Queen Mary
University of London

HIGH-ACCURACY NUMERICAL METHODS IN GENERAL RELATIVITY

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(Part III)

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SUMMARISE II

- **Derivatives:** loss of accuracy (~2 digits) with each derivative

(spectral representation)

$$f'(x) \approx \sum_{i=0}^{N} c_i'^{(N)} \phi_i(x_j)$$

(derivative matrices) $\partial_\mu \to \hat{D}_\mu$

$$\vec{f'} = \hat{D}\vec{f}$$

- **Solution algorithm:**

  - Vector $\vec{X}$ stores all unknown of the system
  - Algebraic system $\vec{F}(\vec{X})$ is composed by the partial differential equations, boundary conditions and subsidiary conditions
  - Solution is found via Newton-Raphson scheme

EXAMPLES

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- Identify the physical / mathematical problem of interest
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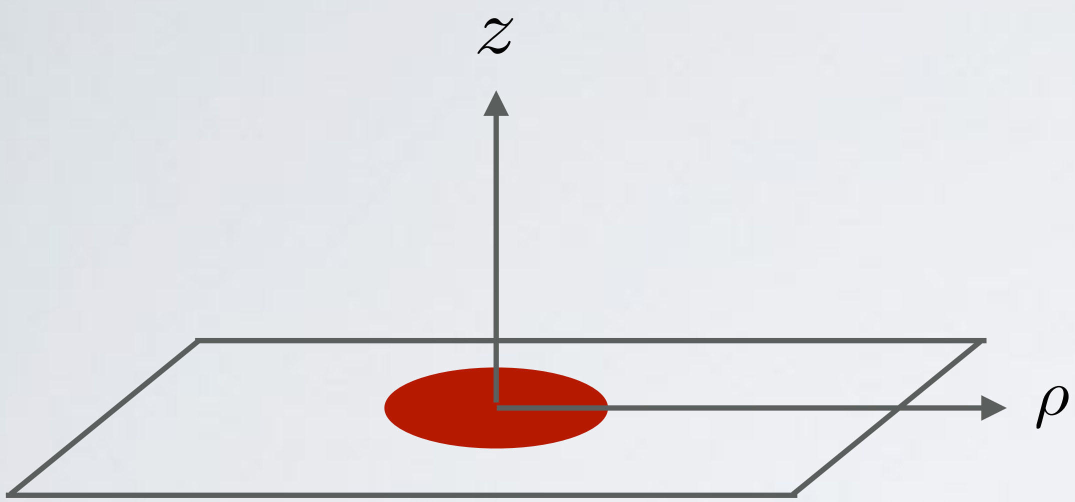
- Identify extra unknowns in the system (if any)
- Derive complementary equations (if any) to close the system
- Adapt the coordinate system to the geometry/causal structure of the problem
- Learn (formal theorems, Taylor expansions, linearisation, trail-and-error…) specific properties of the solution: strong gradients, irregularities….

ELLIPTIC EQUATION

- **Disk of charged dust in General Relativity**

Y.C.Liu, RPM, M. Breithaupt, S. Palenta, R. Meinel. PRD 94 104035 (2016)



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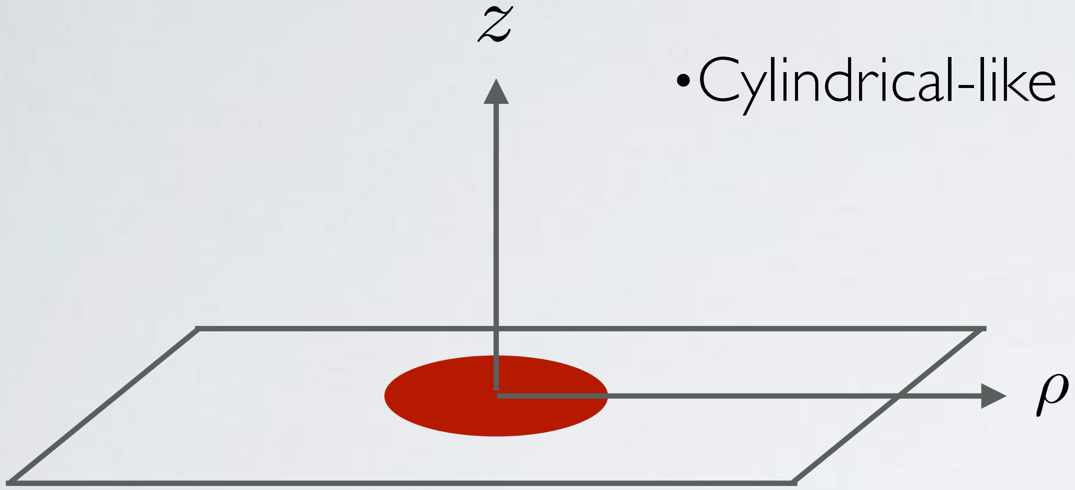

- Cylindrical-like coordinates (Weyl-Lewis-Papapetrou) (t, ρ, ϕ, z)
- Line Element and Maxwell Field

$$ds^2 = \alpha(\rho,z)^2\left[d\rho^2 + dz^2\right] + \frac{\rho^2}{\nu(\rho,z)^2}\left[d\phi - \omega(\rho,z)\,dt\right]^2 - \nu(\rho,z)^2 dt^2$$

$$F_{ab} = \nabla_a A_b - \nabla_b A_a \qquad \mathbf{A} = A_t(\rho,z)\,dt + A_\phi(\rho,z)\,d\phi$$

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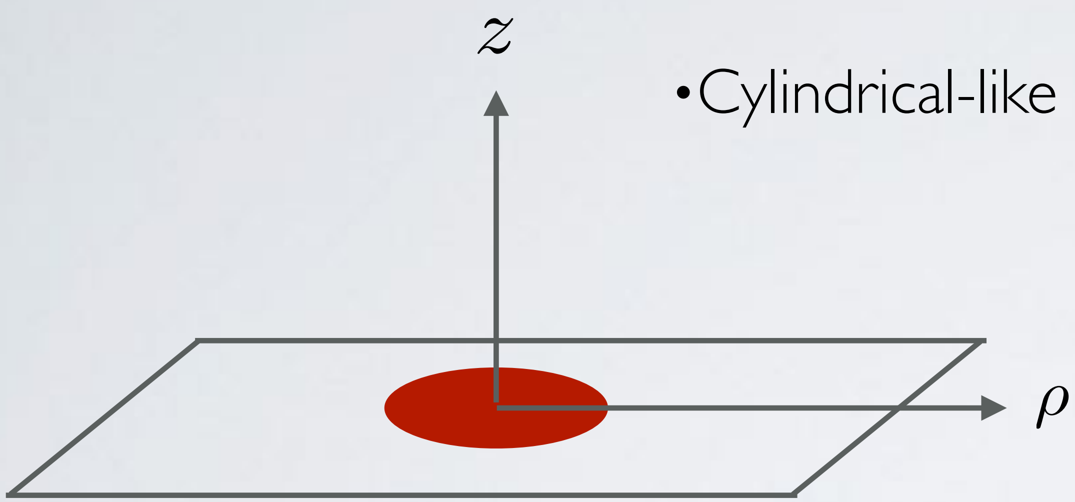

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•Dust properties

Baryonic mass density

$$\mu = \frac{\sigma_{\mathrm{P}}}{\alpha} \delta(z)$$

4-velocity

$$u^a = \frac{1}{\nu\sqrt{1 - V^2}} \left[\delta^a_t + \Omega\, \delta^a_\phi\right], \quad \text{with} \quad V = \frac{\rho}{\nu^2} (\Omega - \omega)$$

4-current density

$$j^a = \varrho_{\mathrm{el}} u^a \quad \text{with} \quad \varrho_{\mathrm{el}} = \epsilon \mu$$

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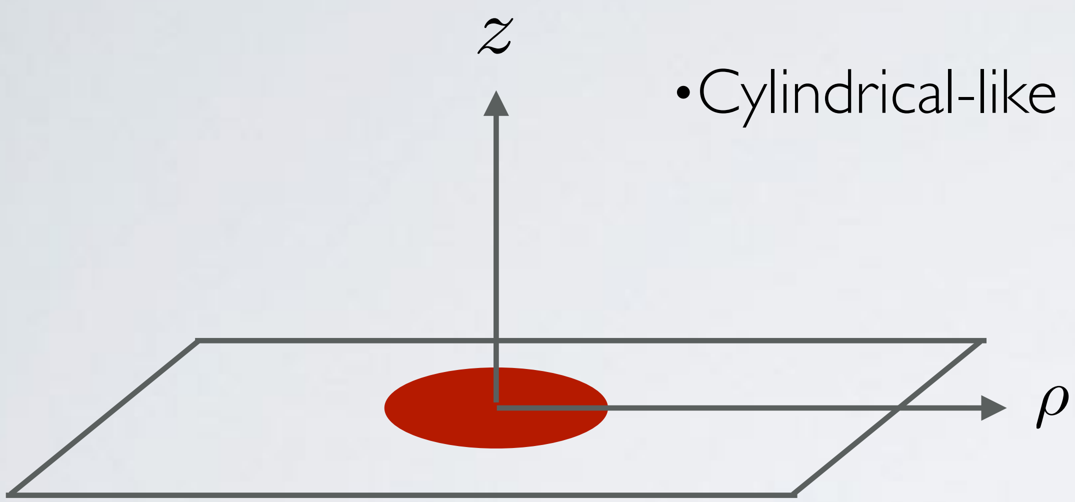

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•Energy-Momentum Tensor $T_{ab} = T^{\mathrm{dust}}_{ab} + T^{\mathrm{EM}}_{ab}$

$$T^{\mathrm{EM}}_{ab} = \frac{1}{4\pi}\left(F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}\right) \qquad T^{\mathrm{dust}}_{ab} = \mu u_a u_b$$

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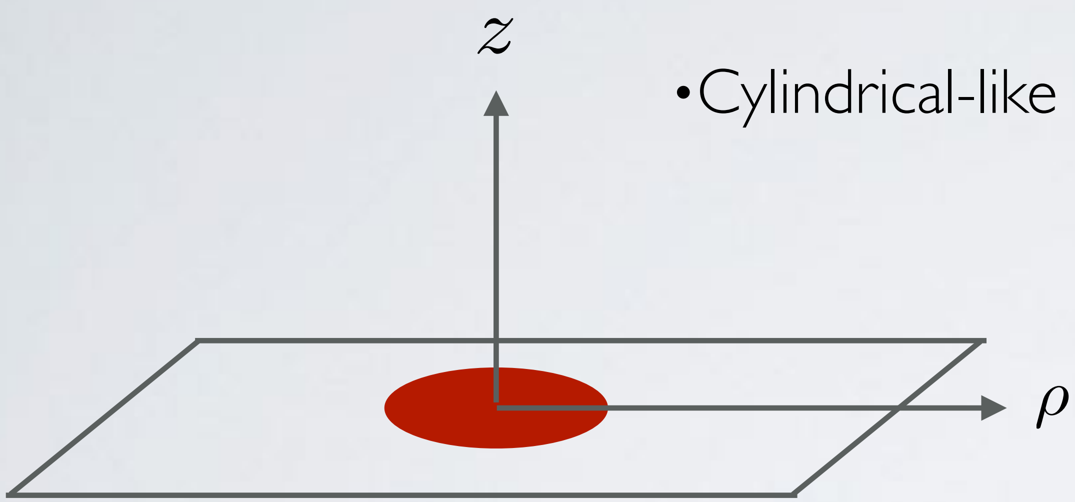

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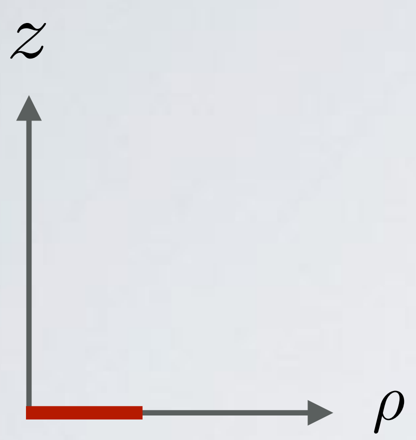
•Equations $R_{ab}=8\pi\left[T_{ab}-\frac{1}{2}g_{ab}T\right]\quad\nabla_bT^{ab}=0$

$$\nabla_bF^{ab}=4\pi j^a$$

ELLIPTIC EQUATION

- **Variables of the system:** $\nu(\rho, z)$ $\omega(\rho, z)$ $A_\phi(\rho, z)$ $A_t(\rho, z)$ Ω



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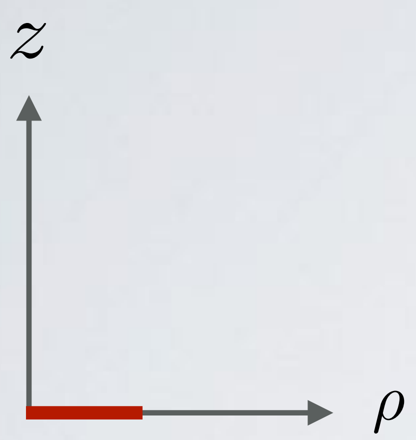

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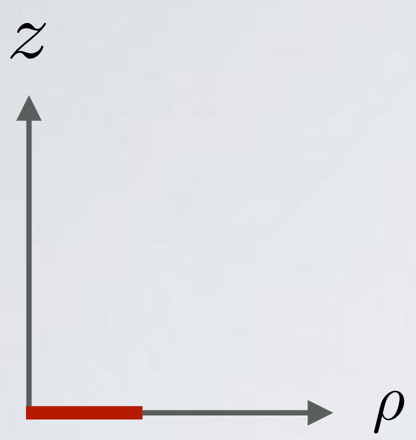

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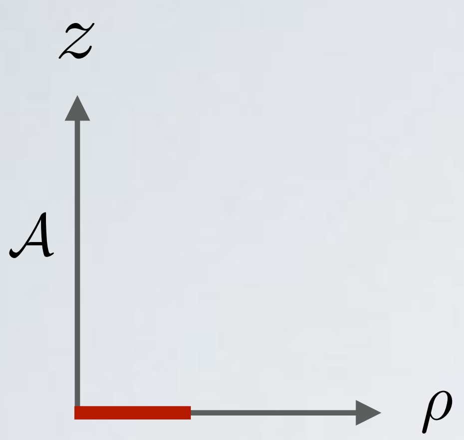

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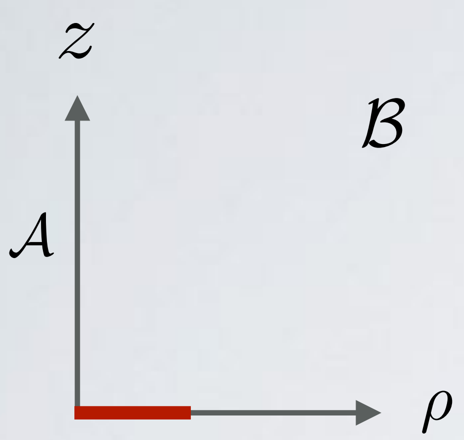

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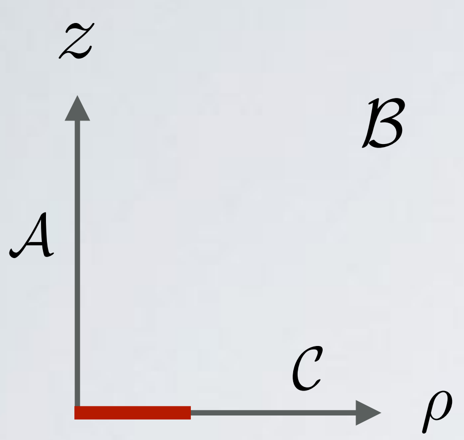

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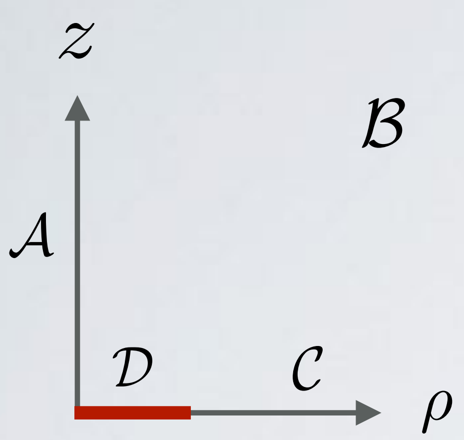

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Lorentz force $\nabla_b T^{ab} = 0$

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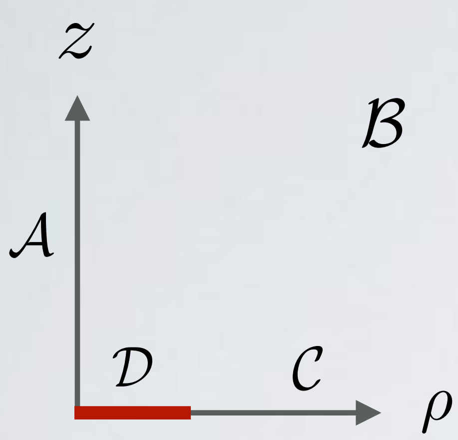

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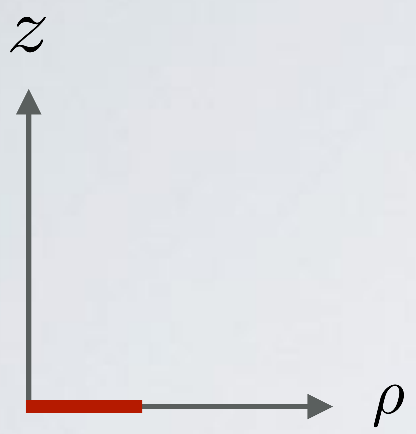
- **Parameter Space:**

