In retrospect:
A personal view on Moshé Flato's scientific legacy

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When a talented sculptor is confronted with a block of marble, he is using tools that come from centuries of tradition and the training which he received from his masters. But from the first stroke he already has, in the back of his mind, the design of the sculpture which he wants to extract from that block. Creating a masterpiece is a long and painful process, interspersed with surprises due to the irregularities of the physical properties of the noble material he is using, and the design continues to evolve with the artist's maturing and subject to external influences. Only gradually does the shape emerge, at first visible to experts, then to the general public.

Beyond the technicalities, however important they may be, Moshé's scientific life was a perfect illustration of this idea. His masterpiece is not finished. Much polishing remains to be done, new elements still need to be added to the main themes which he conceived, and entirely new themes are now beginning to emerge from them. But a very partial view of what he had in mind can already described. This is what I shall attempt here, based on my experience of having, for about 35 years, participated with others in the elaboration of his ideas. All those who have been close to him know that he used to call me affectionately an "idiot" more often than he used that expression for anybody else; this usually happened when my formulation of one of his ideas was not to his taste and needed to be corrected, either because he did not like some of the ingredients I was adding or when, while striving for too precise a formulation, I obscured an essential point which he had purposedly stated ambiguously. François Mitterrand used to say, paraphrasing the Cardinal de Retz, "On ne sort de l'ambiguïté qu'à son détriment". In his case, this was smart politics, in Moshé's, using seemingly ambiguous statements was the only way by which he could transmit the deep feeling which he had without the distortion of too precise, confining words. I hope that now, with him not around, my formulation will be faithful enough to the main ideas which he developed and often initiated. My purpose here is to show how, in retrospect, one can look at the genesis and evolution of ideas and concepts in some of the fields where Moshé made an impact that has only begun to be fully recognized. I shall refrain from entering into technicalities, because they are covered in part in the present volume and in recent review papers, and in order to make the ideas understandable to a general, scientifically educated audience.

Towards the end of the 19$^{\text{th}}$ century, physics seemed to have achieved a complete understanding of the world, with classical mechanics describing the motion of rigid bodies, electromagnetism waves and charged particles, and the Lorentz force their interactions. It was only a plateau because then the 'deformation dae-

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mon' struck. In 1881 an American physicist, Albert Michelson, came to Berlin and tried to observe the movement of the earth in the 'ether' by measuring the velocity of light in different directions, using the interferometer which he had developed. He found no differences, which did not conform to classical mechanics. In such a case, in Europe, the first reaction was and remains – as Charles Schulz wrote in a 'Peanuts' cartoon –, "my mind is made up, don't bother me with facts". That is how the first observation of neutral currents at the CERN, ten years before their theoretical prediction, was deemed a mistake and never published. However facts and some American scientists are stubborn. In 1887, with Edward Morley, Michelson performed a more refined experiment in Cleveland and the fact was proved. So there was a *paradox*, because the speed of light is a limit. The so-called black-body radiation provided another paradox until in 1900, as a last resort, Max Planck proposed the quantum hypothesis, the energy of light is not emitted continuously but in quanta, proportional to its frequency. He wrote h for the proportionality constant which bears his name.

At the same time the notion of *symmetry* started to appear in mathematics and in physics, developments in both sciences being very closely interrelated all through the twentieth century. In physics, equations and quantities are usually invariant under some transformation groups. The theory of the latter was systematized by Sophus Lie and developed further by many, among them Élie Cartan and Henri Poincaré, to remain within that time range and in mathematics. Emphasizing this invariance property has many advantages, but we shall not elaborate on that aspect here because it has been developed in many books. In any case, this is a concept that Moshé learned from his teacher Giulio Racah, and first applied to molecular spectroscopy in his celebrated M.Sc. thesis, completed in 1960 under Racah but only published in 1965 [13], and in his unpublished Ph.D. thesis on applications of group theory to nuclear physics, a problem involving the restriction of representations from unitary to symplectic groups, which Moshé also completed under Racah before coming to France in 1963 but never defended. As a consequence of the expertise which he gained in these applications of group theory to physics, he took with more than a grain of salt the quite naive applications that were fashionable among particle physicists in the sixties and, in a much more elaborate manner, still influence the present interpretation of experimental data. This can already be seen in his 1965 French D.Sc. thesis [14]. He never hesitated to express these views, which many considered blasphemous, in public, and eventually his outspokenness obliged him to seek "scientific asylum" in the French mathematical community while staying in contact with physicists around the world who dared and could afford to hold independent opinions.

