are the following:

(i) The approximation of the true distribution of the average A_T by the log-normal distribution of S_T, with adjustment of the first two moments.

(ii) The use of Monte-Carlo simulations: Different paths of the stock price are generated, providing for each of them the values $S(t_1), \dots, S(t_n)$ which are averaged to exhibit one outcome O_i of $\max(0, A_T - k)$, where k is the strike price of the Asian call.

From the law of large numbers, we know that the quantity $\frac{1}{N}\sum_{i=1}^{i=N} O_i$ will converge to the expectation of the pay-off which, discounted, gives the price of the Asian call at time zero.

(iii) An exact approach to the Asian option valuation and hedging: Geman and Yor [7] used sophisticated mathematical tools to compute the Laplace transform of the call price with respect to maturity.

Geman and Eydeland [6] on one hand, Fu, Madan, and Wang [3] on the other hand, apply different algorithms to invert the Geman-Yor formula but come up with remarkably close results. The former authors exhibit the superiority of their results over those obtained through Monte Carlo simulations, in particular in terms of the accuracy of delta hedging.

b) Barrier and double — barrier options

They have become increasingly popular in equity markets, and even more so, in FX and commodity markets. A barrier call option provides at maturity the classical Black-Scholes pay-off $\max(0, S_T - k)$ only if (unless) the underlying asset S has touched a prespecified barrier L over the lifetime $[0, T]$ of the option. For instance, if a non-zero pay-off is contingent on the fact that the value L is reached at some date t, the call is said to be "up and in" or "down and in".

It appears immediately that:

- The price of a barrier call at time 0 is smaller than the standard call with the same strike and maturity (since it pays zero more often).
- Hence, a barrier option is an inexpensive hedge against extreme events. An "up and in" option will be activated only if an insurance index, a foreign currency or the spot price of electricity reaches a very high level (defined as the barrier in the option contract).

A vast body of literature has been recently dedicated to barrier options. The pricing of "single barrier" options is not extremely difficult in the classical Black-Scholes-Merton framework and explicit formulas have existed for some time (see [11, 9]). For instance, if the barrier L is less than or equal to the strike price, the value at date 0 of a down-and-in call written on a non-dividend paying stock is

$$C(0) = S(0)\Big(\frac{L}{S(0)}\Big)^{2\alpha} N(d_1) - ke^{-rT}\Big(\frac{L}{S(0)}\Big)^{2\alpha} N(d_2),$$

where $\alpha = \frac{r+\frac{\sigma^2}{2}}{\sigma^2}$, d_1 and d_2 have the same expressions as in Formula (6).

Double-barrier options yield the Black-Scholes payout $\max(0, S_T - k)$ at maturity under the condition that S_t has not touched either of two barriers, denoted

L (lower barrier) and U (upper barrier), during the lifetime of the option $[0,T]$; they are then called corridor options and allow traders or investors *to take a view* on the range of a stock, a commodity or an index price. Another type of double barrier option is activated by the fact that one (at least) of the two barriers has been crossed by the trajectory of S_t between 0 and T. All types of double barrier options are more difficult to price than simple barrier options. In both cases, Monte Carlo methods may be dangerous: The discrete trajectories which are simulated may misrepresent the continuous possibility of (des)activation of the barrier option over the interval $[0,T]$.

Geman and Yor [8] provide the Laplace transforms of the exact prices of double barrier options and of their deltas, gammas, and other sensitivities as well .

c) Options on exotic underlyings

Over the last six years, a variety of non financial assets have become the source of risk on which standard and path-dependent options are written. Insurance derivatives were launched in December 1993 by the Chicago Board of Trade and are traded today both on exchanges and in OTC markets. An index, precluding manipulations and opacity, needed to be constructed for its value at maturity T to determine the option payout; this index is currently the PCS (Property Claim Services) index. Since, it is defined as an aggregate amount of losses, i.e., an average (up to constant), the insurance derivatives give rise to Asian-type problems. At the same time, the nature of the underlying instrument — an index which spikes in the case of an earthquake or a hurricane — requires the presence of a jump process in its modeling.

More recently, with all the credit events which shook the financial markets, from the Asian crisis to the Russian turmoil, credit derivatives developed as hedging instruments not only for defaultable bonds but also swaps and transactions with counterparties bearing default, i.e., all OTC transactions. Again, credit not being a traded instrument, a precise definition of the option contract and of the *insured risk* is necessary to avoid possible litigation later on.

Lastly, with deregulation of electricity on its way worldwide and in the United States in particular, weather derivatives have been introduced to allow utilities to hedge their electricity risk since, in the U.S., spikes in electricity price coincide very much with extreme weather conditions. Very quickly, these weather derivatives have appeared as a great instrument to farmers, construction companies or even clothe manufacturers. Also weather derivatives raise two types of difficulties: The first one is technical, since the pay-off of the option is related to an average of temperature spreads (65° Fahrenheit representing the "ideal" temperature), hence again Asian-type. A much bigger difficulty however is of an economic nature and resides in the fact that the seller of a weather derivative cannot delta-hedge since of his position this procedure would involve *owning* the underlying source of risk, i.e., the temperature index.

Hence, the pricing and hedging of weather derivatives require new answers, much closer to actuarial and insurance techniques than to the *dynamic hedging* of options written on traded assets.

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