$\gamma = 1 - \nu_{\rm c} \in (0, 1]$ Relativistic parameter

$\epsilon \in [0, 1]$ Charge density

ELLIPTIC EQUATION

- **Adapted coordinates:** map into elliptic coordinates:



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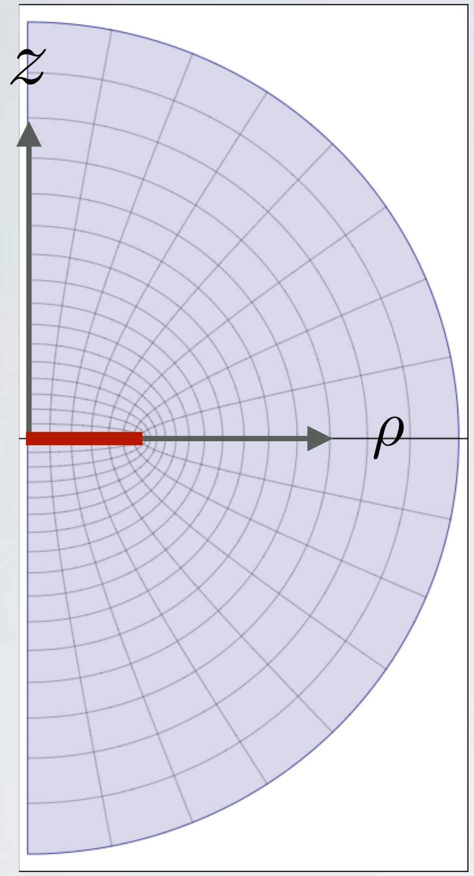

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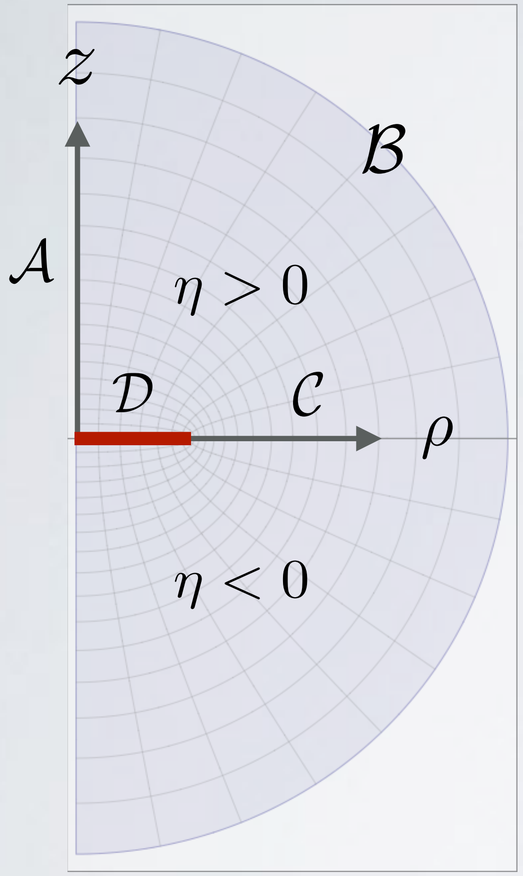

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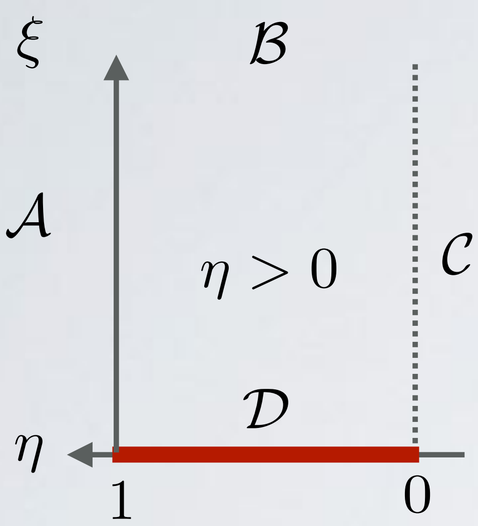

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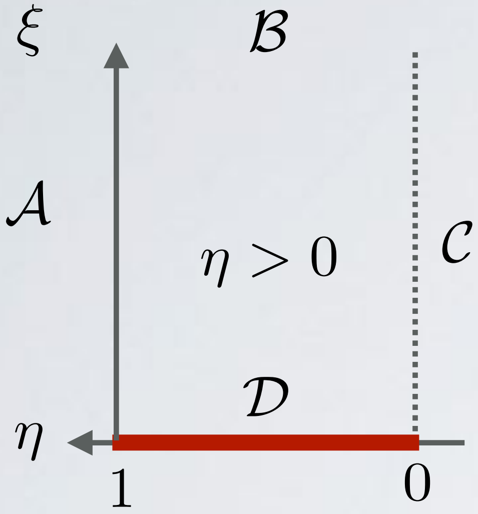

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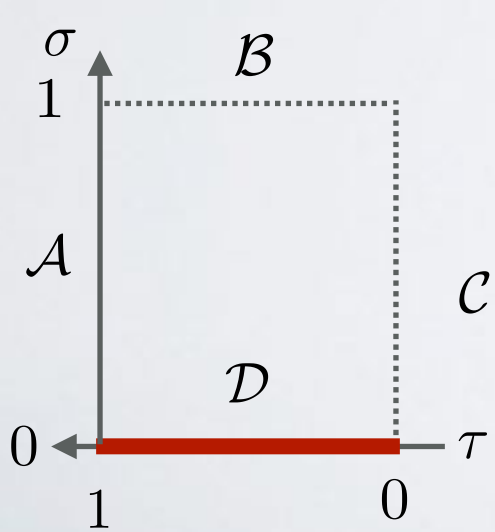
- **Spectral Coordinates:**

Compactify infinity and restrict to upper half plane

$$\sigma = \frac{2}{\pi} \arctan \xi \qquad \tau = \eta^2$$

$$\{\sigma, \tau\} \in [0, 1]^2$$



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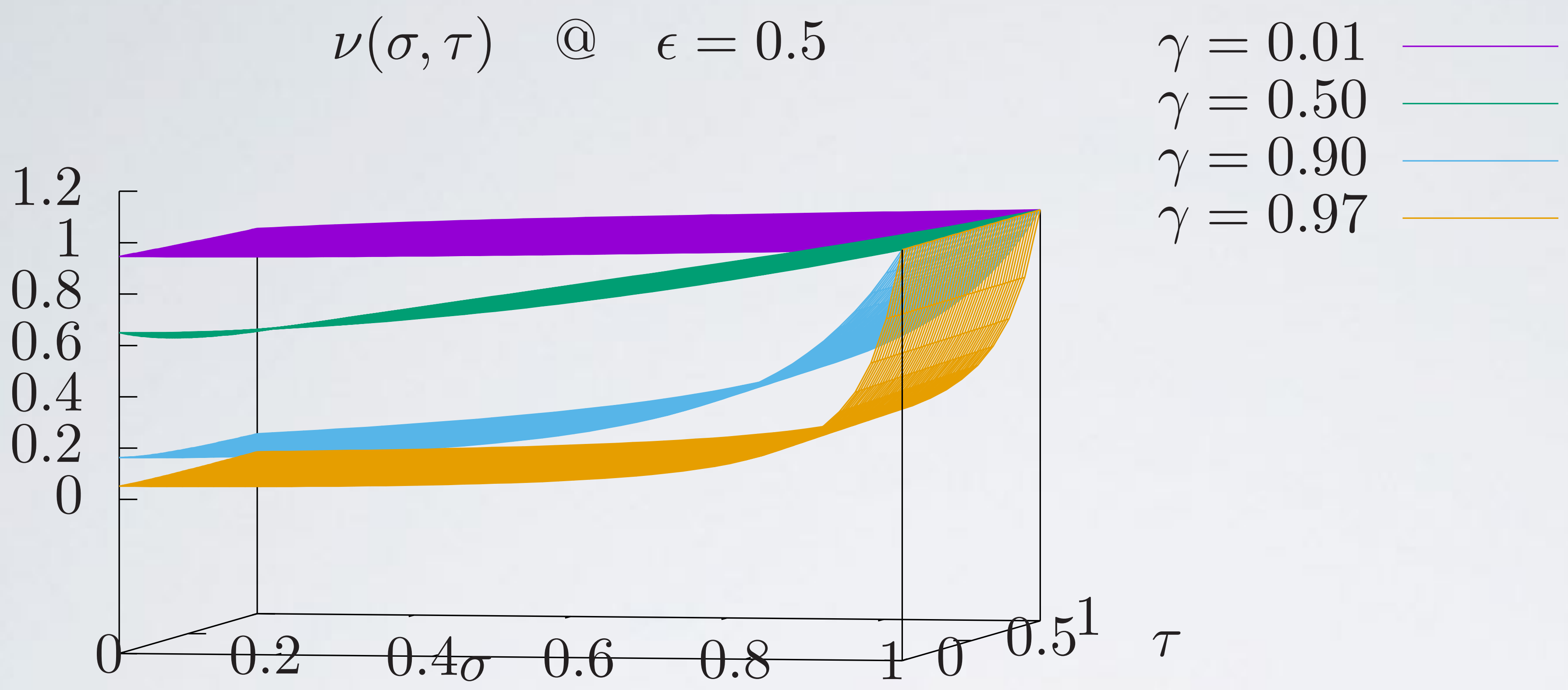
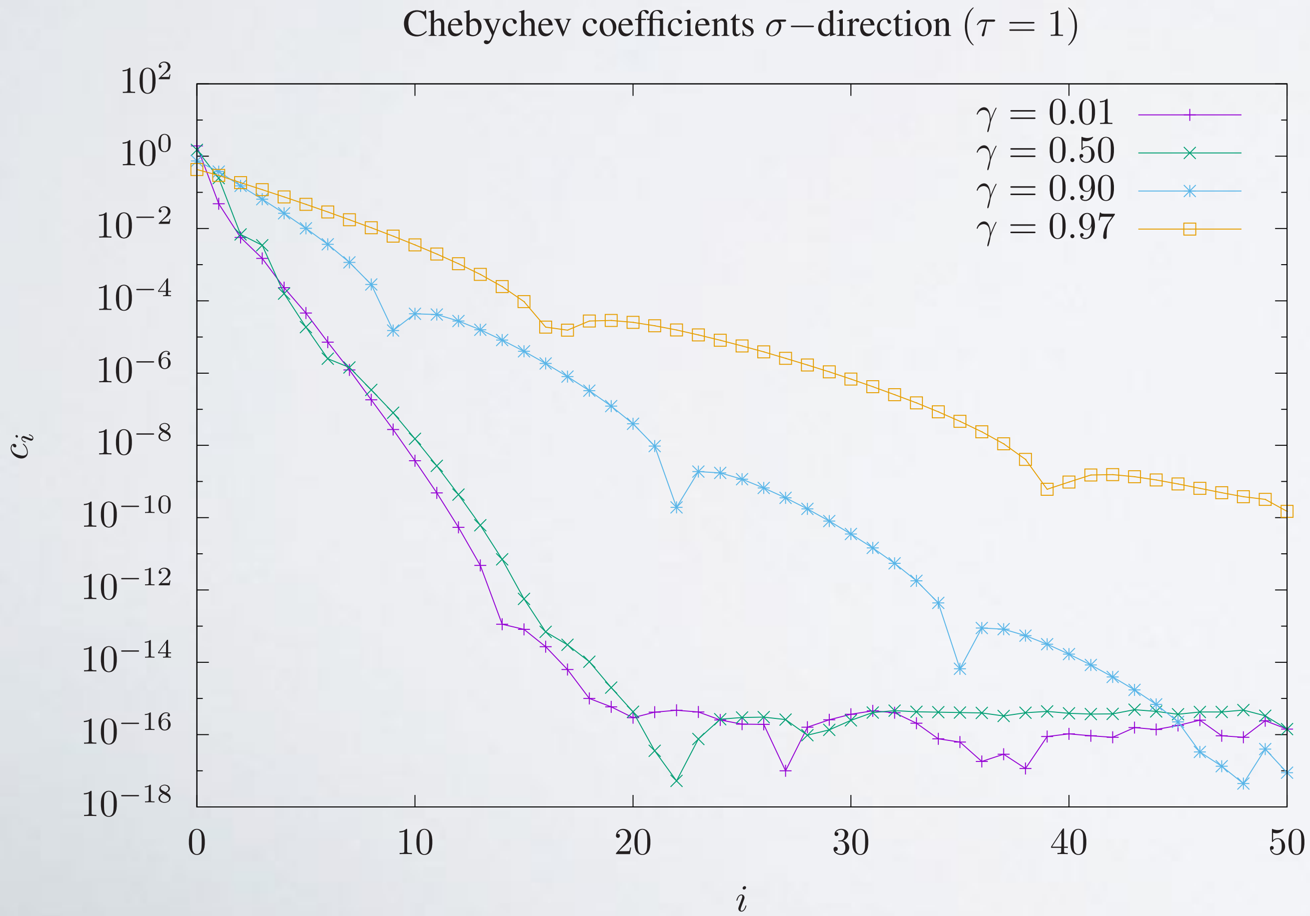
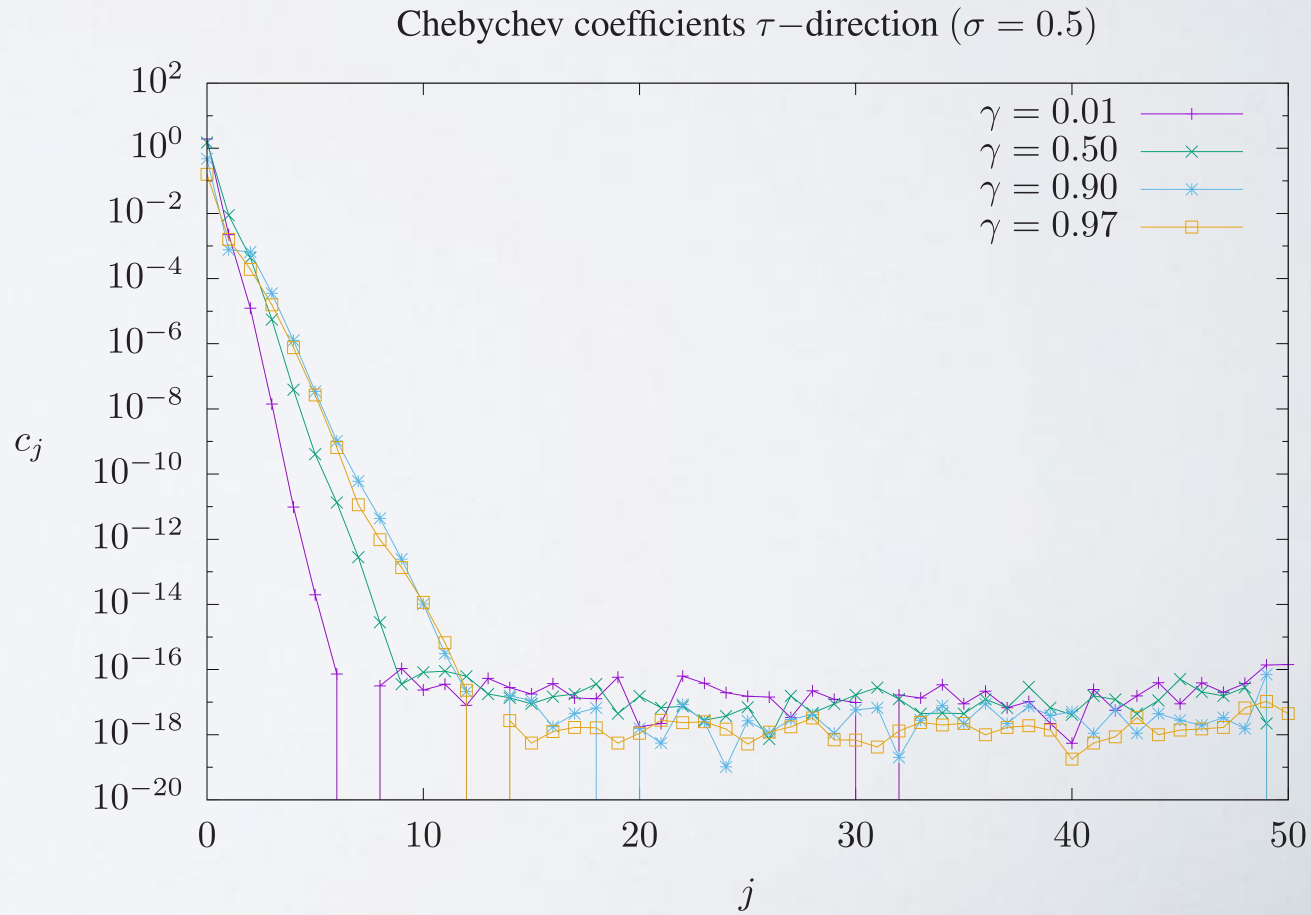


