A new method for optimal equity allocation and portfolio selection‡

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Abstract. We review, for an audience of mathematically minded listeners, but non specialists on insurance and finance, some recent research results on optimal equity allocation and portfolio selection. These results, first developed in the domain of reinsurance, are also applicable to insurance, banking and more generally to corporate finance. A multi-time stochastic model, is presented within the context of a company with several portfolios (or subsidiaries), representing both liabilities and assets. The model has solutions respecting constraints on ROE's, ruin probabilities, non-solvency probabilities and market shares currently in practical use. The solutions, which give global and optimal risk management strategies of the company, also generalize VaR (Value at Risk) conditions.

Keywords: equity allocation, portfolio selection, risk management, value at risk

Mathematics Subject Classifications (2000): 91Bxx, 49xx, 60Gxx

1. Introduction and mathematical model

To fix the ideas and for simplicity, in this talk, we shall think of the equity of a company, at a given time, as the value of its accumulated wealth at that time. We consider an idealized situation, where the variation of the equity over a given time period is the difference of the result of the company and the dividends, paid to the shareholders, during that period. The result is the difference of all the incomes and expenses (dividends excluded). By basic economic and financial hypotheses, it is expected that the ROE (Return On Equity), i.e., the result per equity, is sufficiently high. For example it should, in some sense, be higher than the return on equally liquid values, with lower risks. The problem we shall consider in this talk is to find a satisfactory global solution on the balance between Low Equity Risk and High ROE. We formulate it as an optimization problem involving essentially three different kind of variables: portfolios, equities and dividends, which are discrete time stochastic processes. Acceptable upper bounds of risks, lower bounds of annual ROE's and positivity of portfolios are introduced as constraints and we optimize the expected final utility, being the wealth produced, until no more financial flows are generated. The solution of this problem consists of those stochastic processes, satisfying the constraints, for which the optimum is reached. A large class of *Equity Allocation and Portfolio Selection Models*, for which solutions exist in a certain sense, was introduced in references [5] and [6], where also a constructive approach to the solutions was given. Portfolios representing both assets and liabilities

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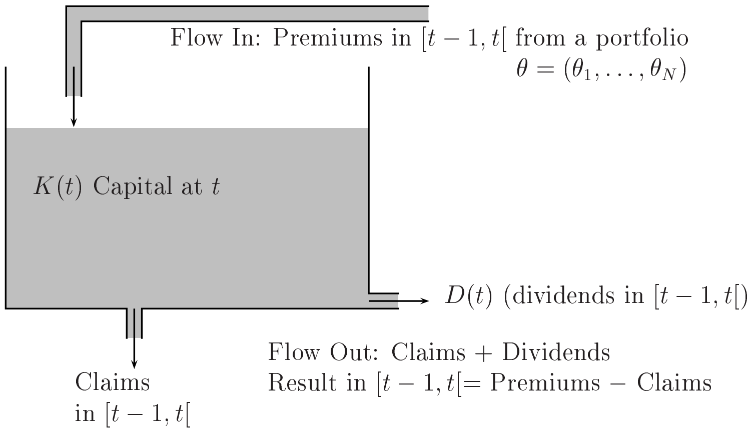

Figure 1. Ordinary reservoir model of an insurance company. All flows are stochastic.

are admitted in these references. The very popular VAR conditions are particular cases of constraints in [6]. These models therefore also give, in the context under consideration, a framework for *Optimal ALM* (Asset and Liability Management) and *Optimal Risk Management* of a company. In this talk, I will mainly review these results. The mathematical model as well as results are outlined in the sequel of this introduction. Somewhat more detailed results, an idea of their proofs and some supplementary mathematical points are given in §2.

We shall consider companies of two different levels of complexity. Let S be a company with only one portfolio and let H be a company having several subsidiaries. Each subsidiary has one portfolio and can, in certain situations, ceases its activity. For S we shall study the problem of a optimal portfolio selection. Its result is a basic building block for the more complex problem of an optimal equity distribution (between different subsidiaries and shareholders) and portfolio selection of H.

Let us first consider the company S, which for the purpose of this introduction, for example can be a reinsurance company. S has a portfolio $\theta=(\theta_1,\ldots,\theta_N)$ of N different kind of contracts and with an initial capital $K(0)>0$, at $t=0$. It can be thought of, which is frequent, as a reservoir (see Figure 1), filled with capital $K(t)$ at time t, with premiums as in flows and claims and dividends $D(t)$ as out flows during the time interval $[t-1,t[$.

The in flow is a (discrete time) stochastic process. In fact in this model, the amount $\theta_i(t)$ of contract i written at time t, is known for past times, $t<0$, is determined from past information at the present, $t=0$, and the amount written at future times $t>0$ will depend on information collected until that instant. Similarly, the dividends are given by a stochastic process. The claims are of course stochastic. It is supposed that the amounts $\theta_i(t)$ vanish after a sufficiently large time $\bar{T}$ (as well as before a sufficiently past time). Similarly, it is also supposed that the flows vanish after a sufficiently large time $\bar{T}+T$. The portfolio can be decomposed into $\theta = \xi + \eta$, where ξ is the run-off, at time $t = 0$, concluded at a finite number of past times $t < 0$ and where η is the (present and future) underwriting portfolio, to be concluded at a given finite number of times $0, \ldots, \bar{T}$, $\bar{T} \geq 1$.