The notion of symmetry guided Moshé throughout his scientific career, both as a mathematician and as a theoretical physicist, and led him to make a number of significant contributions. Even on the mathematical side, the motivation for the study of symmetry comes from physics and the mathematical theory has physical applications. In this connection I can mention two directions which his work, in collaboration with (among others) his D.Sc. student and close friend since 1968, Jacques Simon, took. The first is the theory of analytic vectors in Lie group representations [25], suggested by the problems that arise in passing from Lie algebras to Lie groups. The second is the cohomological study of nonlinear representations [24] of covariance groups of nonlinear partial differential equations, which leads to important mathematical developments with nontrivial physical consequences such as the "tour de force" on Maxwell-Dirac equations [26], the very basis of classical electrodynamics. A lesser known, third direction, is the notion of supersymmetry which one can find already in an embryonic form – close to the Wess-Zumino Poincaré supersymmetry – in his 1970 paper [18].

In 1905 Einstein solved the first paradox that we described above when he showed, in our terminology, that the Galilei invariance group $SO(3)\cdot\mathbb{R}^3\cdot\mathbb{R}^4$ (space rotations, addition of velocities and space-time translations) of Newtonian mechanics had to be *deformed* into the Poincaré group $SO(3,1)\cdot\mathbb{R}^4$, where velocities do not add linearly, of relativistic mechanics. That same year Einstein contributed, in addition to a third basic paper, on the Brownian motion, to the solution of the second paradox by his theory of the photoelectric effect. Incidentally, that theory was the main contribution recognized by the Nobel Prize in Physics which was awarded to him only in 1922. Revolutionary ideas take time to be recognized. Around 1920, an "agrégé d'histoire", Prince Louis de Broglie, was introduced to the photoelectric effect, together with the Planck-Einstein relations and the theory of relativity, in the laboratory of his much older brother, Maurice duc de Broglie. This led him, in 1923, to his discovery of the duality of waves and particles, which he described in his celebrated 1925 thesis. After Davisson and Germer's observation of the diffraction of electrons in 1927 confirmed his theory, Louis de Broglie was awarded the Nobel Prize in 1929.

Moshé had been sent to Louis de Broglie when he came to Paris in October 1963. There was immediate sympathy between these two very different but profound personalities. Moshé used to say that Louis de Broglie was an extraordinarily well-adjusted schizophrenic, and that without pejorative connotation, because he could see single physical phenomena as dualities. The wave-particle duality is one example, but there are others, like his theory of fusion, in particular, his short-lived neutrino theory of light. As de Broglie formulated that theory, with photons composed of two neutrinos, it was soon 'shot down', and justly so, but the general idea was vindicated, in a way, by Moshé when he showed in 1988 [19] that it was possible to formulate quantum electrodynamics with photons composed of two scalar singletons. The latter 'particles', in fact, massless particles in a 2+1 dimensional space-time at infinity in the anti-de Sitter universe, are just one step below neutrinos (at the limit of unitarity) in the diagram of representations of the anti-de Sitter group $SO(3,2)$, which is the last possible deformation of the Poincaré group in the category of Lie groups when, due to a tiny negative curvature *in the microworld*, space-time translations do not add linearly. The idea that singletons, with Higgs particles, should be considered as the ultimate constituents of matter at our level of understanding and experimental results, is an example of the interaction between Moshé's basic *deformation view of physical theories* [16] and some of the concepts which he learned when he arrived in France.