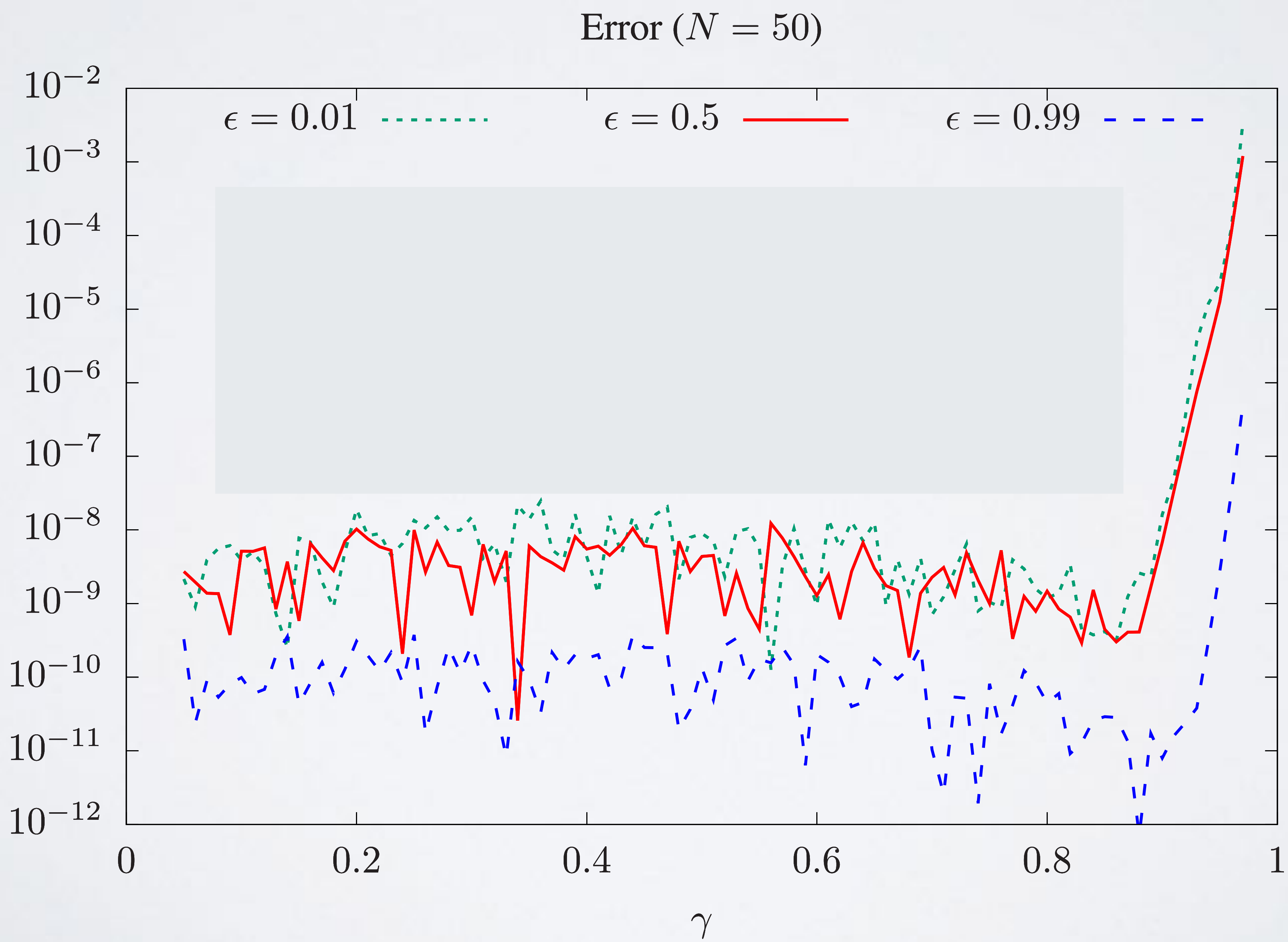




ELLIPTIC EQUATIONS REMARKS

Error: Physical parameters are related via $M = 2\Omega J + \left[\frac{1-\gamma}{\epsilon} - A_t^{\mathrm{c}}\right] Q,$

$$\mathrm{Error} = \left| 1 - \left[\frac{2\Omega J}{M} + \left(\frac{1-\gamma}{\epsilon} - A_t^{\mathrm{c}} \right) \frac{Q}{M} \right] \right|$$



ELLIPTIC EQUATIONS REMARKS

In general:

- Spectral methods are well stablished for elliptic equations in many communities
- Introduction of adapted coordinates to map the coordinates into a squared domain
- Treatment of unknown functions and unknown parameters in equal stage

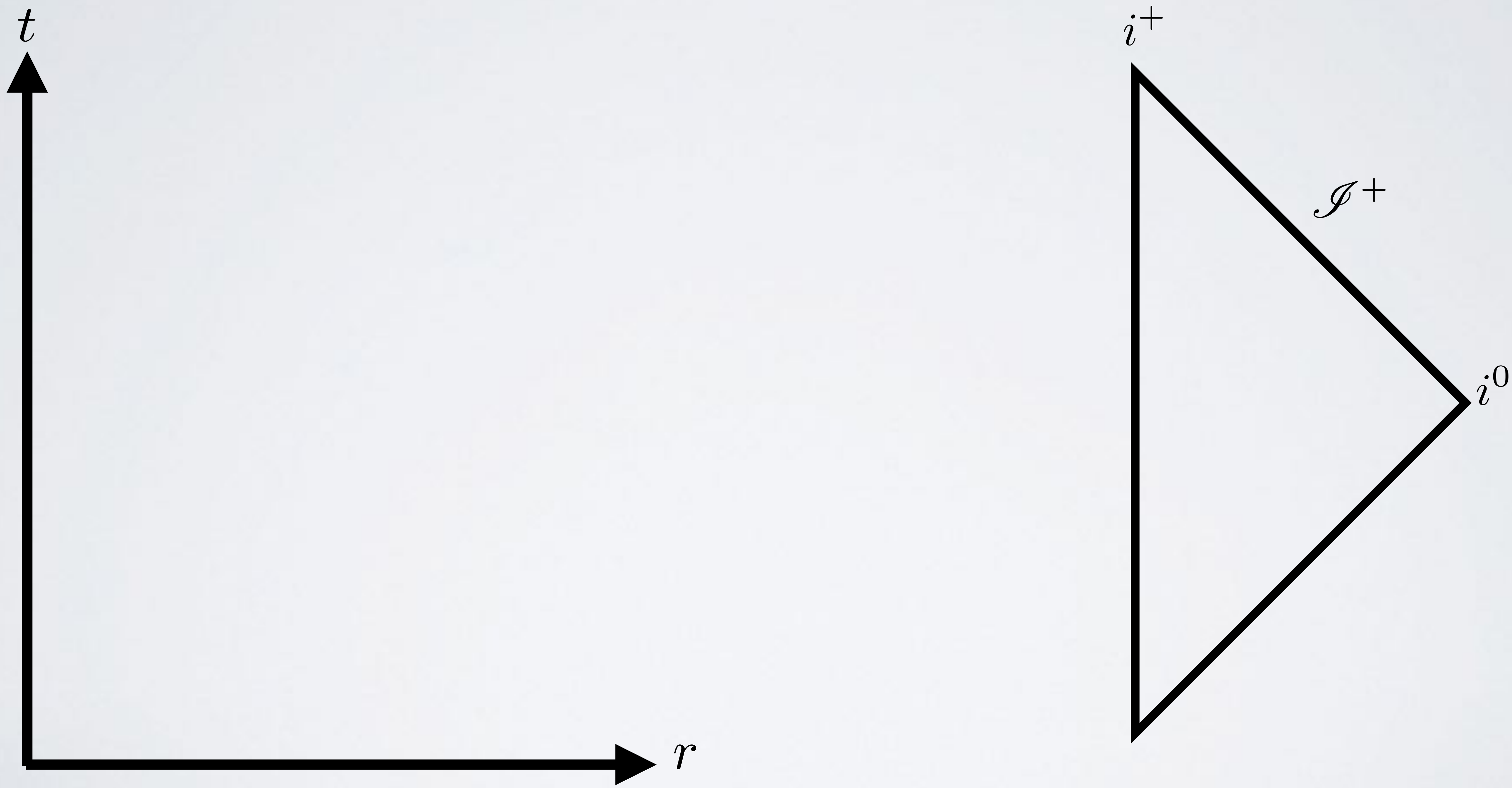
(Treatment of a priori unknown boundaries - example surface of a star)

Here:

- Solution is analytic, exponential decay of Chebyshev coefficients
- Regions in the parameter space with strong gradient (careful treatment)

HYPERBOLIC EQUATION

- **Conformal compactification:** bring infinities of the spacetime into finite regions, while preserving causal structure



Minkowski Spacetime

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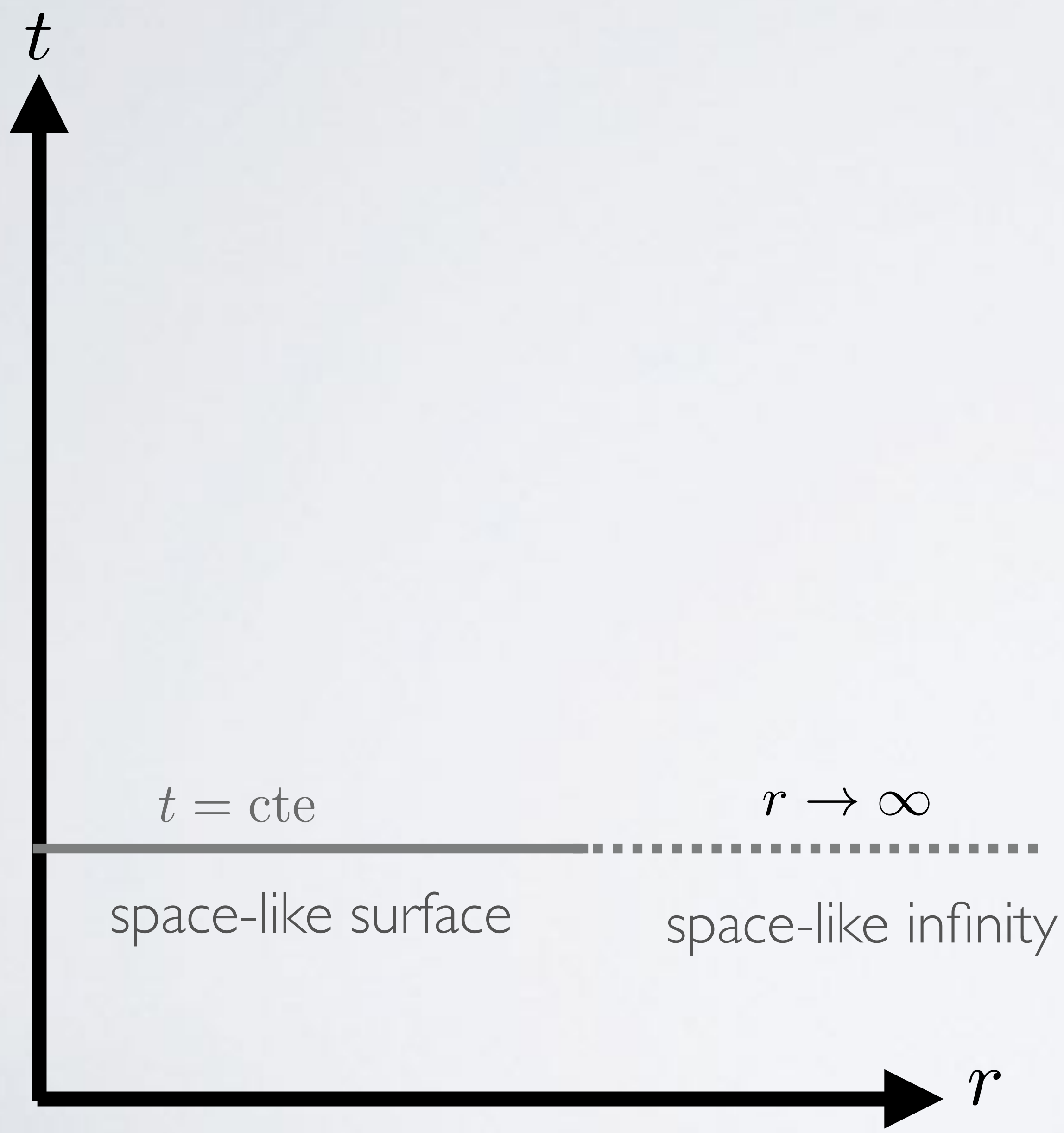

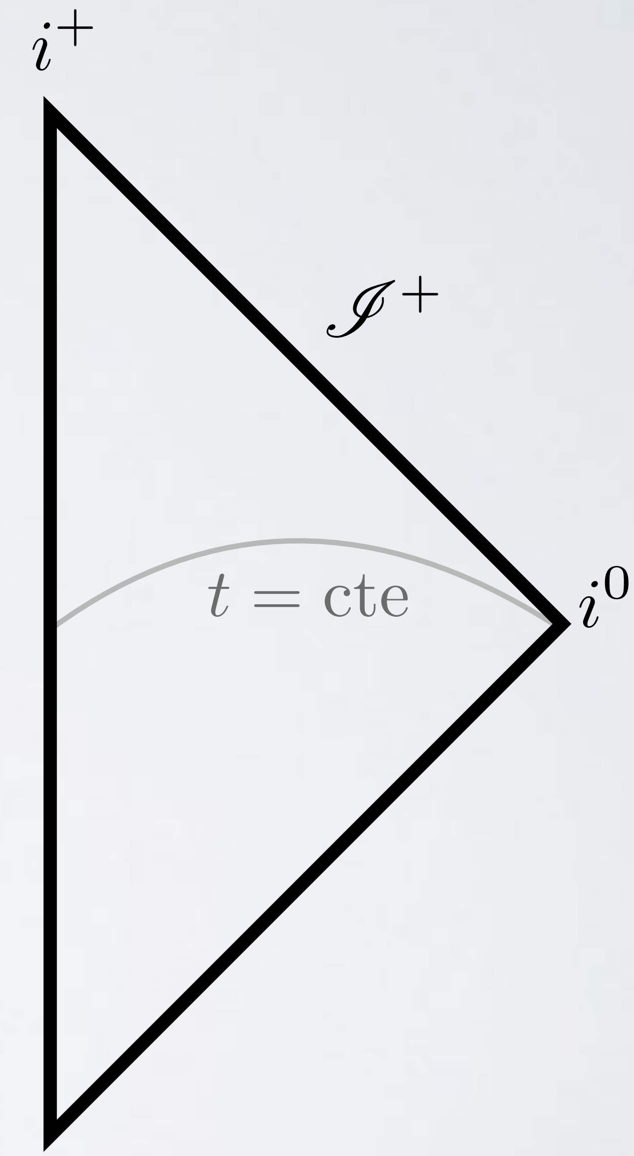

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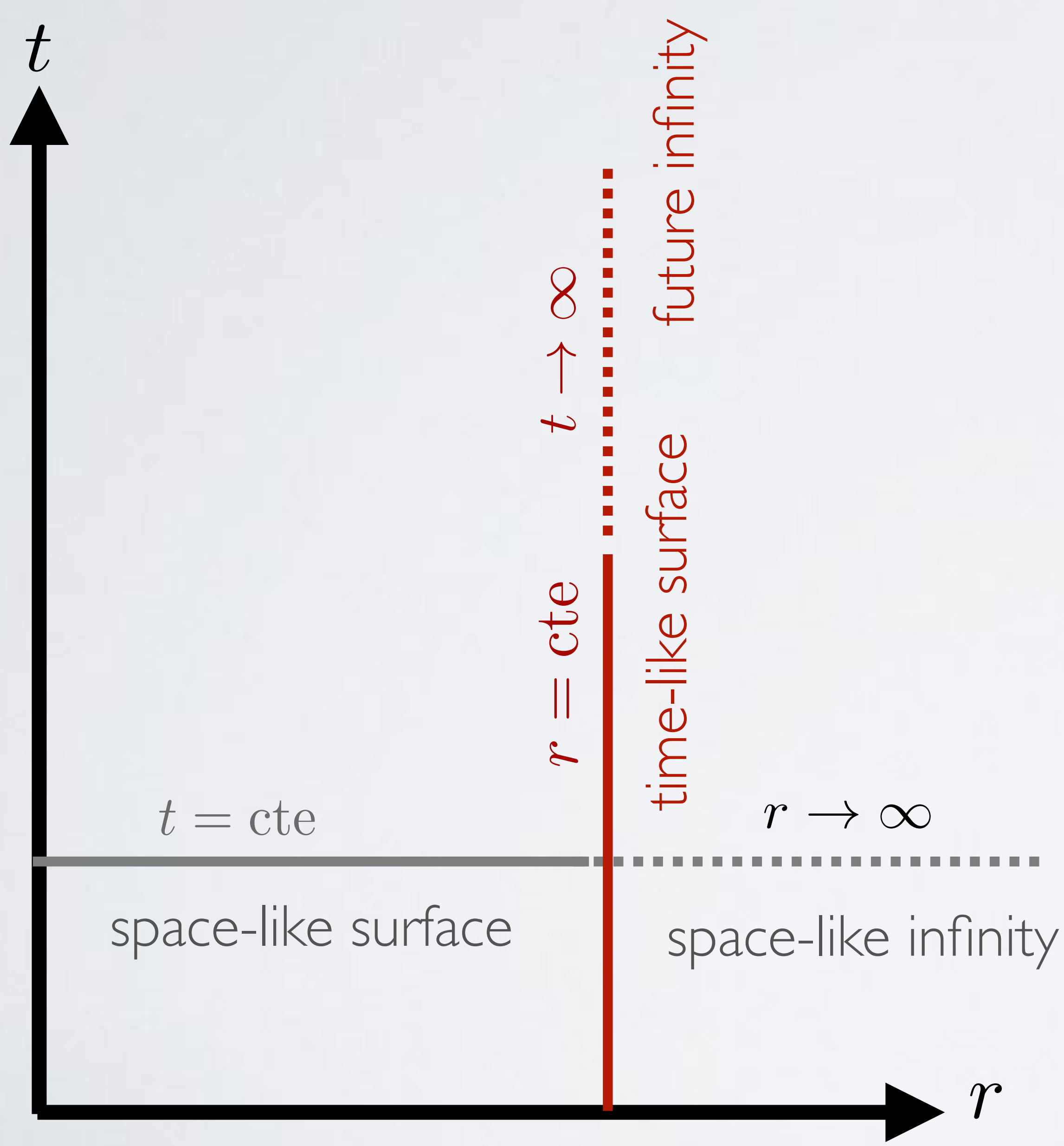