We introduce a Cramér–Lundberg like *utility function* U of the portfolio θ (cf. [2]), whose value at time t is

$$U(t,\theta) = \textit{accumulated premiums in } [0,t[- \textit{accumulated claims in } [0,t[. \quad (1.1)$$

The *final utility* $U(\infty,\theta) = \lim_{t\to\infty} U(t,\theta)$ is simply the wealth produced in $[0,\bar{T}+T[$. The *capital* is then given by

$$K(t) = K(0) + U(t,\theta) - \sum_{1\leq k\leq t} D(k). \quad (1.2)$$

It is constant for $t \geq \bar{T}+T$. A negative value of the capital corresponds to ruin. The ROE at time t is given by $(U(t,\theta) - U(t-1,\theta))/K(t-1)$. According to what has been said above we want, for given initial capital $K(0)$ and run-off ξ, to find a underwriting portfolio η, such that the capital is maintained non-negative with a high probability, such that the annual ROE's are bounded below by given positive constants and such that the final utility $U(\infty,\theta)$ is optimal. Generically such an optimal portfolio does not exist, so we shall give a weaker formulation useful in practical applications and where non-existence is not the rule. Tentatively we shall try, for given $K(0)$ and ξ, to select η as to achieve the following ($E(X)$ and $V(X)$ are respectively the expected value and the variance of the random variable X):

- A) Optimize the final expected utility $E(U(\infty,\theta))$, i.e., the expected wealth produced in $[0,\bar{T}+T[$;

- B) The expected result satisfies $E(U(t,\theta) - U(t-1,\theta)) \geq c(t)E(K(t-1))$, for $0 \leq t \leq \bar{T}+T$, where $c(t)$ are pre-determined positive lower limits of ROE's;

- C) Sufficiently high probability that $K(t) \geq 0$, for $t \geq 0$, so the company will reasonably not make ruin and insurance claims can be paid;

- D) Present and future underwriting levels are positive, $0 \leq \theta_i(t)$, where $1 \leq i \leq N$ and $0 \leq t \leq \bar{T}$.

We have here also introduced a constraint, corresponding to the more difficult case of positive underwriting levels.

In order to have a well-defined problem, we at least need to know the function $\theta \mapsto U(t,\theta)$. To this end, let a *unit contract* be an insurance contract whose total premium is one currency unit. The utility $u_i(t,t')$, at $t' \in \mathbb{N}$ of the unit contract i,

$1 \leq i \leq N$, concluded at $t \in \mathbb{Z}$, is by definition:

$$u_i(t,t') = \begin{cases} \textit{accumulated result in the time interval } [0,t'[, \textit{ if } t < 0, \\ \textit{accumulated result in the time interval } [t,t'[, \textit{ if } 0 \leq t \leq t', \\ 0, \textit{ if } 0 \leq t' \leq t. \end{cases} \tag{1.3}$$

The utility function of the portfolio θ at time t is then, according to (1.1) given by

$$U(t,\theta) = \sum_{k \leq t} \theta(k) \cdot u(k,t), \tag{1.4}$$

where the dot denotes the scalar product in $\mathbb{R}^N$. The final utility of θ is

$$U(\infty,\theta) = \sum_{k \in \mathbb{Z}} \theta(k) \cdot u^\infty(k), \tag{1.5}$$

where only a finite number of terms are non vanishing and $u^\infty(k) = u(k,\infty)$ exists since all flows vanish from time $\bar{T} + T$.

The random variables $u_i(t,t')$, are essential inputs of the model. They determine its probability space $(\Omega, P, \mathscr{F})$ and its information structure by a filtration $\mathscr{A} = \{\mathscr{F}_t\}_{t \in \mathbb{N}}$, of sub σ-algebras of the σ-algebra $\mathscr{F}$, i.e., $\mathscr{F}_0 = \{\Omega, \emptyset\}$ and $\mathscr{F}_s \subset \mathscr{F}_t \subset \mathscr{F}$ for $0 \leq s \leq t$. For given $k \in \mathbb{N}$ and $0 \leq i \leq N$, the utility $u_i(k,t)$ is the difference of accumulated premiums and claims in the time interval $[k,t[$, $k \leq t$. The accumulated premiums are usually a deterministic function of t, equal to 1 for sufficiently large times. The probability space $(\Omega, P, \mathscr{F})$ and the filtration $\mathscr{A}$ are then given by claims processes of unit contracts. We here suppose that $u_i(k,t)$ is $\mathscr{F}_t$-measurable (i.e., its value is known at time t), so the processes $(u(k,t))_{t \geq 0}$ is $\mathscr{A}$-adapted. Loosely speaking, since we have not defined what is meant by a claim, this amounts to neglect IBNR (Incurred But Not Registered claims) effects. The amount $\theta_i(t)$ of contract i written at time t, is known at t, so the process $(\theta(t))_{t \geq 0}$ is $\mathscr{A}$-adapted.