Moshé's deformation philosophy was inspired by Murray Gerstenhaber's mathematical theory of deformations of algebras [29], joined to Moshé's deep analysis of the evolution of physics. That analysis had epistemological consequences going beyond its mathematical formulation in terms of deformations in a well-chosen category. In particular, it tells us that there can be no "theory of everything". Our representation of physical reality is only a mathematical idealization of that reality in a given range, and there are several ranges without hermetic border between them. One has to be modest and realize that there are limitations to the capacities of the human brain, however brilliant and profound it may be. Only God, if one is willing to add to our representation of the universe what is, in my view, an axiom that is not contradictory to science but independent of it, may possibly go further. Moshé was a pantheist, à la Spinoza, and considered the universe itself to be the manifestation of what theologians call God.

Moshé used to discuss physical concepts and their philosophical implications with Louis de Broglie, Hideki Yukawa (who spent some time with de Broglie in Paris in 1964), David Bohm, Roger Penrose, and André Lichnerowicz. This led him in the mid-sixties to stress, sometimes provocatively [27], that one should not decouple *space-time* from the so-called 'internal symmetries' of elementary particles. Rather, the conformal group $SO(4,2)$ of (flat or de Sitter) space-time has a role to play [5]. Reality is probably much more complex than any of us could have foreseen at that time, but this gives even greater value to Moshé's strong warnings about the dangers of some of the simplistic and incorrectly formulated 'no-go' theorems that were then popular.

I am confident that the first half of the 21st century will show how fundamental is the singleton hypothesis. After formulating the singleton hypothesis around 1978, a hypothesis which he considered "too beautiful not to be true", Moshé never ceased trying to prove it. At this stage it is not yet a complete theory and, so far, it has not yielded a verified model or new predictions (see however [28]). But after all, in a different context, quarks are still only a hypothesis without a complete theory, and yet the quark hypothesis has given rise to a succession of models, the last of which is considered successful because a wide array of experimental data is consistent with it. Logically, a hypothesis is a sufficient condition, not a necessary one because data might have a different explanation. Moshé was fully aware of the delicate work remaining to be done and even discussed a few related mathematical questions with some of his students. With the confirmation of neutrino oscillations in 1998, Moshé saw the 'promised land' not too far away. He died before reaching it, but he has brought us through the desert to a point where we can cross the Jordan river, at least, and sound the trumpets around the walls of Jericho. The contribution of our long time friend and collaborator Christian Frønsdal in these *Proceedings*

[28], along the lines indicated in Moshé's last paper [20], is a further step in this direction.

More will come, ranging from *phenomenology* where a lot of work and possibly reinterpretation of data could be required, through subtle *theoretical physics* developments needed for the new theory (including considerations of flavor symmetry, *PC* violation, the number of generations of leptons, conformal covariance and the new light on renormalization which emerges from the latest works of Alain Connes and Dirk Kreimer [10]), to pure *mathematics*, including groups (their indecomposable representations and associated Gupta-Bleuler triplets), infinite-dimensional algebras, possibly with a notion of "square root of a superalgebra" (which would develop in a general context the construction introduced in [19], where the generators of the canonical commutation relations associated with photon fields are obtained from the commutators of singleton field creation and annihilation operators), deformation quantization in infinite-dimensional manifolds of initial data of nonlinear classical field equations (which will require a study of the latter, e.g., along the lines developed by Moshé in [26] for the Maxwell-Dirac equations of classical electrodynamics), all the way to the very abstract ideas and methods which are now being developed by Maxim Kontsevich [33]. This is a simplistic sketch of a program which will occupy a generation of mathematicians and physicists.

De Broglie had another basic idea, what he and Yukawa had called the 'bilocal model', which had been formulated – as an unintended caricature – by some of his collaborators before Moshé arrived in Paris. It did not take long for Moshé, already an expert in group theory, to realize how naive the formulation was. But the basic idea, that points in space-time are not one, but two, is not so far fetched. Just pull a string between those two points and you can get into string theory, a major trend in modern theoretical physics which, though so far lacking experimental verifications, has inspired some very beautiful mathematics. Or look at the $\mathbb{Z}_2$-doubling of space-time, as in the Connes and Lott interpretation of the standard model of elementary particles within noncommutative geometry [9].