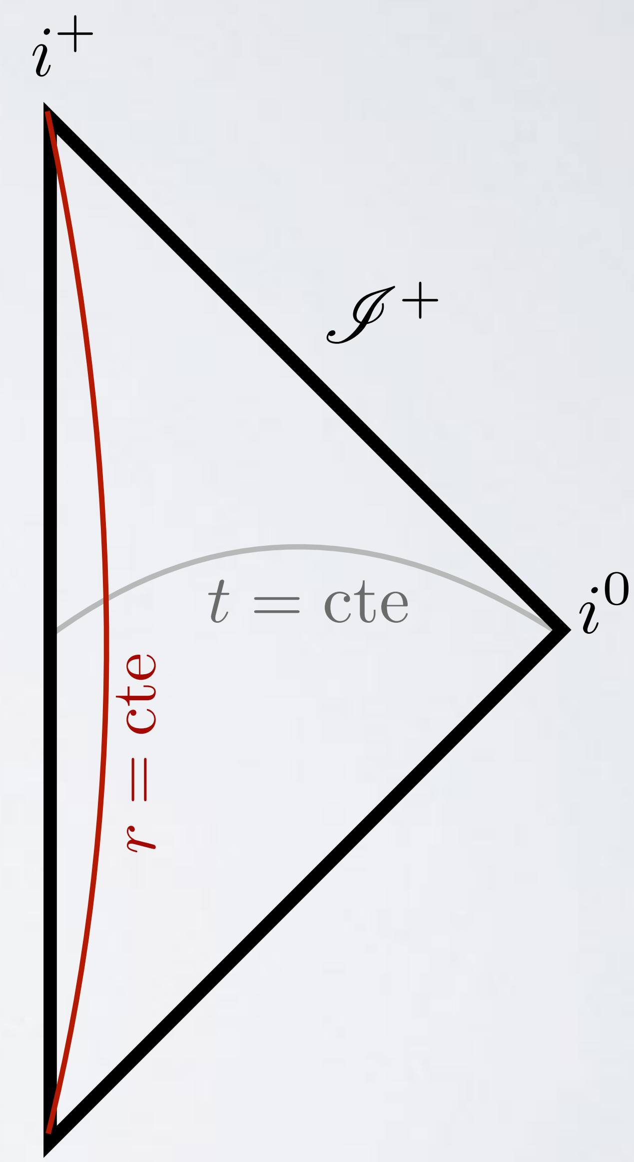

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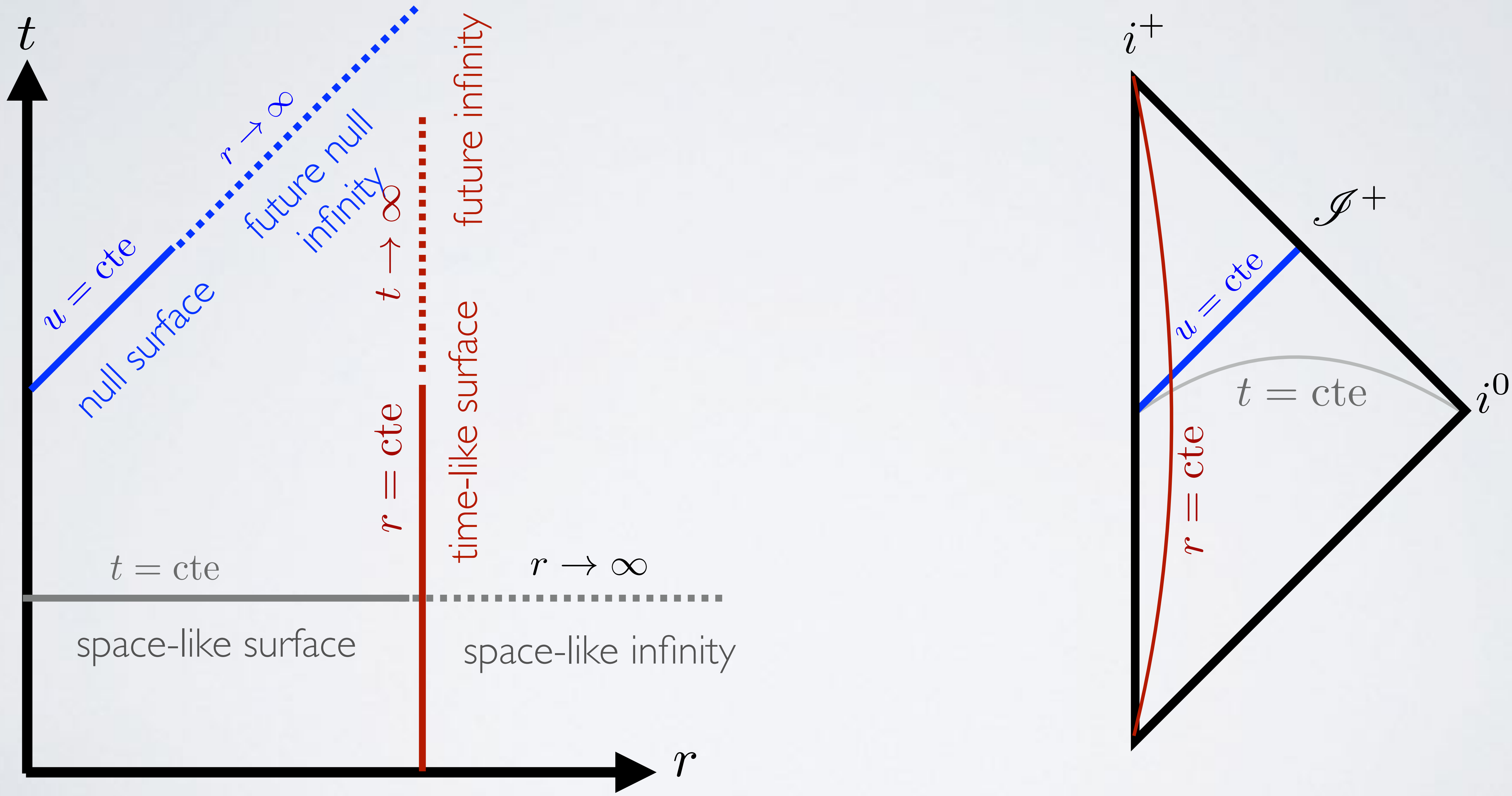

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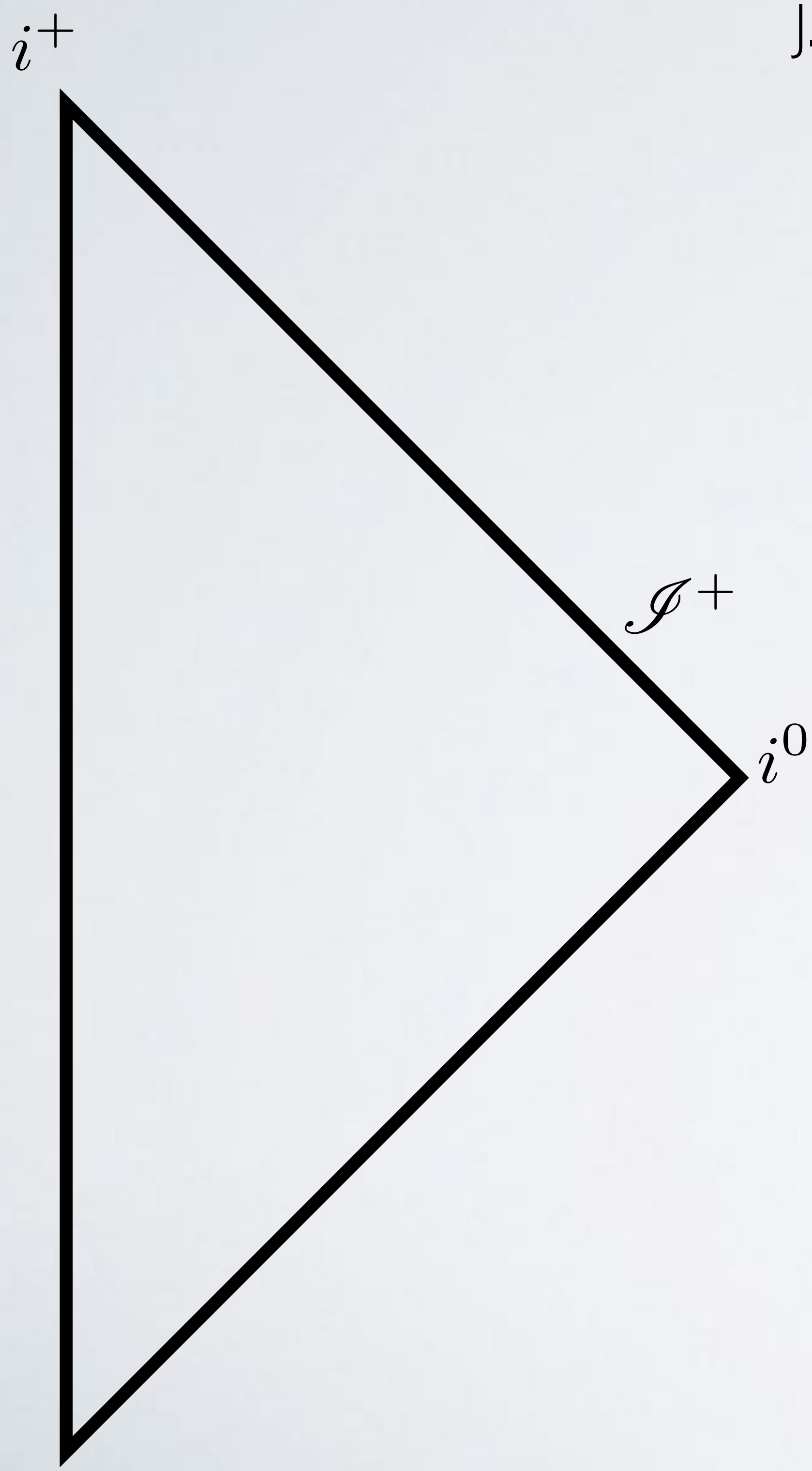
J. Valiente, RPM. In preparation (2017)



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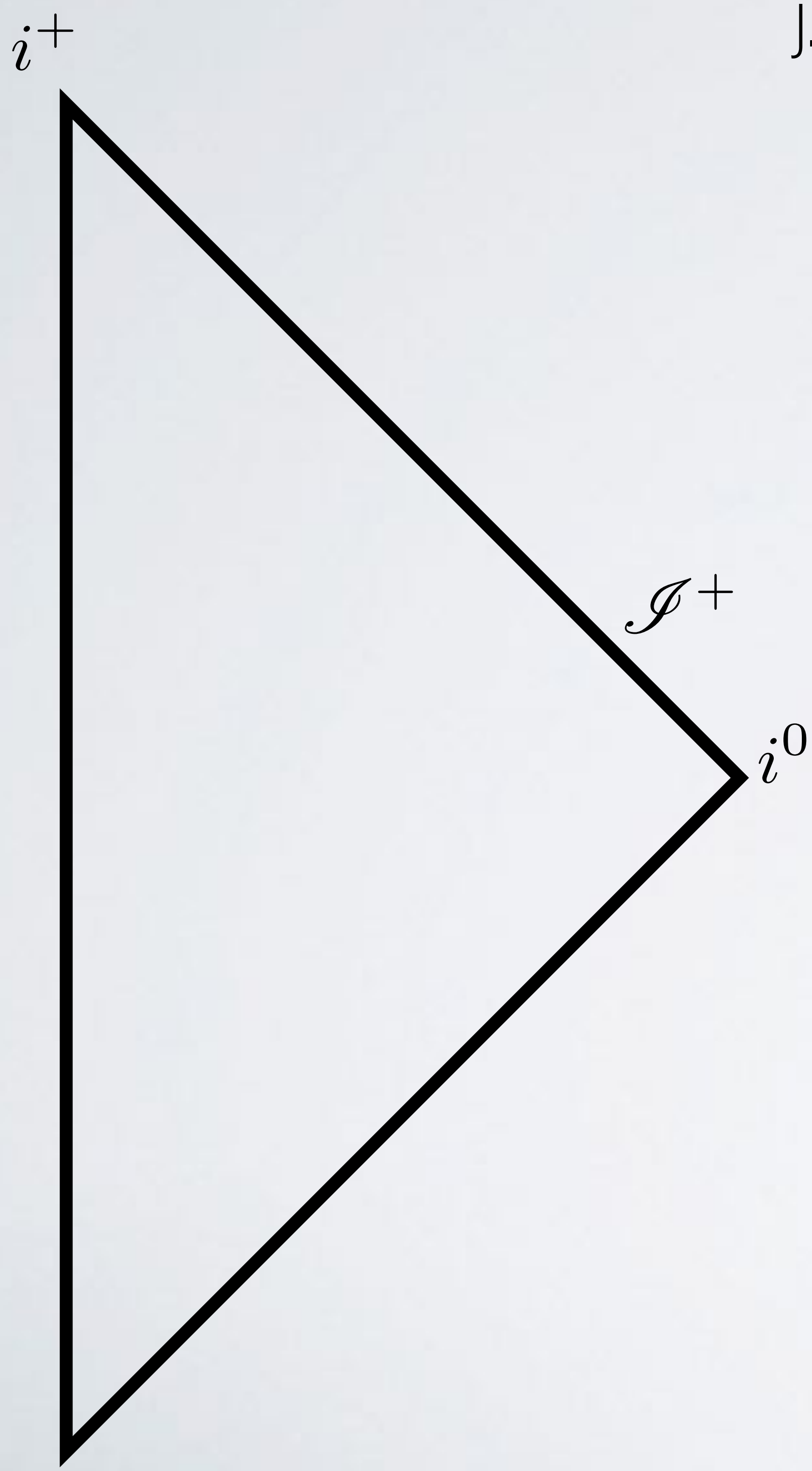

Gravitational degrees of freedom: components of the Riemann Tensor (Newman-Penrose scalars)

ϕ_0 ϕ_4 ingoing/outgoing gravitational waves

ϕ_2 "Coulomb" field

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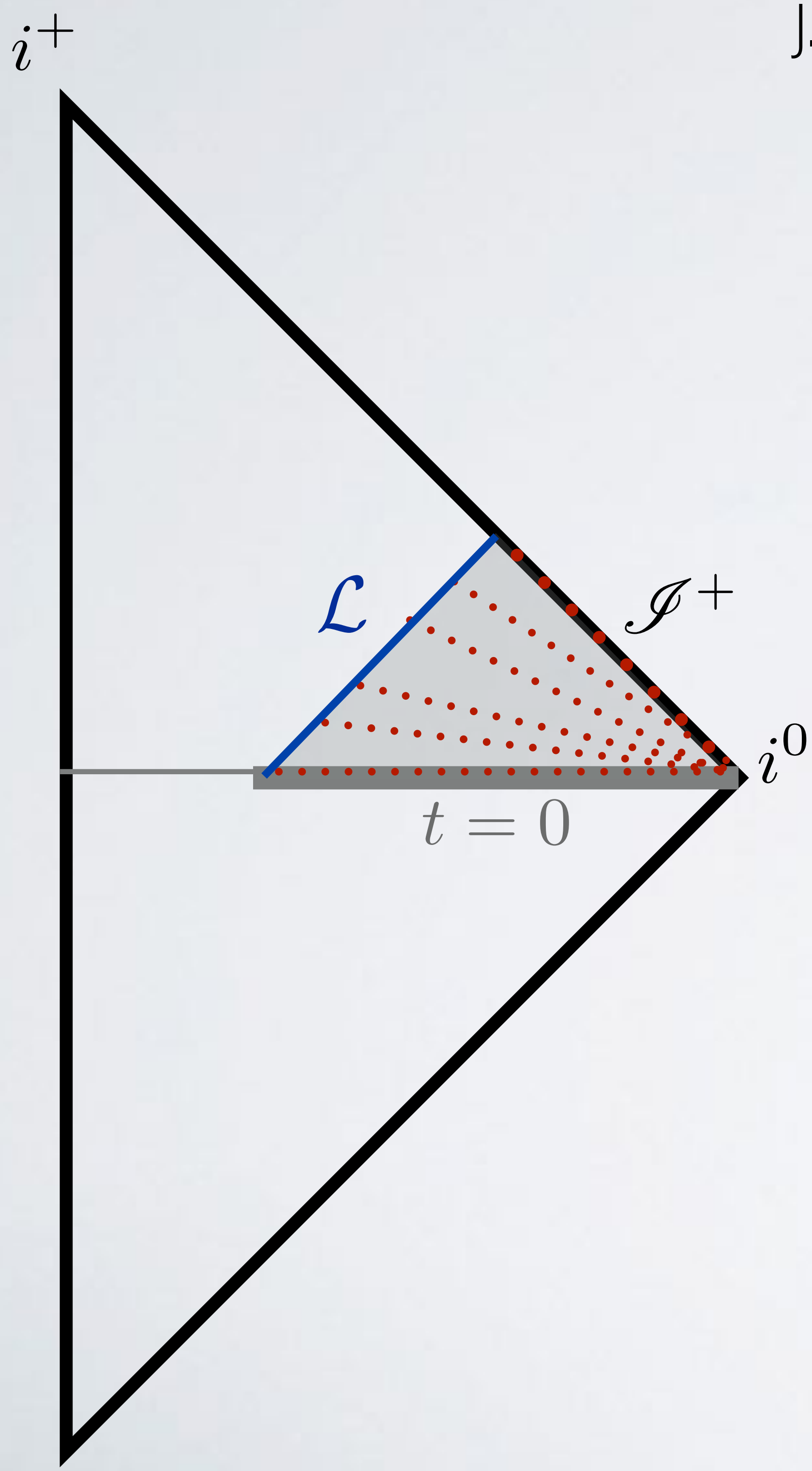