Secondly, let us consider the company H, which as already said is more complex than S. We shall study the capital structure and dividend flows for H, in addition to the portfolio selection. H is organized as a holding with several subsidiaries $S^{(1)}, \ldots, S^{(\aleph)}$, where $\aleph \geq 1$ is an integer (see Figure 2). The companies $S^{(i)}$ here only correspond to a division of the activities of H into parts, whose equity, profitability, portfolio selection and certain other properties need to be considered individually. This allows localization of capital flows and results. In general $S^{(i)}$ does not have to have a legal existence. Each company $S^{(i)}$ is similar to S, with the exception that its activity can cease. We introduce a supplementary value τ_f, which is a final state of the process $(\theta^{(i)}(t))_{t \in \mathbb{Z}}$, reached when the activity of the company $S^{(i)}$ ceases. $\theta^{(i)}(t)$ is then an $\mathbb{R}^{N^{(i)}} \cup \{\tau_f\}$-valued random-variable. We modify the definition of the utility function for $S^{(i)}$, such that the already written contracts, but no new, continue to be honored. The utility of $\theta^{(i)}$ at time t can then

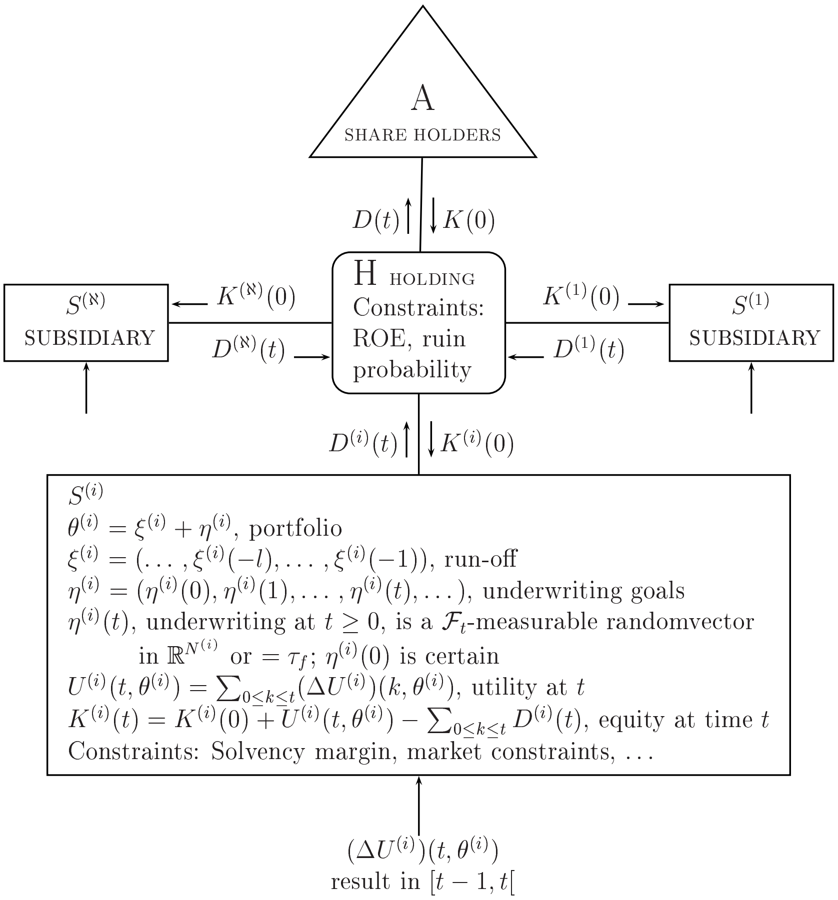

Figure 2. Flows, stocks and constraints.

be written in the following form:

$$U^{(i)}(t, \theta^{(i)}) = U^{(i)}(t, p \circ \theta^{(i)}) = \sum_{k \leq t} (p \circ \theta^{(i)}(k)) \cdot u^{(i)}(k, t), \tag{1.6}$$

where $p : \mathbb{R}^N \cup \{\tau_f\} \to \mathbb{R}^N$ is defined by $p(\tau_f) = 0$ and $p(x) = x$, for $x \in \mathbb{R}^N$, where $U^{(i)}$ in the second member is defined as for S in (1.4) and where the second equality follows from (1.4). The result in the time interval $[t, t+1[$ is

$$(\Delta U^{(i)})(t+1, \theta^{(i)}) = U^{(i)}(t+1, \theta^{(i)}) - U(t, \theta^{(i)}), \tag{1.7}$$

for $t \geq 0$.

The utility of an aggregate portfolio

$$\theta = (\theta^{(1)}, \dots, \theta^{(\aleph)}) \tag{1.8}$$

is defined by

$$U(t,\theta) = \sum_{1\le i\le \aleph} U^{(i)}(t,\theta^{(i)}). \tag{1.9}$$

The dividends D paid to the shareholders by the company H and the equity of H are now given by $D = \sum_{1\le i\le \aleph} D^{(i)}$ and $K = \sum_{1\le i\le \aleph} K^{(i)}$ respectively.

It is supposed that the dividend paid to the shareholders by the company H at time t only depends of the aggregate portfolio θ. As matter of fact, it is usually a function of the aggregate results $(\Delta U)(1,\theta),\dots,(\Delta U)(t,\theta)$ and often we simply have $(D(\theta))(t) = ((\Delta U)(t,\theta)-c)_+$, for some $c \ge 0$, where $(x)_+ = 0$ if $x<0$ and $(x)_+ = x$ if $x \ge 0$.

To formulate the equity allocation and portfolio selection problem in the context of the company H, let us use the decomposition of $\theta = \xi + \eta$, into its run-off ξ (at time $t=0$) and into its underwriting portfolio η and let us introduce the notation $\vec{K}(0) = (K^{(1)}(0),\dots,K^{(\aleph)}(0))$ and $\vec{D} = (D^{(1)},\dots,D^{(\aleph)})$.