The duality between waves and particles had brought de Broglie to what he called the 'mécanique ondulatoire'. When Moshé came to Lyon in 1965, that was still what quantum mechanics was called there, and it was only taught at the graduate level. It had taken some German and Austrian physicists, in particular, Hermann Weyl – less well-known to the physics public because he was mainly a mathematician –, Werner Heisenberg and Erwin Schrödinger, to transform it into quantum mechanics, based on operators in spaces of wave functions, endowed with a structure invented some years before by David Hilbert. Goethe used to say that "Mathematicians are like Frenchmen: they translate everything into their own language and henceforth it is something entirely different". Those German physicists did just that with de Broglie's approach! A consequence of the operatorial formulation is the 'Copenhagen' probabilistic interpretation systematized by Niels Bohr, which Moshé learned from Giulio Racah. It offers an efficient interpretation of the experimental results, e.g., in terms of the probability of presence of a particle given by some wave function and of expectation values of observables (operators in this case). But Einstein and de Broglie hated the idea.

Einstein would say that "God does not play dice with the universe" and formulated his celebrated paradox with Podolsky and Rosen [11]. De Broglie (and many others) looked for 'hidden variables', a classical mechanics underlying quantum mechanics. Eventually it was experimentally proven [1] that hidden variables do not exist, or rather – as Moshé pointed out [21] when the first experimental indications appeared – that if they exist they cannot be local. Deformation quantization, Moshé's main scientific contribution, the most striking example of his deformation philosophy and a seminal idea in mathematics, solves the mystery in a way that reconciles Einstein and de Broglie with Bohr. God does not play dice with the universe, 'it just feels like it' when you consider only the operatorial formulation.

Around 1960 there occurred two seemingly unrelated mathematical events to which Moshé and I were exposed. Some mathematicians (Kunihiko Kodaira and Donald Spencer [32] in 1958 for complex analytic structures, and Murray Gerstenhaber [29] in 1963-64 for algebras) developed a cohomological theory of deformations. At the symmetry group level it became immediately clear to us, and many others, that the passage from the Galilei to the Poincaré group (with deformation parameter c^{-1}, where c is the velocity of light) and then to the de Sitter group (with curvature as the deformation parameter) is a typical example of a deformation.

At about the same time other mathematicians introduced pseudodifferential operators (called singular integral operators in 1957 by Alberto Calderón and Antoni Zygmund [7] who defined their symbols), which provided a rigorous framework for notions such as square roots of partial differential operators (of the Laplacian for instance). They gained celebrity outside the world of partial differential equations by their use, in 1963, by Michael Atiyah and Isadore Singer [2] and subsequently by many other mathematicians, in the proof of index theorems, a problem that had been posed by Israel Gelfand, how to express the analytic index of an operator in topological terms. I had personally taken an active part in a weekly seminar [8], led by Henri Cartan and Laurent Schwartz in Paris, which studied index theorems for elliptic operators on a compact manifold from both the analytical and topological points of view, and shaped the future of many of the most promising young mathematicians of our generation. Moshé did not attend the seminar but was exposed to it by me.

Around 1974 Moshé came to the conclusion that physics evolves in stages, when it hits a paradox [17], the passage from one level of scales, e.g., velocities and distances, to another being mathematically described by a deformation, in an appropriate category. This led us in 1976 – 1978 [23, 3] to the formulation of quantum mechanics and quantum theories on the same observables as classical mechanics, functions on a phase space, but with a deformed composition law which we called a star-product. The first term in the deformation series beyond the usual product of functions was the Poisson bracket of classical mechanics and the deformation parameter $\frac{i\hbar}{2}$, the Planck constant being $h = 2\pi\hbar$. Quantization should therefore be understood as a deformation, what is now called *deformation quantization*. This in effect reconciles Einstein and de Broglie with Bohr because it shows that, while there cannot be an underlying classical level with hidden variables, classical and quantum theories can be formulated on the same algebras of observables, and physical results can be described by appropriate mathematical notions, equivalent to what is given by the operatorial formulation when both theories apply, but without need for the latter's explicit probabilistic interpretation. Moreover deformation quantization can be applied to cases not covered by the existing operatorial formulation [36].