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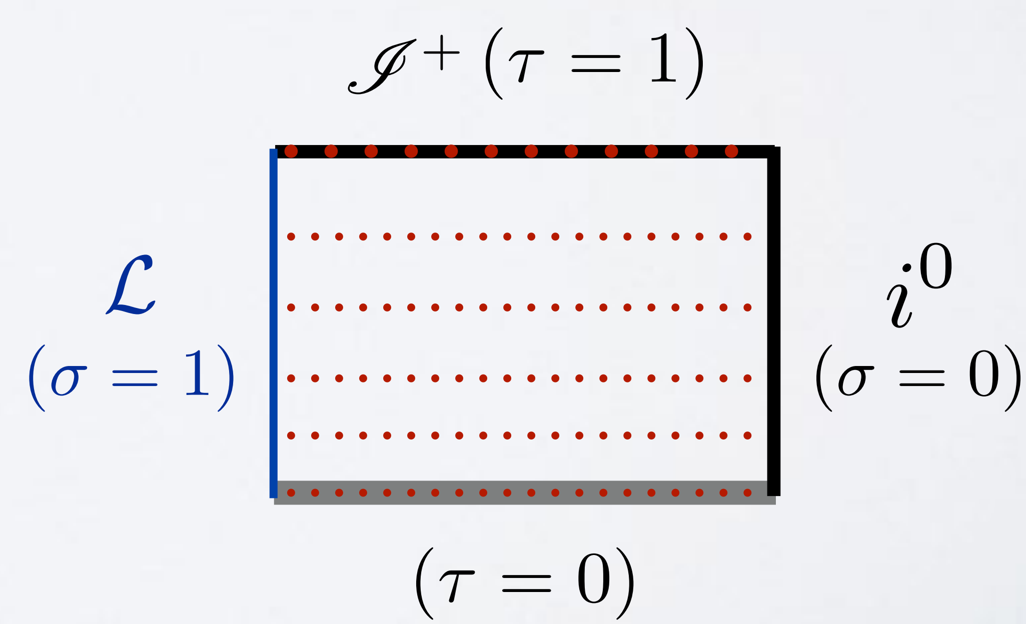
ϕ_3 ϕ_1 gauge freedom

Near station infinity: introduce coordinates (τ, σ) that "inflates" the point i^0 into a surface



HYPERBOLIC EQUATION

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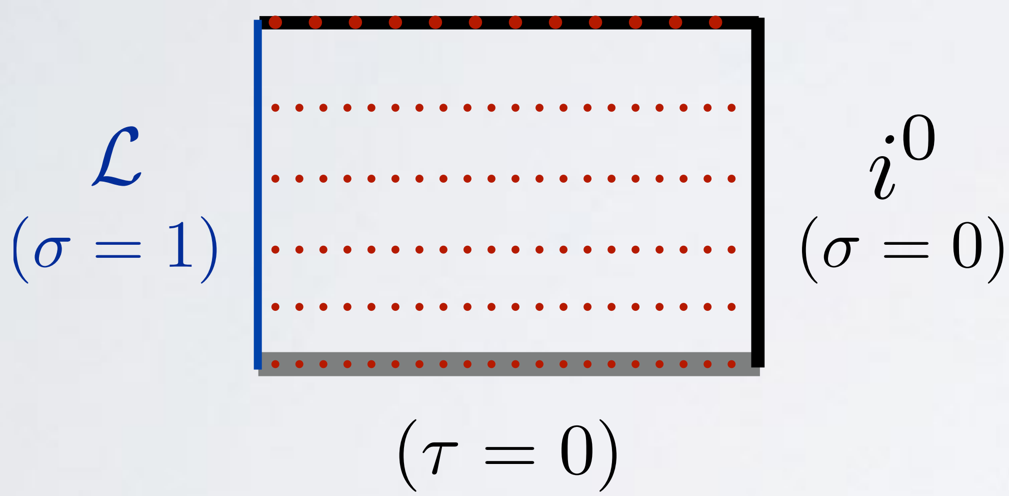

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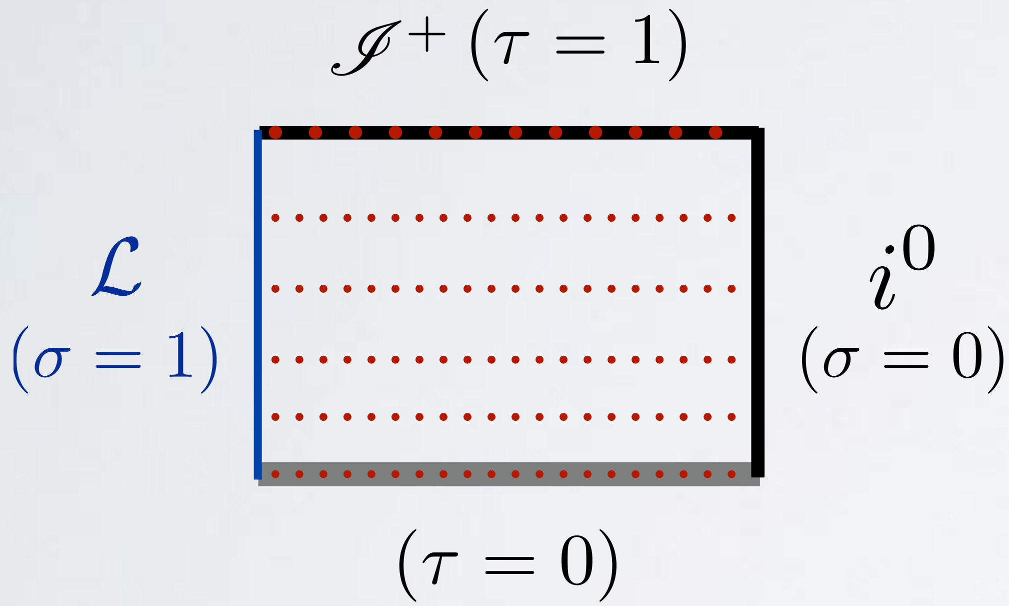

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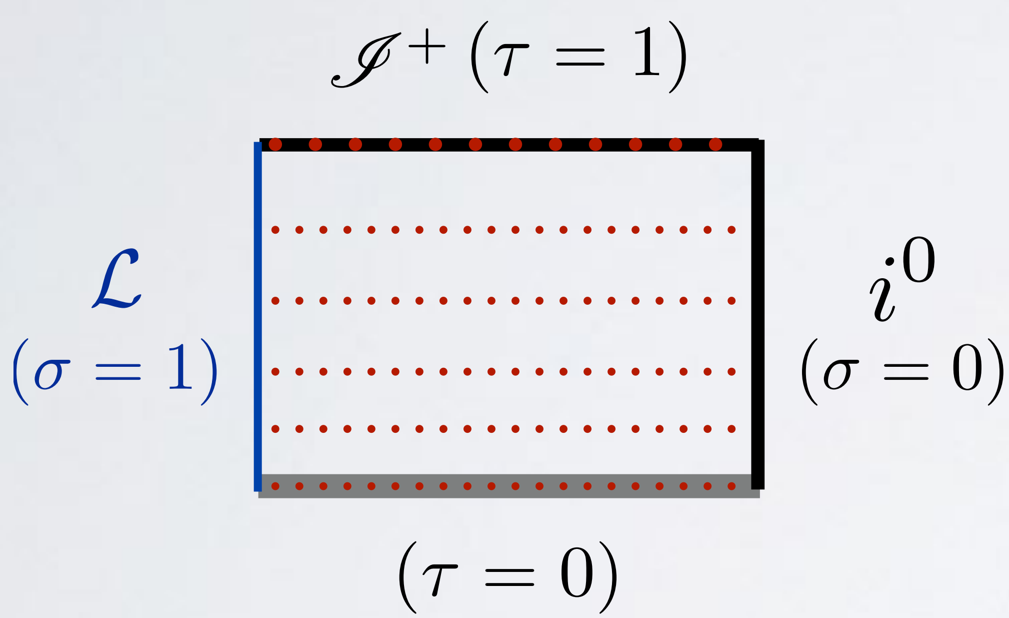

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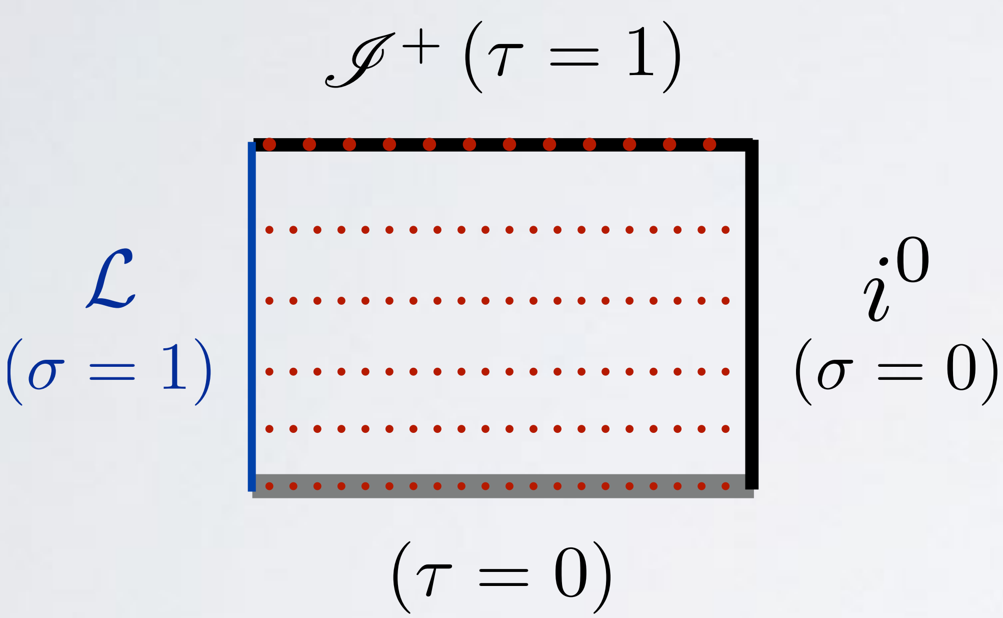

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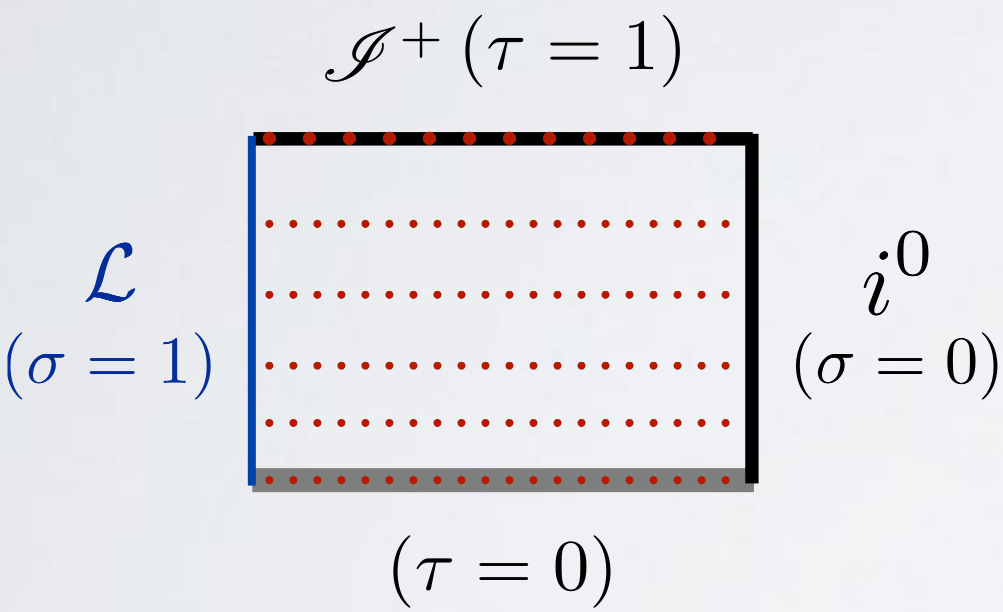

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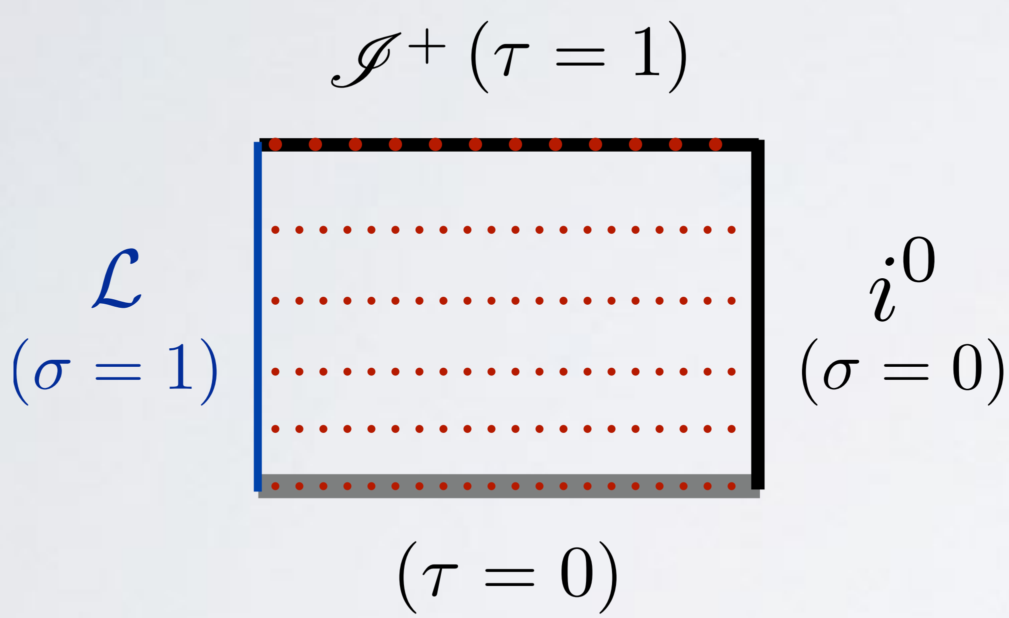

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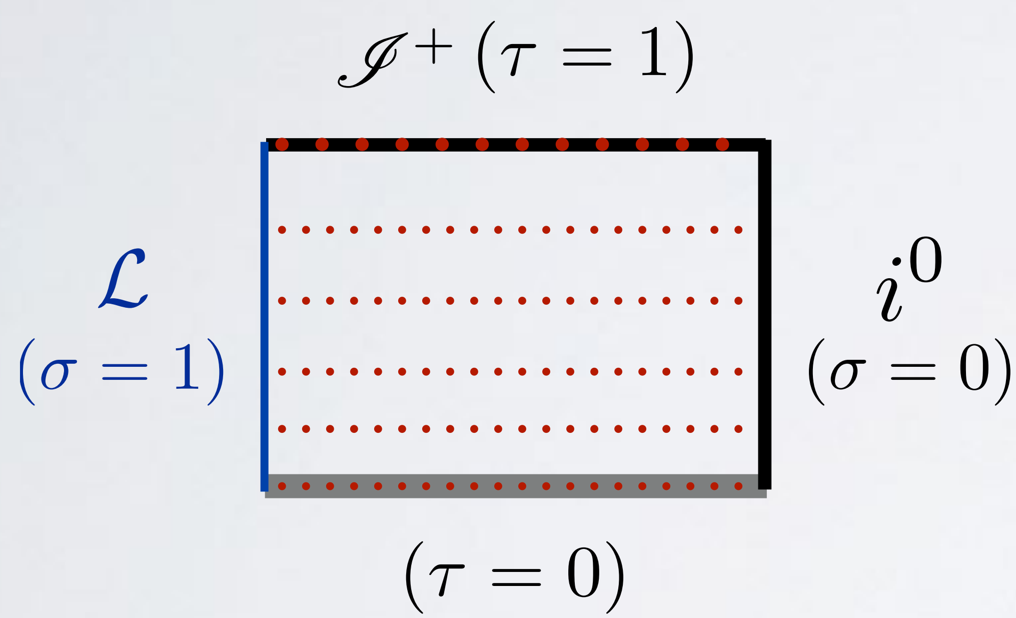

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In fact: study of expansion around $\tau = 1$ reveals

$$\phi_0 \sim K \ln(1-\tau)$$

HYPERBOLIC EQUATION

Solution: introduce auxiliary variables that remove leading order singular terms

$$C_0(\tau,\sigma) = \phi_0(\tau,\sigma) - K\left[1 - \frac{\lambda_1^2}{2}(1-\tau)\right]\ln(1-\tau)$$

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$(*)|_{\tau=1,\sigma=0}$ $$\lambda_1 C_1 = -K \qquad (\tau = 1, \sigma = 0)$$

(subsidiary equation)

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$$\partial_\tau(*) \qquad \partial_\tau C_0 - \sigma\partial^2_{\tau\sigma}C_0 + \frac{\sigma}{2}\partial_\sigma C_0 + \lambda_1\partial_\tau C_1 = -K \qquad (\tau = 1)$$