We shall try, for given $K(0)$ and ξ, to determine η, $\vec{K}(0)$ and $\vec{D}$ as to achieve the items (A)–(D) (with (D) slightly modified as follows) and certain supplementary constraints:

- D) Present and future underwriting levels are positive, $0 \le \theta^{(i)}_k(t)$, where $1 \le k \le N$ and $0 \le t \le \bar{T}$ and there are upper and lower limits on $\theta^{(i)}_k(t)$;
- E) $K(0) = \sum_{1\le i\le \aleph} K^{(i)}(0)$;
- F) $D = \sum_{1\le i\le \aleph} D^{(i)}$;
- G) Sufficiently high probability that $K^{(i)}(t) \ge m^{(i)}(t)$, for $t \ge 0$, where $m^{(i)}$ is a given solvency margin depending of $\theta^{(i)}$;
- H) The activity of $S^{(i)}$ ceases just after that the solvency margin is not satisfied, i.e., the final state τ_f is reached.

The modification of item (D) corresponds to the introduction of upper and lower market limits. Items (E) and (F) are just budget constraints. In item (G), the solvency margin $m^{(i)}$ is often determined by stronger conditions than the usual ruin condition (i.e., $m^{(i)} = 0$). For example, it can be the case that a company $S^{(i)}$ has to be highly quoted (AAA, ...) on the market. Item (H) is a constraint defining the dynamics of how the final state τ_f of ceased activity of $S^{(i)}$ is reached. We remind that the portfolio $\theta^{(i)}$ is managed as a run-off from the instant when τ_f is reached.

We next sum up the results, which are given in more detail in §2. The optimization problem for the company S (resp. H) given by (A)–(D) (resp. (A)–(H)) gives rise to highly non-linear equations, due to the non-solvability probabilities in the constraints. In order to construct approximate solutions, *but satisfying the original constraints*, the constraints with non-solvability probabilities can be replaced by stronger quadratic variance constraints (cf. Theorem 2.4 of reference

[6]). We note that the portfolio of S is then an extension of Markowitz portfolio [4] to a multiperiod stochastic portfolio, as suggested by [3] (cf. also [1]). In this situation and under certain mild conditions, (H_1), (H_2) and (H_3) of §2, on the result processes of the unit-contracts, there is a unique solution of the problem for S, in the simplest case when the run-off vanishes, $\xi = 0$ (Theorem 2.2). Moreover the solution, is derived from a Lagrangian formalism (2.9), and is given by formula (2.12). The inverse of the integral operator, defined by the quadratic part of the Lagrangian, can be obtained explicitly (Corollary 2.2; for details see Proposition A.3 and Proposition A.4 of [5]). Condition (H_1) says that the final utility (sum of all results) of a unit contract, written at time k, is independent of events occurred before k. In practice, this is generally not true, among other things, because of feed-forward phenomena in the prizing. Condition (H_2) is equivalent to the statement that no non-trivial linear combinations of final utilities, of contracts written at a certain time, is a certain random-variable. This can also be coined, in more financial terms: An underwriting portfolio $\eta(t)$, constituted at time t, cannot be risk-free. Condition (H_3) says that the final utility of unit contracts, written at different times are independent. The conditions (H_1), (H_2) and (H_3), which exclude many interesting situations, like cyclic markets, have here been chosen for mathematical simplicity. They also constitute an approximation of real situations and can largely be weakened, without altering the results of this paper. They reflect to some extent how reinsurance underwriters tempt to build portfolios. In the case of the company H a solution exists (Theorem 2.5), under similar conditions. However it is usually not unique (see Theorem 2.8 of [6]). There is also a Lagrangian formalism, obtained to the price of some complications. An important point, in both cases S and H, is that no particular distributions (statistical laws) are required.

2. Results and mathematical complements

We will here first give results (§2.1) in the simplest case, i.e., the basic model of S with vanishing run-off, $\xi = 0$, and dividends $D = 0$ and then (§2.2) in the general case of H. But first let us introduce some notations.

We define spaces $\mathscr{E}^q(\mathbb{R}^N)$ of discrete and $\mathscr{E}^q(\mathbb{R}^N, \mathscr{A})$ of discrete adapted processes, for $1 \le q \le \infty$. Let $1 \le q < \infty$. Then $(X_i)_{0 \le i} \in \mathscr{E}^q(\mathbb{R}^N)$ if and only if $X_i : \Omega \to \mathbb{R}^N$ is $\mathscr{F}$-measurable and $\|X_i\|_{L^q(\Omega,\mathbb{R}^N)} = (E(|X_i(\omega)|^q_{\mathbb{R}^N}))^{1/q} < \infty$ for $i \ge 0$, where $|\ |_{\mathbb{R}^N}$ is the norm in $\mathbb{R}^N$. Let $\mathscr{E}^q(\mathbb{R}^N, \mathscr{A})$ be the subspace of $\mathscr{A}$-adapted processes in $\mathscr{E}^q(\mathbb{R}^N)$. Similarly $\mathscr{E}^\infty(\mathbb{R}^N)$ and $\mathscr{E}^\infty(\mathbb{R}^N, \mathscr{A})$ are defined by replacing the L^q-norm by the L^∞-norm. We define $\mathscr{E}(\mathbb{R}^N) = \cap_{q \ge 1} \mathscr{E}^q(\mathbb{R}^N)$ and $\mathscr{E}(\mathbb{R}^N, \mathscr{A}) = \cap_{q \ge 1} \mathscr{E}^q(\mathbb{R}^N, \mathscr{A})$ (where of course the intersection is over only finitely many q's). Let $\mathscr{E}^q_{\bar{T}}(\mathbb{R}^N, \mathscr{A})$ and $\mathscr{E}_{\bar{T}}(\mathbb{R}^N, \mathscr{A})$ be the subspace of elements $\eta \in \mathscr{E}^q(\mathbb{R}^N, \mathscr{A})$ and $\mathscr{E}(\mathbb{R}^N, \mathscr{A})$ respectively, with $\eta(t) = 0$ for $t > \bar{T}$.