The way things developed is interesting in itself. Since it is at present the most recognized of Moshé's contributions and mathematically the most seminal, I shall elaborate a little. Technically, it started with the renewal of interest, in the late sixties, in infinite-dimensional symmetries, which date back to the work of Élie Cartan at the beginning of the century. Among them were what are now called Kac-Moody algebras and algebras of vector fields (first order differential operators) on some manifolds, with their Gelfand-Fuks cohomology. Our friend André Lichnerowicz who, among his many talents, was a master of tensor calculus on manifolds, had computed what he called the 1-differentiable cohomologies of such algebras [34]. Since deformations, in Gerstenhaber's sense, are governed by second and third cohomologies, it was natural to apply Lichnerowicz's results to deformations, which we did in the 1-differentiable case [22] because we did not yet know how to handle the more complicated differentiable case. A remarkable example stood out, the algebra of differentiable functions over a symplectic manifold endowed with the Poisson bracket, which is at the base of the geometric, Hamiltonian formulation of classical mechanics. There we discovered that infinitesimal deformations of the Poisson bracket algebra are, subject to some natural technical conditions, classified by the deformations of the symplectic structure, which are themselves governed by the second de Rham cohomology of the manifold.

From the beginning, Moshé had in mind that the deformation approach should provide a clue to a true understanding of quantization. As evidence I shall translate here an excerpt from his introduction to our *Compositio Mathematica* paper, submitted November 1, 1974. The introductory and concluding remarks in papers by Moshé are very inspiring, often going beyond the strict purpose of the paper. In retrospect it is clear that, beyond the technical difficulty which we encountered at that time, he had already foreseen deformation quantization 'around the corner':

> The study of deformations of those algebras is in part motivated by theoretical physics which appears as a generalization of classical mechanics where those algebras are naturally introduced. For instance, in quantum mechanics, the Poisson brackets of functions are replaced by commutators of operators in Hilbert space; in classical and quantum field theory, the symplectic manifold becomes infinite dimensional; in various branches of statistical mechanics, phase space plays an important role. ...

Our work inspired Jacques Vey [37] to pass in 1975 from 1-differentiable deformations to general differentiable deformations of the Poisson bracket algebra, at least subject to a technical assumption on the topology of the manifold, which was waived much later. His work relied on the determination of the differentiable Hochschild cohomology of the algebra of functions, itself related to a previous important algebraic result, well hidden in a now famous paper [31] by Hochschild, Kostant and Rosenberg. In passing he derived, on flat space $\mathbb{R}^{2n}$, a bracket which in fact had been obtained in England in 1949, in a completely different perspective related to the hidden variables question, by a student of Maurice Bartlett named José Enrique Moyal [35], as the inverse image, in the sense of Wigner, of the commutator of operators in the Weyl quantization procedure. Vey also considered the corresponding associative deformed product, that we now call star-product. In the Spring of 1975, I lectured about our work on the 1-differentiable case in Austin, Texas. Our friend Jerzy Plebański, who had mentioned Moyal brackets in lectures given in 1969 in Poland – much later we learned that another friend, Ludwig Faddeev, had also done so in Leningrad around the same time – was in the audience and said that it reminded him of Moyal. I could not find the paper [35] in the library there and *Mathematical Reviews* made no mention of the aspect for which that paper is now widely quoted, but Moshé decided that we should order a photocopy from the CNRS. At the same time we had received Vey's preprint. The bracket was in both places, independently, and that is exactly what Moshé had intuited in the passage we just quoted! This led us to our first Note in 1976 [23] and, with Frønsdal and Bayen who joined us, to the papers [3] where we showed, among many other things, that it is possible to obtain the most fundamental results of quantum mechanics, such as the description of the harmonic oscillator and the determination of the spectrum of the hydrogen atom, by considering the same algebra of observables as classical mechanics, but with a deformed product.