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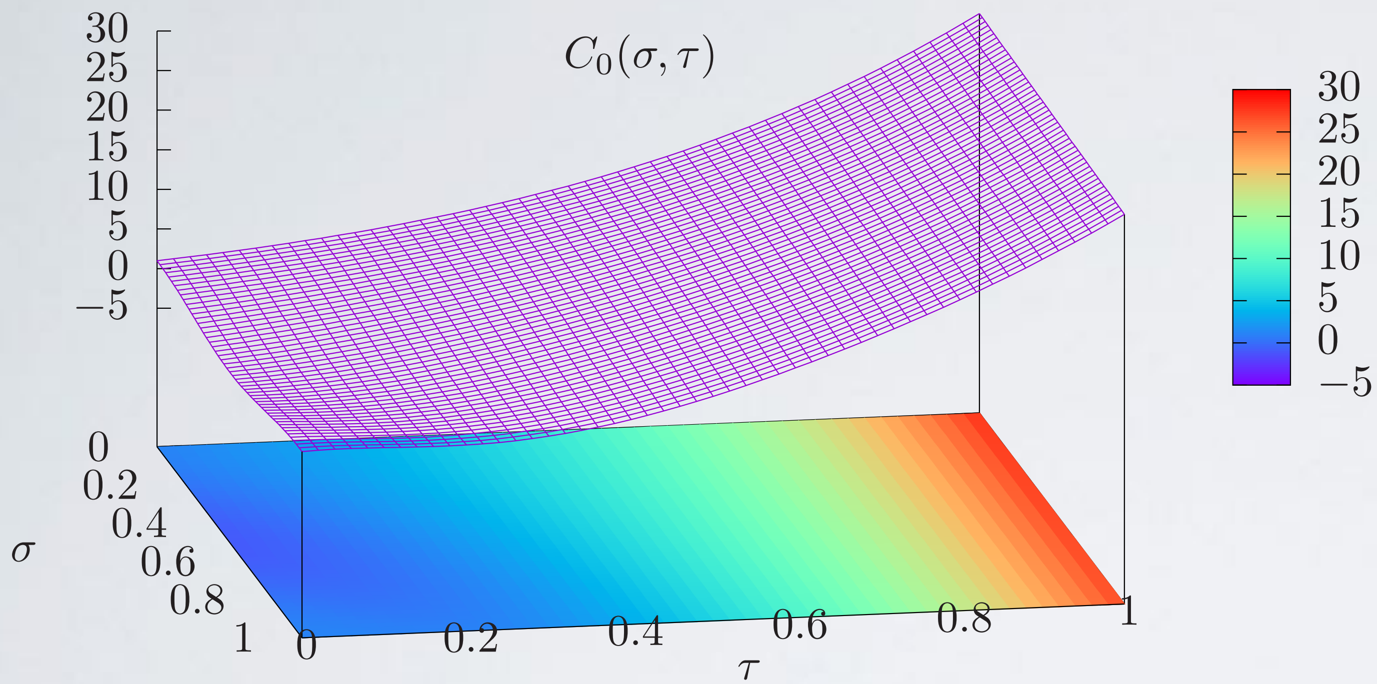