2.1. Basic model

We shall here sum up certain results obtained in reference [5] in the case of the company S with vanishing run-off and dividends. The portfolio η is an element of the Hilbert space $\mathscr{H} = \mathscr{E}^2_T(\mathbb{R}^N, \mathscr{A})$. We remind that, in this situation, the equity is according to formula (1.2) given by

$$K(t) = K(0) + U(t,\eta), \tag{2.1}$$

where $K(0) \geq 0$ is the initial equity at $t = 0$.

In the sequel of this paragraph, we closely follow reference [5]. The constraints (B), (C) and (D) on the portfolio η take, after the introduction of variance constraints in §1, the following form:

- C_3) $E((\Delta U)(t+1,\eta)) \geq c(t)E(K(t))$, $c(t) \in \mathbb{R}^+$ is given (constraint on profitability);
- C_4) $E((U(\infty,\eta) - E(U(\infty,\eta)))^2) \leq \sigma^2$, where $\sigma^2 > 0$ is given (acceptable level of the variance of the final utility);
- C_6) $0 \leq \eta_i(t)$, where $1 \leq i \leq N$ (only positive subscription levels).

Let $\mathscr{C}_0$ be the set of portfolios $\eta \in \mathscr{H}$ such that constraints (C_3), (C_4), and (C_6) are satisfied. This is well-defined. In fact the quadratic form

$$\eta \mapsto \mathfrak{a}(\eta) = E((U(\infty,\eta))^2), \tag{2.2}$$

in $\mathscr{H}$, has a maximal domain $\mathscr{D}(\mathfrak{a})$, since for each $\eta \in \mathscr{H}$, the stochastic process $(U(t,\theta))_{t\geq 0}$ is an element of the space $\mathscr{E}^p(\mathbb{R},\mathscr{A})$, for $1 \leq p < 2$ (which follows directly from the Schwarz inequality). The optimization problem is now, to find all $\hat{\eta} \in \mathscr{C}_0$, such that

$$E(U(\infty,\hat{\eta})) = \sup_{\eta\in\mathscr{C}_0} E(U(\infty,\eta)). \tag{2.3}$$

The solution of this optimization problem is largely based on the study of the quadratic form

$$\eta \mapsto \mathfrak{b}(\eta) = E((U(\infty,\eta) - E(U(\infty,\eta)))^2), \tag{2.4}$$

in $\mathscr{H}$, with (maximal) domain $\mathscr{D}(\mathfrak{b}) = \mathscr{D}(\mathfrak{a})$.

We make certain (technical) hypotheses on the claim processes:

- H_1) $u^\infty(k)$ is independent of $\mathscr{F}_k$ for $k \in \mathbb{N}$;
- H_2) for $k \in \mathbb{N}$ the $N \times N$ (positive) matrix $c(k)$ with elements
  $c_{ij}(k) = E((u_i^\infty(k) - E(u_i^\infty(k))(u_j^\infty(k) - E(u_j^\infty(k)))$ is strictly positive;
- H_3) $u_i^\infty(k)$ and $u_j^\infty(l)$ are independent for $k \neq l$.

The next crucial result (Lemma 2.2 and Theorem 2.3 of [5]) show that (the square root of) each one of the quadratic forms $\mathfrak{b}$ and $\mathfrak{a}$ is equivalent to the norm in $\mathscr{H}$.

Theorem 2.1 *If the hypotheses (H_1), (H_2), and (H_3) are satisfied, then the quadratic forms* $\mathfrak{b}$ *and* $\mathfrak{a}$ *are bounded from below and from above, by strictly positive numbers c and C respectively, where* $0 < c \leq C$, *i.e.,*

$$c\|\eta\|^2_{\mathscr{H}} \leq \mathfrak{b}(\eta) \leq \mathfrak{a}(\eta) \leq C\|\eta\|^2_{\mathscr{H}}, \tag{2.5}$$

for $\eta \in \mathscr{H}$.

Since there is no risk for confusion, the bilinear form corresponding to the quadratic form $\mathfrak{b}$ (resp. $\mathfrak{a}$) will also be denoted $\mathfrak{b}$ (resp. $\mathfrak{a}$). The operators B (resp. A) in $\mathscr{H}$, associated with $\mathfrak{b}$ (resp. $\mathfrak{a}$), by the representation theorem, i.e.,

$$\mathfrak{b}(\xi,\eta) = (\xi, B\eta)_{\mathscr{H}} \quad \text{(resp.} \quad \mathfrak{a}(\xi,\eta) = (\xi, A\eta)_{\mathscr{H}}), \tag{2.6}$$

for $\xi \in \mathscr{H}$ and $\eta \in \mathscr{H}$, are strictly positive, bounded, self-adjoint operators onto $\mathscr{H}$ with bounded inverses. There exist $c \in \mathbb{R}$, such that $0 < cI \leq B \leq A$, where I is the identity operator. It follows from formula (2.6), that an explicit expression of A is given by

$$(A\eta)(k) = E(U(\infty,\eta)u^\infty(k)|\mathscr{F}_k) \tag{2.7}$$

and that an explicit expression of B is given by

$$(B\eta)(k) = E((U(\infty,\eta) - E(U(\infty,\eta)))u^\infty(k)|\mathscr{F}_k), \tag{2.8}$$

for $\eta \in \mathscr{H}$, where $0 \leq k \leq \bar{T}$.