It turned out that the composition of symbols of pseudodifferential operators, a key ingredient in the proofs of the index theorems, is exactly a star-product, related to what physicists called the standard ordering, which consists, for differential operators, in writing first the coefficients and then the differentiations. So the two 'unrelated' events from the early sixties were related after all, and we, like Molière's M. Jourdain, had been 'speaking prose' for over twelve years without knowing it! Looking at the footnote by Louis Boutet de Monvel in [38], p.18, one can see that neither Vey, because of his algebraic approach, nor most mathematicians working in the field of pseudodifferential operators, realized the connection for quite some time. Nevertheless it must have had some impact on Moshé and me at the subconscious level. One should mention, for completeness, what is now called 'geometric quantization', developed around 1970, independently, by Bertram Kostant and Jean-Marie Souriau, related to the orbit method of Alexandre Kirillov and to ideas on quantization formulated at that time by Felix "Alik" Berezin, of which we became aware much later. It proved successful in group representation theory but, due to a too conservative approach which omitted the deformation concept, not equally so in physical applications.

I shall not elaborate further on what is now well-known history. For a more detailed exposition and for references the reader may consult my review [36], the "promenade" by Simone Gutt [30] in these *Proceedings* and the contribution by Louis Boutet de Monvel [6]. Let me just add that star-products gave rise to an 'autonomous' representation theory of Lie groups on spaces of functions over coadjoint orbits with deformed products, and that quantum groups are an avatar of deformation quantization in the category of Hopf algebras, and can, in many interesting cases, be reduced to deformations of the product of functions over Lie groups, leaving the coproduct undeformed. The questions of existence and classification of star-products over symplectic manifolds have been advanced by the more geometrical and algorithmic approach of Boris Fedosov [12], who incidentally came to this branch of mathematics from index theorems. Essentially, one combines our 1-differentiable deformations, classified by formal series in the second cohomology classes of the manifold, with a Moyal product.

In 1997 Maxim Kontsevich put what seemed at first 'the frosting on the cake' that we had baked, by proving his formality conjecture and giving a complete solution to the problem of existence and classification of deformation quantizations on general Poisson manifolds. It now appears [33] that this has roots going very deep into modern mathematics, using in part methods, such as graphs, inspired by physics and extending, in particular via the notion of deformations of algebras over operads which he defines in his contribution to these *Proceedings*, far into seemingly unrelated notions like Feynman path integrals, periods, and Grothendieck's unfinished symphony of algebraic geometry. All this will develop well into the 21$^{\text{st}}$ century.

This is, in my view, part of the kernel of Moshé's scientific legacy. I have not attempted here to develop a final historical description or philosophical judgment on it. I am merely doing some phenomenological work, trying to bring a little order and indicate some guidelines in the scientific production of a brain that was as complex and in constant evolution as life itself. There is a lot which remains to be done, beautiful mathematics for sure, in the interplay with fundamental physical developments. It will require a variety of talents which no single scientist may possess. I shall be very happy if I live to see the beginning of some breakthrough in that direction.

I shall end this overview by some reflection on the man behind it. Moshé used to say that scientists, especially mathematicians, can be divided into two categories, "problem solvers" and "theory makers". But where does he stand in this dichotomy? He was a theory maker who could solve problems and inspire others to solve problems, certainly in mathematics and in theoretical physics, but also in epistemology, and he had many other facets as can be seen from these *Proceedings* and the personal recollections by some of his numerous friends [38]. As three friends say there, it was impossible to define him when he was alive, and it would be presumptuous to attempt now to fix his image: *Personnage inclassable de son vivant, il serait bien présomptueux de vouloir aujourd'hui figer son image.*

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