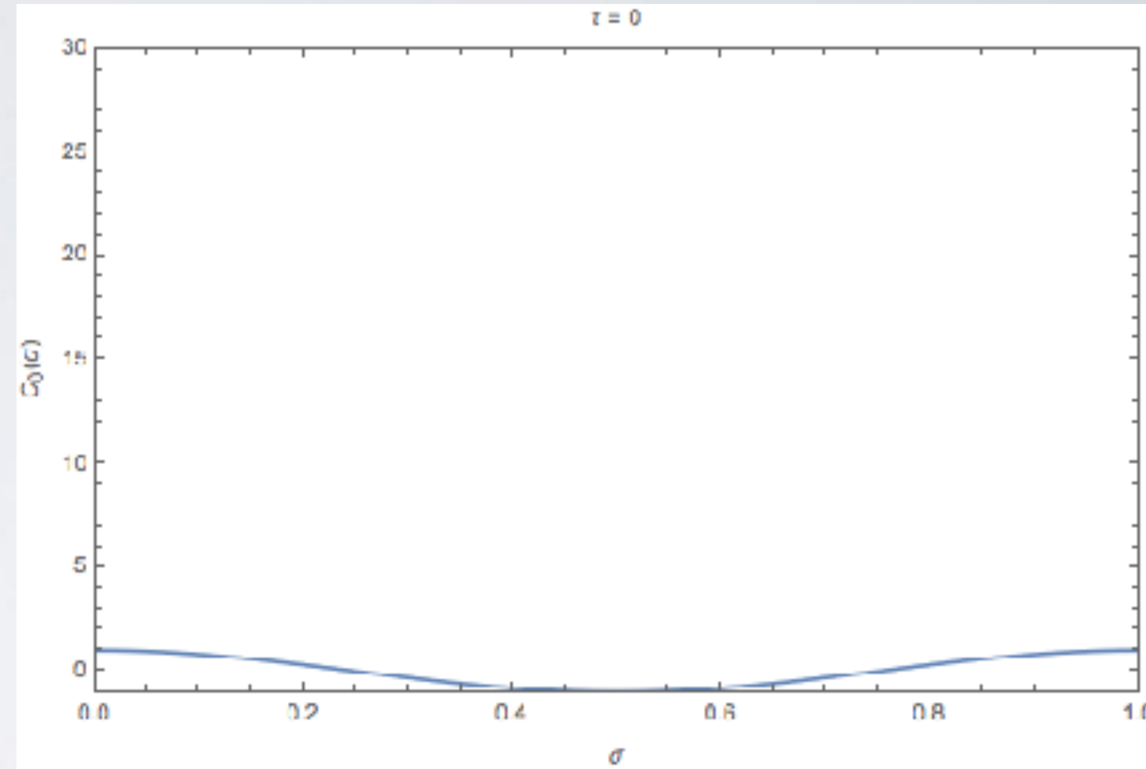

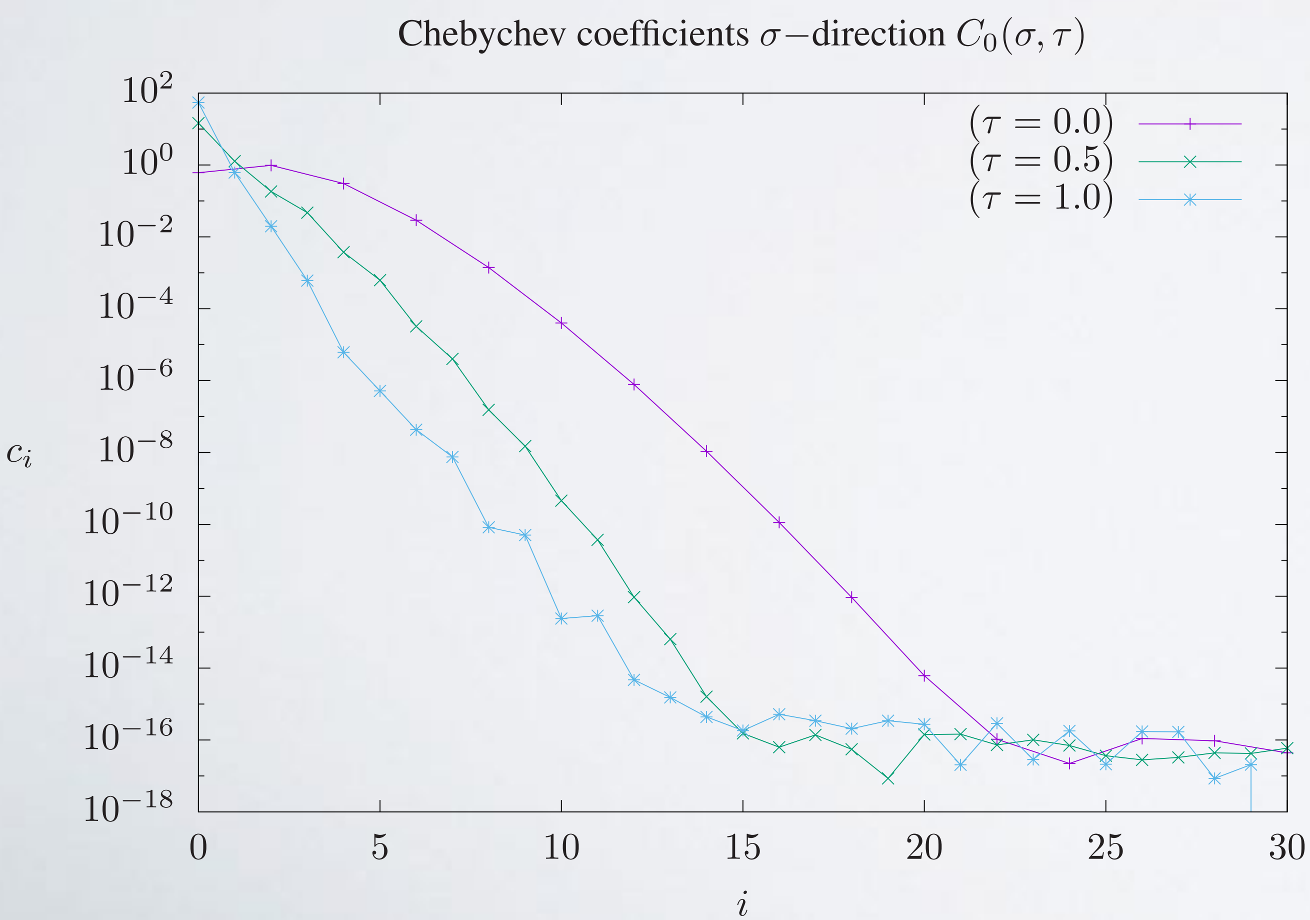

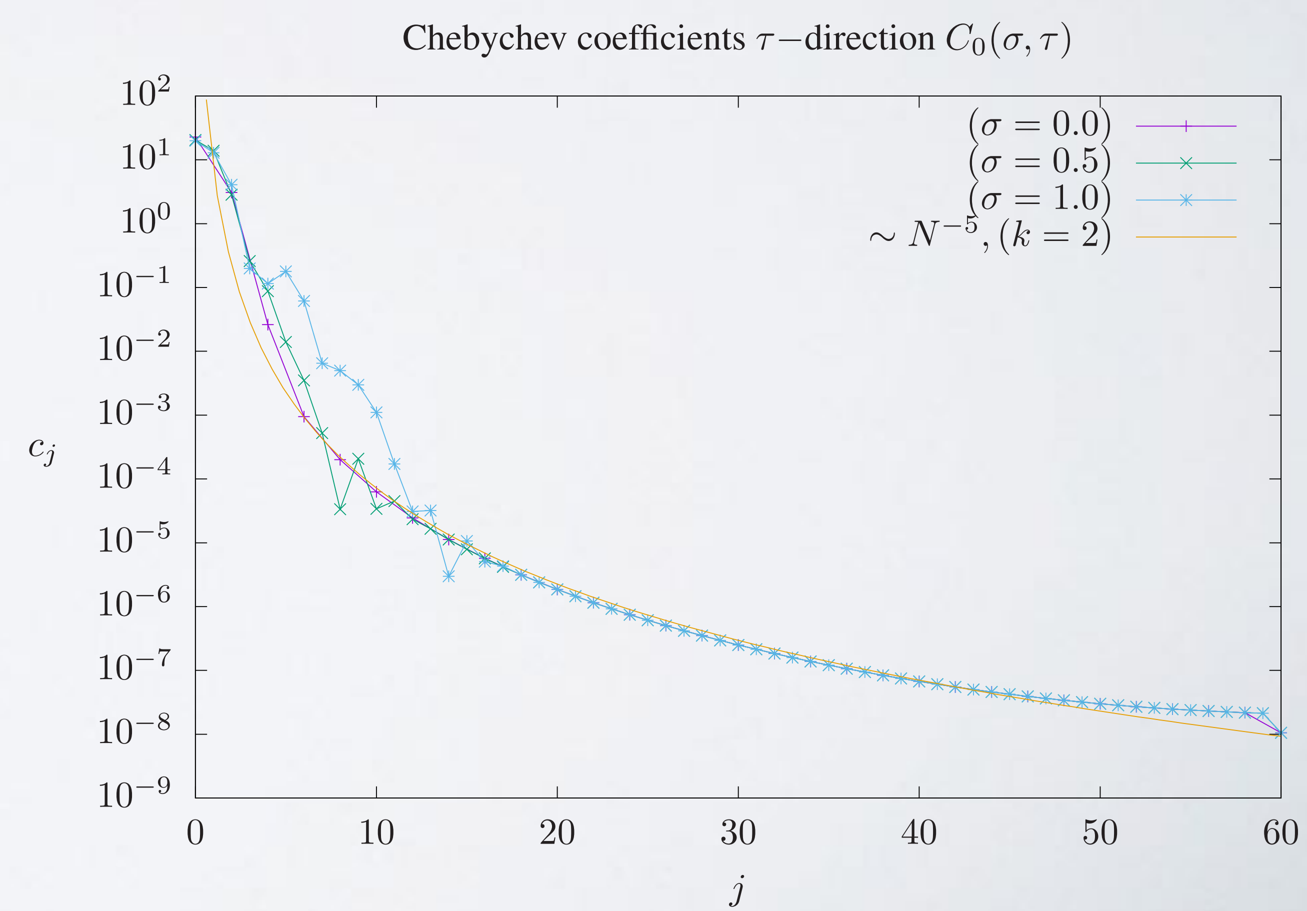

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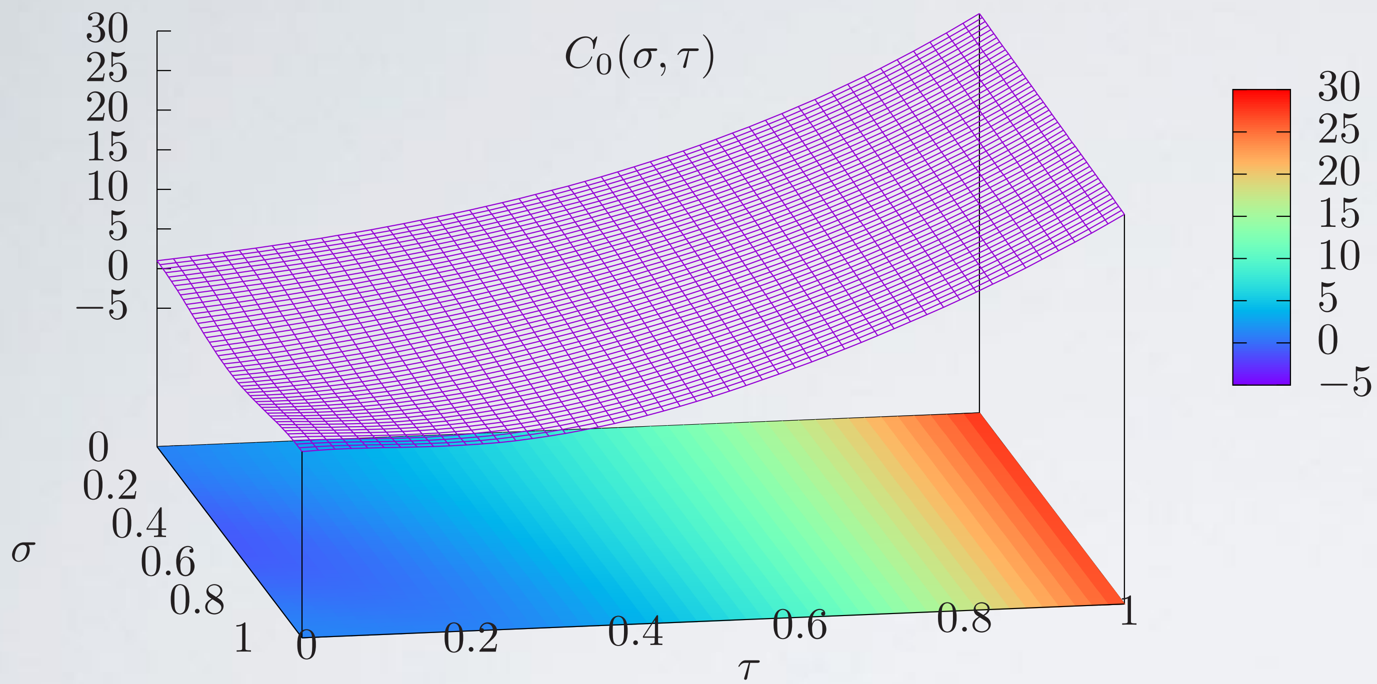
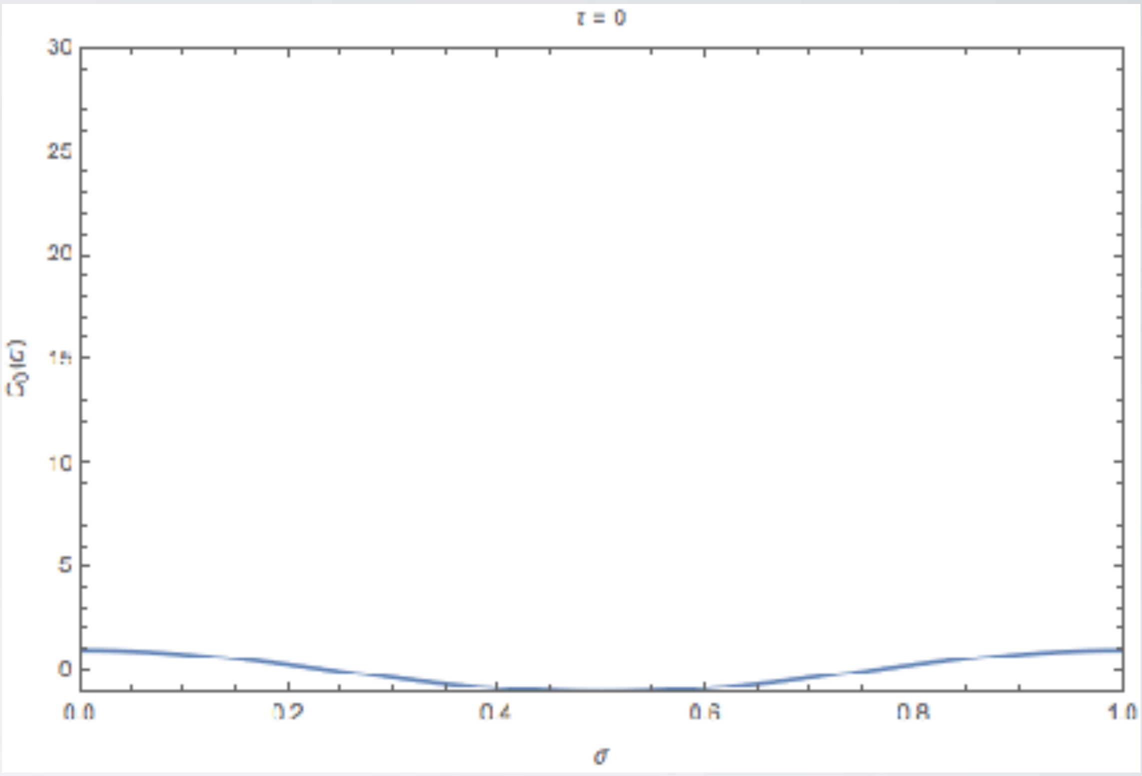
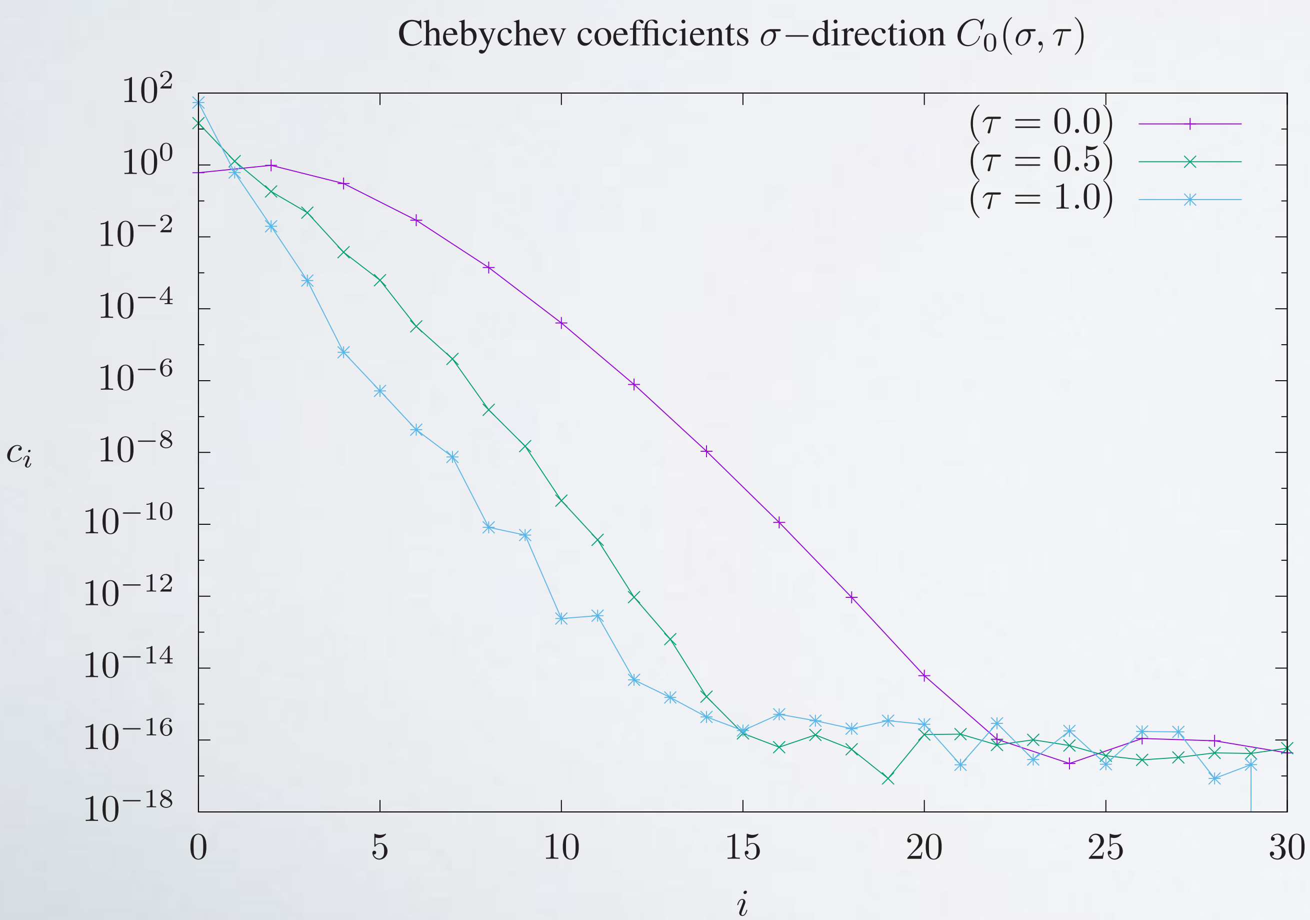
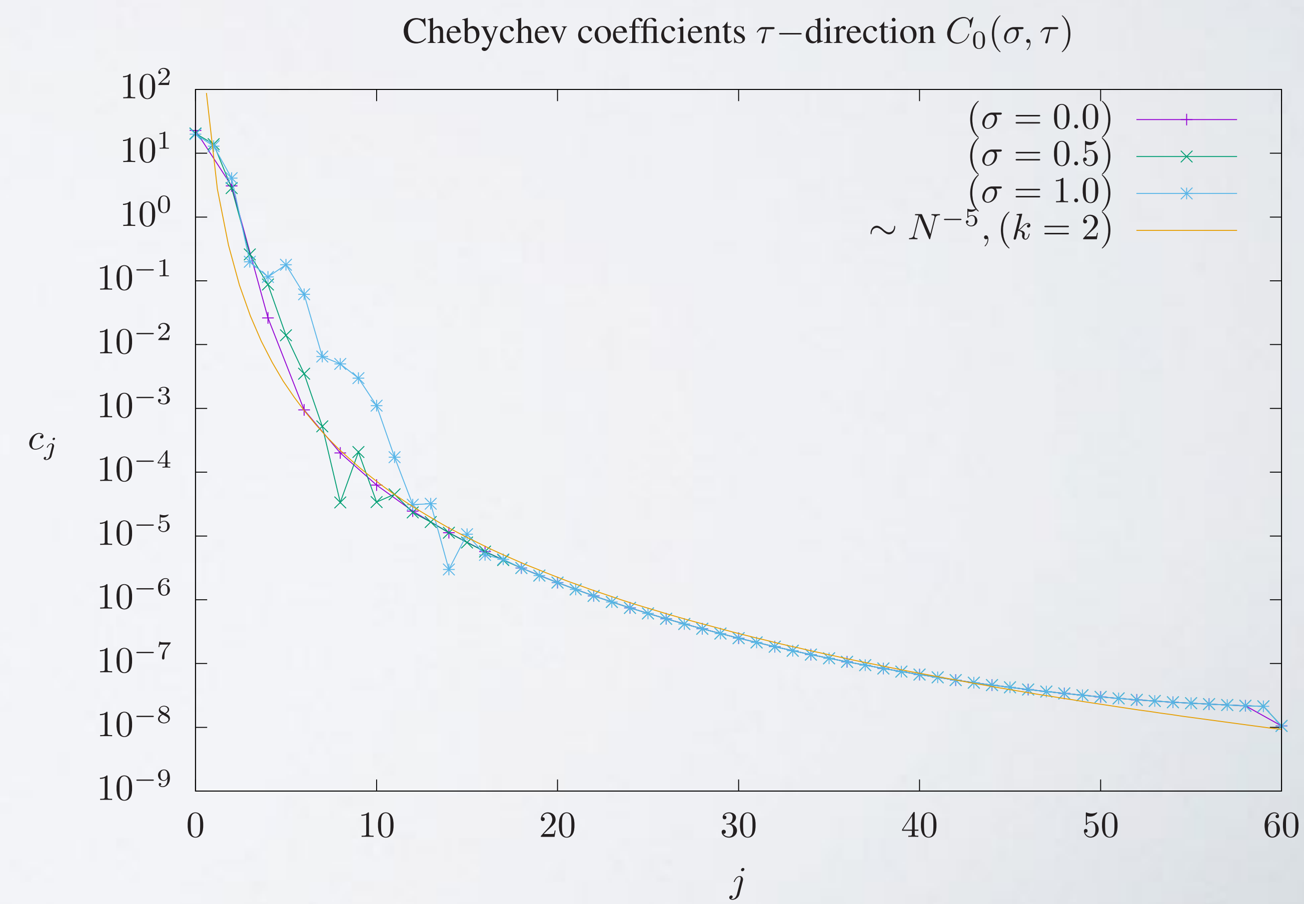








HYPERBOLIC EQUATIONS REMARKS

In general:

- Spectral methods with dynamical equations is an incipient field
- Introduction of adapted coordinates (careful with boundary of causal domain)
- Treatment of unknown functions and unknown parameters in equal stage

  (Treatment of a priori unknown boundaries - evolution of star's surface)

Here:

- Solution is $\mathcal{C}^1$, algebraic decay $\sim N^{-5}$ of Chebyshev coefficients in time direction
- Compatible with the structure of the auxiliary functions $\sim (1-\tau)^2 \ln(1-\tau)$ after removing leading order terms

ENHANCING ACCURACY

Today:

- Softening strong gradients (analytical mesh refinement)
- Introducing $\mathcal{C}^\infty$ functions

Somewhen in the future (lack of time)

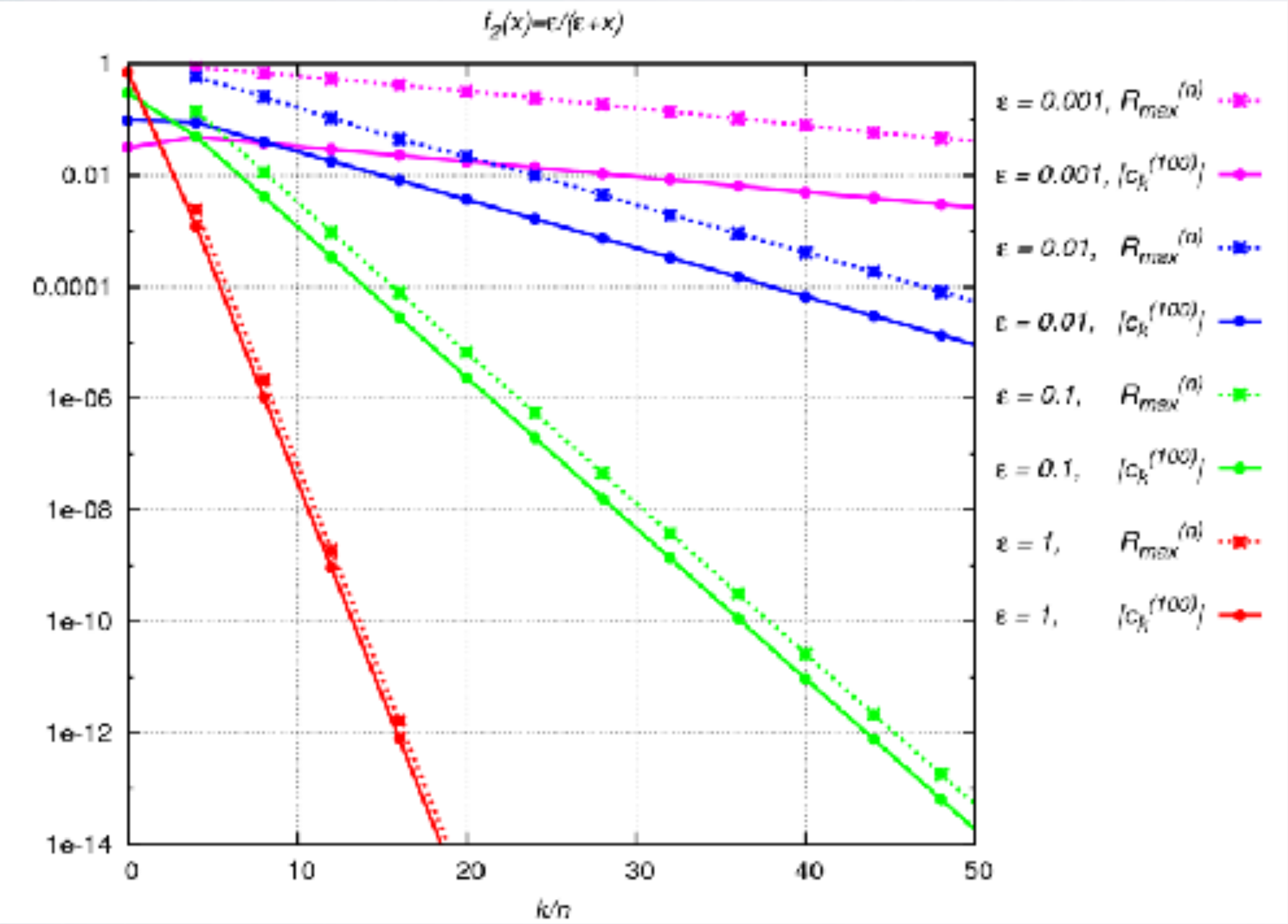
- Discontinuities: Gibbs phenomena and filtering techniques
- Discontinuities: multi-domain approach

ANALYTICAL FUNCTIONS

- **Functions:** $\mathcal{C}^{\omega}$(Taylor expansion around every point in domain)

- **Convergence Rate:** $\mathrm{Error} \sim \varrho^{-N}$

- **Strong gradients:** leads to smaller basis ϱ (slope in the log-linear plot)



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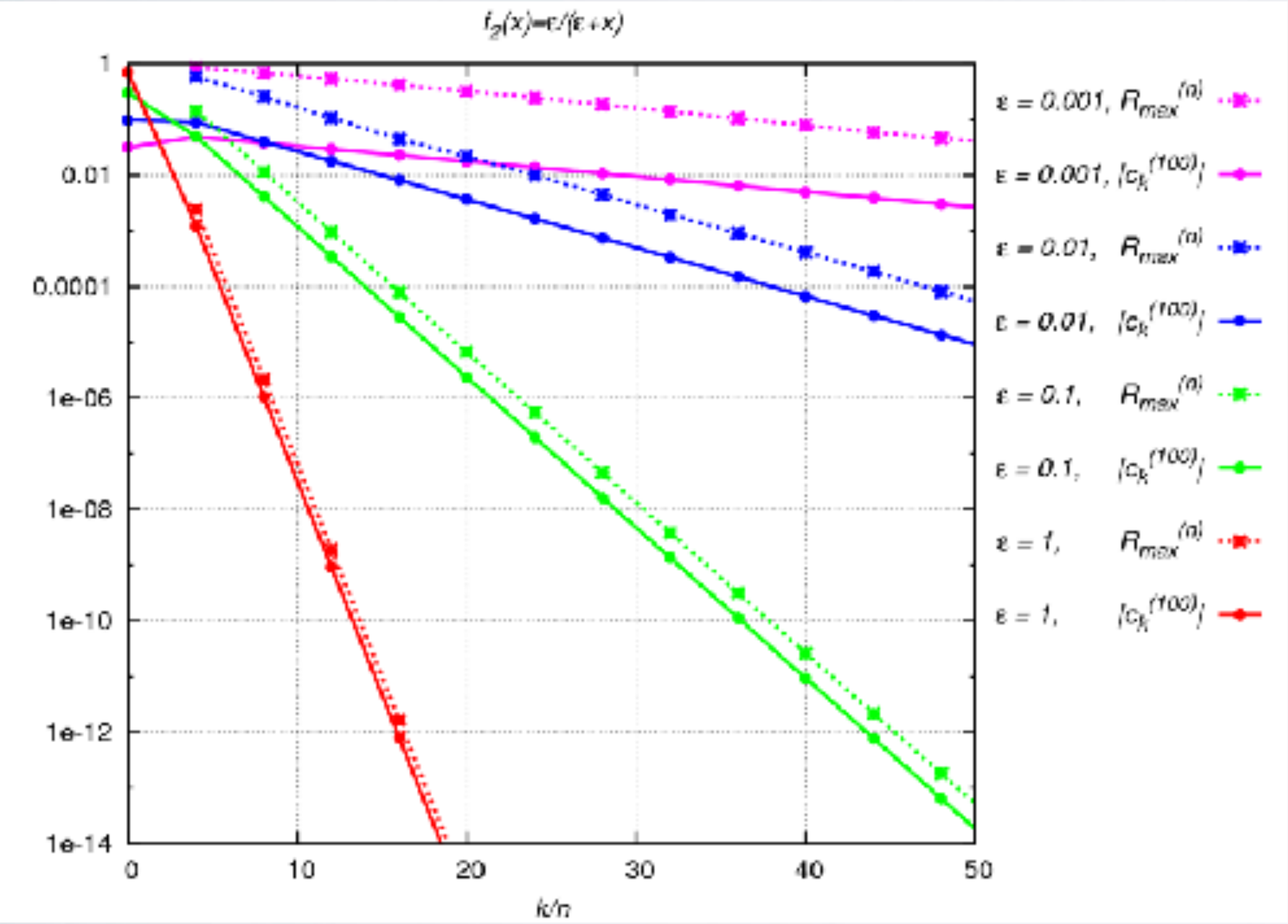

$$f_2(x) = \frac{\epsilon}{\epsilon + x}$$

For a given function, how can we predict the slope ϱ ?