Theorem 2.1 leads to the solution of the optimization problem of this paragraph (see Corollary 2.6 of [5]). In fact, using a simple scaling argument, optimization problem (2.3) can be reformulated (see Lemma 2.5 of [5]) as an elementary Hilbert space problem. Namely, to determine the points closest to the origin, in a closed convex set defined by conditions (C_3), (C_6), and $E(U(\infty,\eta)) \leq e$, for some $e > 0$. As it is well-known, this problem has a unique solution.

Theorem 2.2 *Let hypotheses (H_1), (H_2), and (H_3) be satisfied. If* $\mathscr{C}_0$ *is non-empty, then the optimization problem (2.3) has a unique solution* $\hat{\eta} \in \mathscr{C}_0$.

The solution $\hat{\eta}$ is given by a constructive approach in [5]. In fact, in that reference a Lagrangian formalism, an algorithm to invert the operators A and B and approximation methods for determining the multipliers are given.

To sum up the Lagrangian formalism, let $\lambda_0, \lambda_1, \ldots \lambda_{\bar{T}+T-1}$, and μ be real numbers and let $\nu \in \mathscr{H}$. These are the multipliers. A Lagrangian (slightly different from the Lagrangian (2.13) if [5]) is defined by

$$\begin{aligned} h_{\lambda,\mu,\nu}&(\eta) \\ &= \frac{1}{2}\mathfrak{b}(\eta) - \sum_{0\leq t\leq \bar{T}+T-1} \lambda_t(E((\Delta U)(t+1,\eta)) - c(t)E(U(t,\eta))) \\ &\quad - \mu E(U(\infty,\eta)) - (\nu,\eta)_{\mathscr{H}}, \end{aligned} \tag{2.9}$$

where $\eta \in \mathscr{H}$. We now have to find the critical points in $\mathscr{H}$, for fixed λ, μ and ν and determine the multipliers such that the critical point in fact is an element of $\mathscr{C}_0$. The multipliers shall satisfy

$$\lambda_t \geq 0,\ \lambda_t(E((\Delta U)(t+1,\hat{\eta})) - c(t)E(K(t,\hat{\eta}))) = 0, \tag{2.10}$$

for $0 \leq t \leq \bar{T} + T - 1$ and

$$\nu_i(k) \geq 0,\ (\nu_i(k))(\omega) = 0 \quad \text{a.e. for} \quad \omega \in supp\, \hat{\eta}_i(k), \tag{2.11}$$

$0 \leq k \leq \bar{T}$ and $1 \leq i \leq N$. The solution to the optimization problem is given by the Lagrangian formalism (Theorem 2.7 of [5]):

Theorem 2.3 *Let $\mathscr{C}_0$ be non-empty. Then there exist multipliers, satisfying (2.10) and (2.11), such that the solution $\hat{\eta}$, of the optimization problem (2.3) is given by the unique solution $\hat{\eta}$ of the equation $(Dh_{\lambda,\mu,\nu})(\eta) = 0$. Moreover,*

$$\hat{\eta} = B^{-1}(\mu m + \sum_{0 \leq t \leq \bar{T}+T-1} \lambda_t l_t + \nu), \tag{2.12}$$

where $m \in \mathscr{H}$ is given by the linear functional $(m,\eta)_{\mathscr{H}} = E(U(\infty,\eta))$ and $l_t \in \mathscr{H}$ is given by the linear functional $(l_t,\eta)_{\mathscr{H}} = E((\Delta U)(t+1,\eta)) - c(t)E(U(t,\eta))$, for $0 \leq t \leq \bar{T} + T - 1$. Explicitly

$$m(k) = E(u^{\infty}(k)), \tag{2.13}$$

where $0 \leq k \leq \bar{T}$ and

$$l_t(k) = E(u(k,t+1) - (1+c(t))u(k,t)|\mathscr{F}_k), \tag{2.14}$$

if $0 \leq k \leq t$ and $l_t(k) = 0$, if $t < k$, where $0 \leq t \leq \bar{T} + T - 1$ and $0 \leq k \leq \bar{T}$.

Approximation methods to determine the multipliers (only ν is difficult) are discussed in [5]. A crucial point for the application of formula (2.12) is the inversion of the operators A and B. We developed (Appendix A of [5]) an algorithm giving the inverses in a small number of steps, only involving simple matrix algebra and conditional expectation. In particular we proved (see conclusion after Proposition A.4 of [5]):

Theorem 2.4 *The spectrum of each one of the operators A and B is a finite set of strictly positive real numbers.*

2.2. General case: *H*

In this section we outline results concerning the company H obtained in [6]. The (aggregate) underwriting portfolio $\eta = (\eta^{(1)}, \ldots, \eta^{(\aleph)})$ is supposed to be an element of a space of square-integrable portfolios $\mathscr{P}_{u,\bar{T}}$. This space is defined by the conditions that $p \circ \theta^{(i)} \in \mathscr{E}^2_{\bar{T}}(\mathbb{R}^{N^{(i)}}, \mathscr{A})$ and $\lambda \circ \theta^{(i)} \in \mathscr{E}^2_{\bar{T}}(\mathbb{R}, \mathscr{A})$, for $1 \le i \le \aleph$, where the function p is given before formula (1.6) and where the function λ is given by $\lambda(\tau_f) = 1$ and $\lambda(x) = 0$ for $x \in \mathbb{R}^N$ and $N \in \mathbb{N}$ (for details see [6]). We also suppose that the dividend allocation process $\vec{D}$ is square-integrable, i.e., $\vec{D} \in \mathscr{E}^2_{\bar{T}+T}(\mathbb{R}^{\aleph}, \mathscr{A})$. We remind that the equity allocation $\vec{K}(0) \in \mathbb{R}^{\aleph}$ and that $D^{(j)}(0) = 0$, for $1 \le j \le \aleph$.

Next we shall formulate more precisely the constraints (B)–(H). Reordering them and introducing the variance constraints in §1 (and see Theorem 2.4 of [6]), we obtain the following constraints on $(\eta, \vec{K}(0), \vec{D}) \in \mathscr{P}_{u,\bar{T}} \times \mathbb{R}^{\aleph} \times \mathscr{E}^2_{\bar{T}+T}(\mathbb{R}^{\aleph}, \mathscr{A})$ (We remind that $V(X)$ denotes the variance of the random variable X):

- c'_1) $K(0) = \sum_{1 \le j \le \aleph} K^{(j)}(0)$ (*budget constraint*);
- c'_2) $D = \sum_{1 \le j \le \aleph} D^{(j)}$ and $D^{(j)}(0) = 0$ (*budget constraint*);
- c'_3) $E((\Delta U)(t+1, \xi + \eta)) \ge c(t) E(\sum_{1 \le j \le \aleph} K^{(j)}(t))$, where $c(t) \in \mathbb{R}^+$ is given for $t \in \mathbb{N}$, (*constraint on ROE*);
- c'_4) $V(\sum_{1 \le j \le \aleph} K^{(j)}(t)) \le \varepsilon'(t)(\delta(t)K(0))^2$, $E(\sum_{1 \le j \le \aleph} K^{(j)}(t)) \ge \delta(t)K(0)$, $t \in \mathbb{N}$, where $\varepsilon'(t) \ge 0$ and $\delta(t) > 0$ (*modified ruin constraint on H*);
- c'_5) supplementary constraints, to be specified, on $D^{(j)}$, (e.g. to increase the equity of $S^{(l)}$, one can set $D^{(l)} = 0$). To be general, we only suppose that there are real valued functions F_α, $\alpha \in I$, an index set, such that $F_\alpha(t, \eta, \vec{K}(0), \vec{D}) \le C_\alpha(t, K(0), \xi)$, where $C_\alpha(t, K(0), \xi)$ are constants only depending of the initial equity and the run-off;
- c'_6) if $(\eta_i^{(j)}(t))(\omega) \ne \tau_f$, then $((\underline{c}_i^{(j)}(\eta))(t))(\omega) \le (\eta_i^{(j)}(t))(\omega) < \infty$ and $(\eta_i^{(j)}(t))(\omega) \le ((\overline{c}_i^{(j)}(\eta)))(t))(\omega)$, for $\omega \in \Omega$ (a.e.), $1 \le j \le \aleph$, $1 \le i \le N^{(j)}$ and $0 \le t \le \bar{T}$. Here $\underline{c}_i^{(j)}(\eta) \in \mathscr{E}^2(\mathbb{R}, \mathscr{A})$ and $\overline{c}_i^{(j)}(\eta)$ are given $\mathscr{A}$-adapted processes, which are causal functions of η and which satisfy $((\underline{c}_i^{(j)}(\eta))(t))(\omega) \in [0, \infty[$ and $((\overline{c}_i^{(j)}(\eta))(t))(\omega) \in [0, \infty]$ (*positivity and market constraints*);
- c'_7) $V(K^{(j)}(t) - m^{(j)}(t, \xi^{(j)} + \eta^{(j)})) \le \varepsilon'^{(j)}(t)(\delta^{(j)}(t)K(0))^2$ and $E(K^{(j)}(t) - m^{(j)}(t, \xi^{(j)} + \eta^{(j)})) \ge \delta^{(j)}(t)K(0)$, for $1 \le j \le \aleph$ and $t \in \mathbb{N}$, where $m^{(j)}$ are as in (c_7) and where $\varepsilon'^{(j)}(t) \ge 0$ and $\delta^{(j)}(t) > 0$ (*modified non-solvability constraint*);
- c'_8) if $t > (t_f^{(j)})(\omega)$, then $(\eta^{(j)}(t))(\omega) = \tau_f$, where $(t_f^{(j)})(\omega)$ is the smallest time in $\mathbb{N}$ such that $K^{(j)}(t_f^{(j)}) < m^{(j)}(t_f^{(j)}, \xi^{(j)} + \eta^{(j)})$, and if $t \le (t_f^{(j)})(\omega)$ and $(K^{(j)}(t-1))(\omega) > (m^{(j)}(t-1, \xi^{(j)} + \eta^{(j)}))(\omega)$, then $(\eta^{(j)}(t))(\omega) \in \mathbb{R}^{N^{(j)}}$,

for $\omega \in \Omega$ (a.e.), $1 \leq j \leq \aleph$ and $t \geq 1$ (*the activity of $S^{(j)}$ ceases just after that the solvency margin is not satisfied*).

Let $\mathscr{C}_{c'}$ be the set of all $(\eta, \vec{K}(0), \vec{D})$ satisfying the constraints (c'_1)–(c'_8). Thus we sum up the constraints (c'_1)–(c'_8) in the form:

- c') $(\eta, \vec{K}(0), \vec{D}) \in \mathscr{C}_{c'}$.

The original constraints with non-solvency probabilities are satisfied, when $(\eta, \vec{K}(0), \vec{D}) \in \mathscr{C}_{c'}$, under conditions on the constants $\varepsilon'(t)$ and $\varepsilon'^{(j)}(k)$ in (c'_4) and (c'_7) (see Theorem 2.4 of [6]).

The equity allocation and portfolio selection problem for H is then, for given run-off ξ, initial equity $K(0)$ and functional form $\eta \mapsto D(\xi + \eta)$ of the (total) dividend process for H, to find the solutions $\widehat{\eta} \in \mathscr{C}_{c'}$ of the equation

$$E(U(\infty, \widehat{\eta} + \xi)) = \sup_{(\eta, \vec{K}(0), \vec{D}) \in \mathscr{C}_{c'}} E(U(\infty, \eta + \xi)). \tag{2.15}$$

We need to make some hypotheses on the unit contract result processes. To be as simple as possible, suppose that

$$u^{(j)\infty} \text{ satisfies hypotheses } (H_1), (H_2), \text{ and } (H_3) \text{ of } \S 2.1 \text{ for } 1 \leq j \leq \aleph, \tag{2.16}$$

that

$$\| E((u^{(j)}(k,t))^2 | \mathscr{F}_k) \|_{L^\infty} < \infty \text{ for } k < t \tag{2.17}$$

and that

$$u_i^{(p)\infty}(k) \text{ and } u_j^{(r)\infty}(l) \text{ are independent for } p \neq r \tag{2.18}$$

(see hypotheses (h_1)–(h_4) of [5]). For reference we sum up these hypotheses

- h) Hypotheses (2.16), (2.17) and (2.18) on unit contract result processes.

We can now state, the result on the solutions of optimization problem (2.15) (for details see Theorem 2.5 of [6]). We denote by d-topology, the topology of convergence for distributions.

Theorem 2.5 *Let the utilities $u^{(p)}(k,t)$ and $u^{(p)\infty}(k)$, of unit contracts, satisfy (h) and let the functions $\eta^{(j)} \mapsto m^{(j)}(t, \xi^{(j)} + \eta^{(j)})$, $\eta \mapsto D(\xi + \eta)$ and $\eta \mapsto \underline{c}_i^{(j)}(\eta)$ to $\mathscr{E}^2(\mathbb{R}, \mathscr{A})$ map bounded sets into bounded sets. In the d-topology, let $\eta^{(j)} \mapsto m^{(j)}(t, \xi^{(j)} + \eta^{(j)})$ and $\eta \mapsto D(\xi + \eta)$ be continuous, let $(\eta, \vec{K}(0), \vec{D}) \mapsto F_\alpha(t, \eta, \vec{K}(0), \vec{D})$, $\alpha \in I$, be lower semi-continuous and let $\eta \mapsto (\underline{c}_i^{(j)}(\eta))(t, \omega)$ and $\eta \mapsto -(\overline{c}_i^{(j)}(\eta))(t, \omega)$ be lower semi-continuous (a.e.). If*

$$V\Big(\sum_{1 \leq k \leq \bar{T} + T} D(k, \theta)\Big) \leq c^2 V(U(\infty, \theta)),$$

where $0 \leq c < 1$, and if $\mathscr{C}_{c'}$ is non-empty, then the optimization problem (2.15) has a solution $\widehat{x} = (\widehat{\eta}, \widehat{\vec{K}}(0), \widehat{\vec{D}}) \in \mathscr{C}_{c'}$.

The variance hypothesis of the theorem just states that the total accumulated dividend is less volatile than the accumulated final result (see Remark 2.6 of [6]). The supposed regularity properties are usually satisfied in applications.

The proof of this result in [6] goes along the following lines. Beginning with the fact that $\mathfrak{b}^{1/2}$ is equivalent to the norm in $\mathscr{H}$, one proves that $\mathscr{C}_{c'}$ is d-compact. The function $\eta \mapsto E(U(\infty, \eta + \xi))$ is then proved to be d-continuous, so it takes its maximum in $\mathscr{C}_{c'}$.

References

1. Dana, R. A. and Jeanblanc-Picqué, M.: *Marchés Financiers en Temps Continue: Valorisation et Équilibre*, Economica, 1994.
2. Embrechts, P., Klüppelberg, C., and Mikosh, T.: *Modelling Extremal Events*, Applications of Mathematics, Vol. 33, Springer-Verlag, 1997.
3. Harrison, M. and Pliska, S.: Martingales and Arbitrage in Multiperiod Securities Markets, *J. Econom. Theory* **20** (1979), 381–408.
4. Markowitz, H.: Portfolio Selection, *Jour. Finance* **7** (1952), 77–91.
5. Taflin, E.: Equity Allocation and Portfolio Selection in Insurance: A simplified Portfolio Model, Preprint AXA, March 1998, http://xxx.lanl.gov/abs/math/9907142.
6. Taflin, E.: Equity Allocation and Portfolio Selection in Insurance, Preprint AXA, March 1999, http://xxx.lanl.gov/abs/math/9907160. To appear in *Insurance: Mathematics and Economics*.