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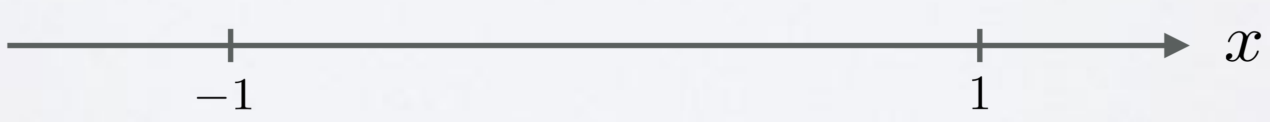

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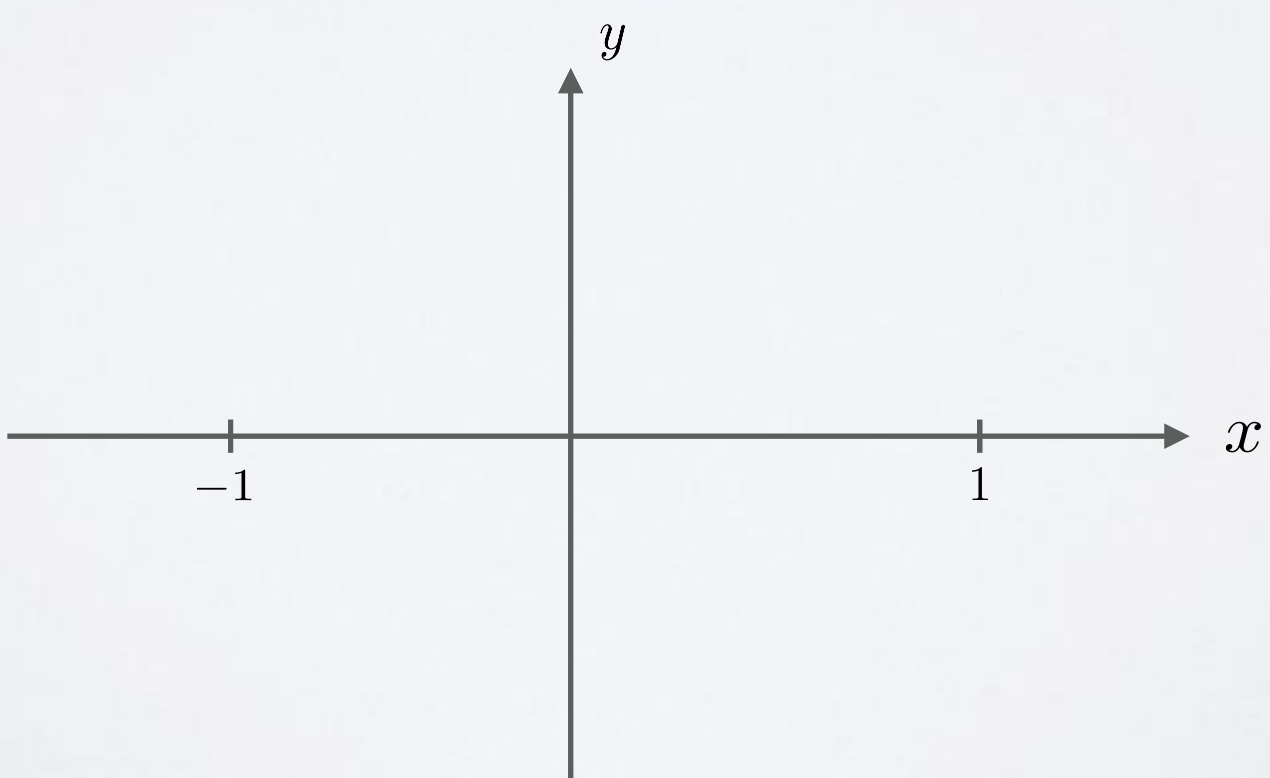
- Consider the analytical extension $f(z)$ into the complex plane with $z = x + iy$



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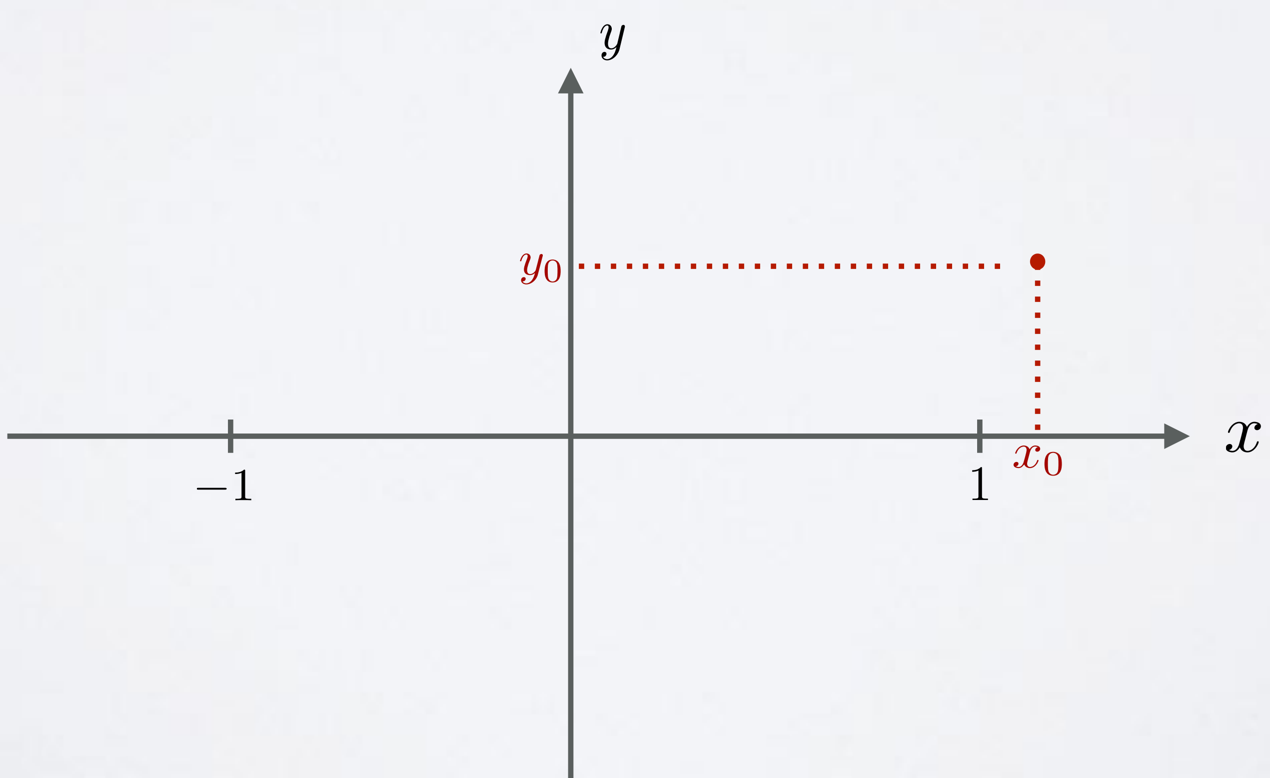

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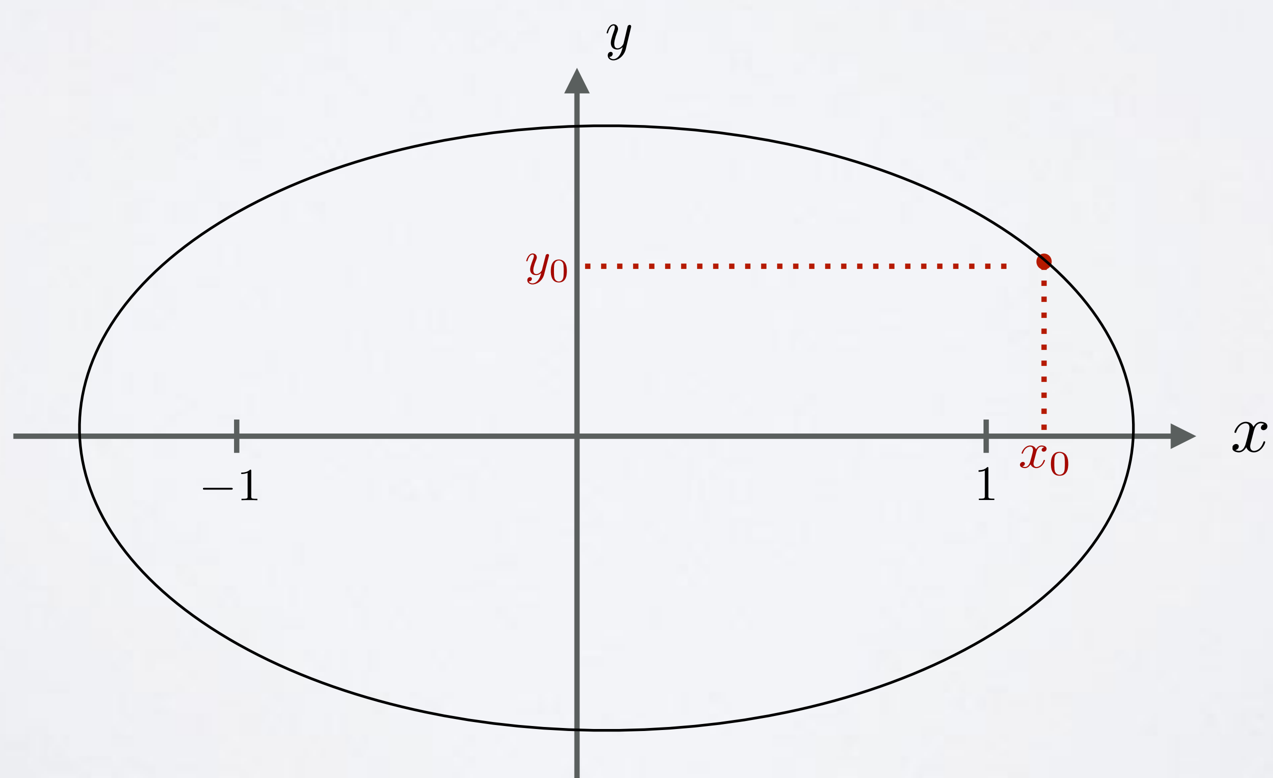

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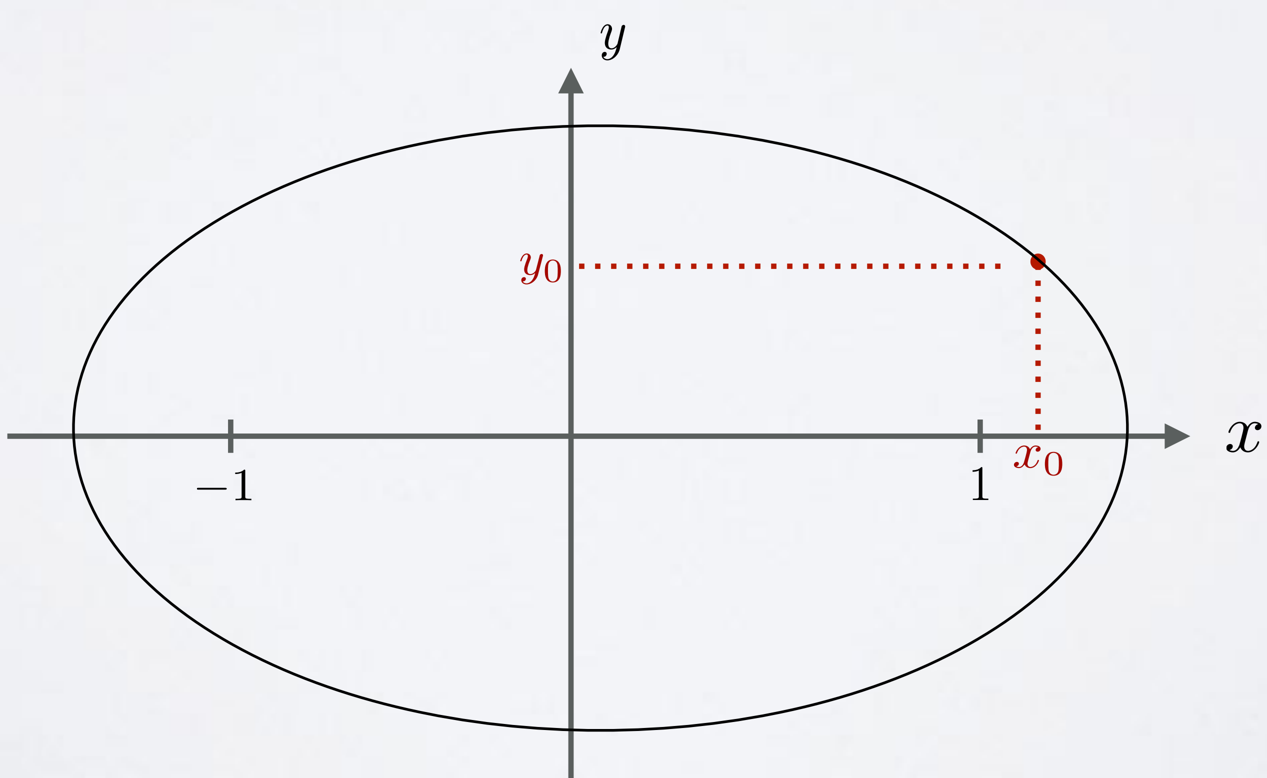

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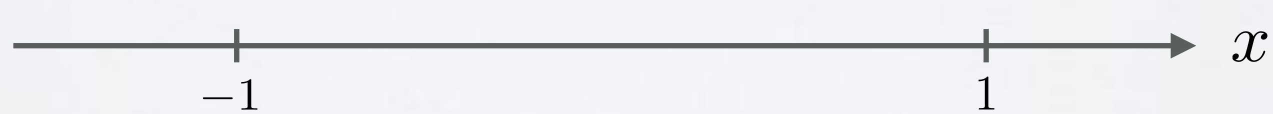

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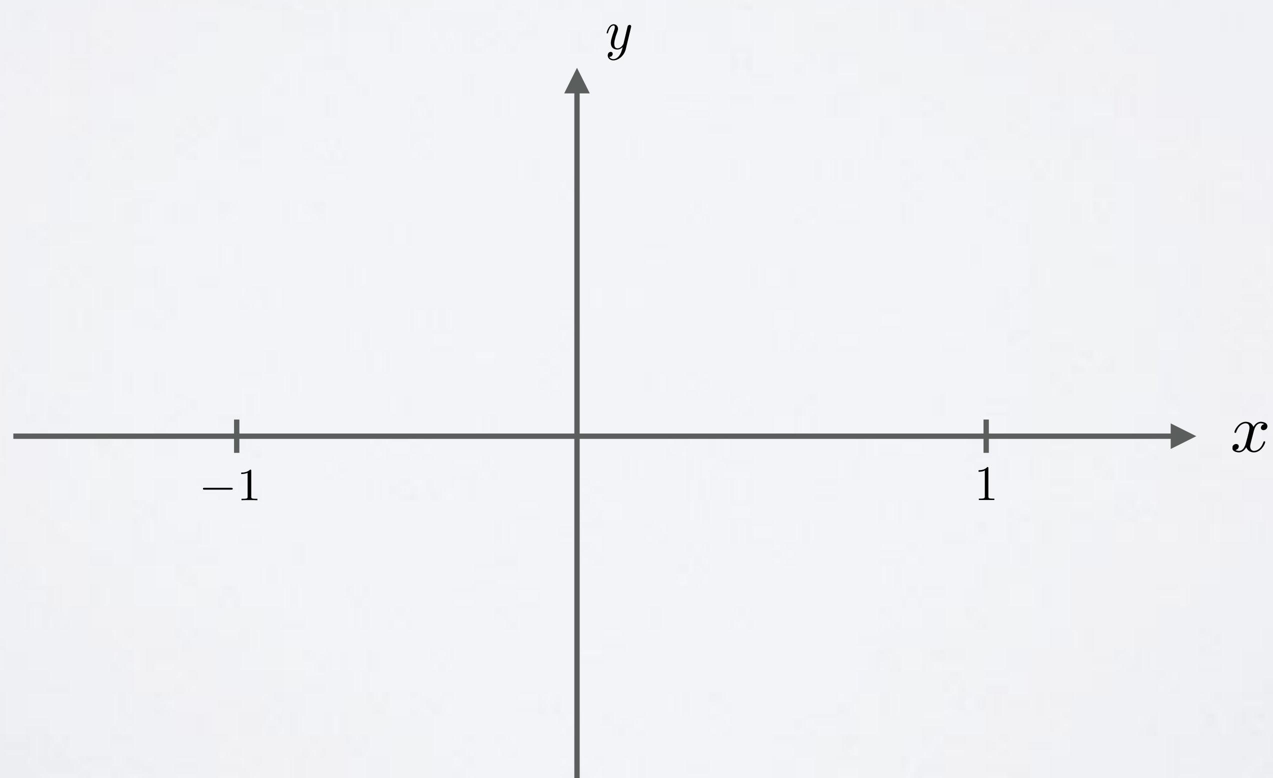

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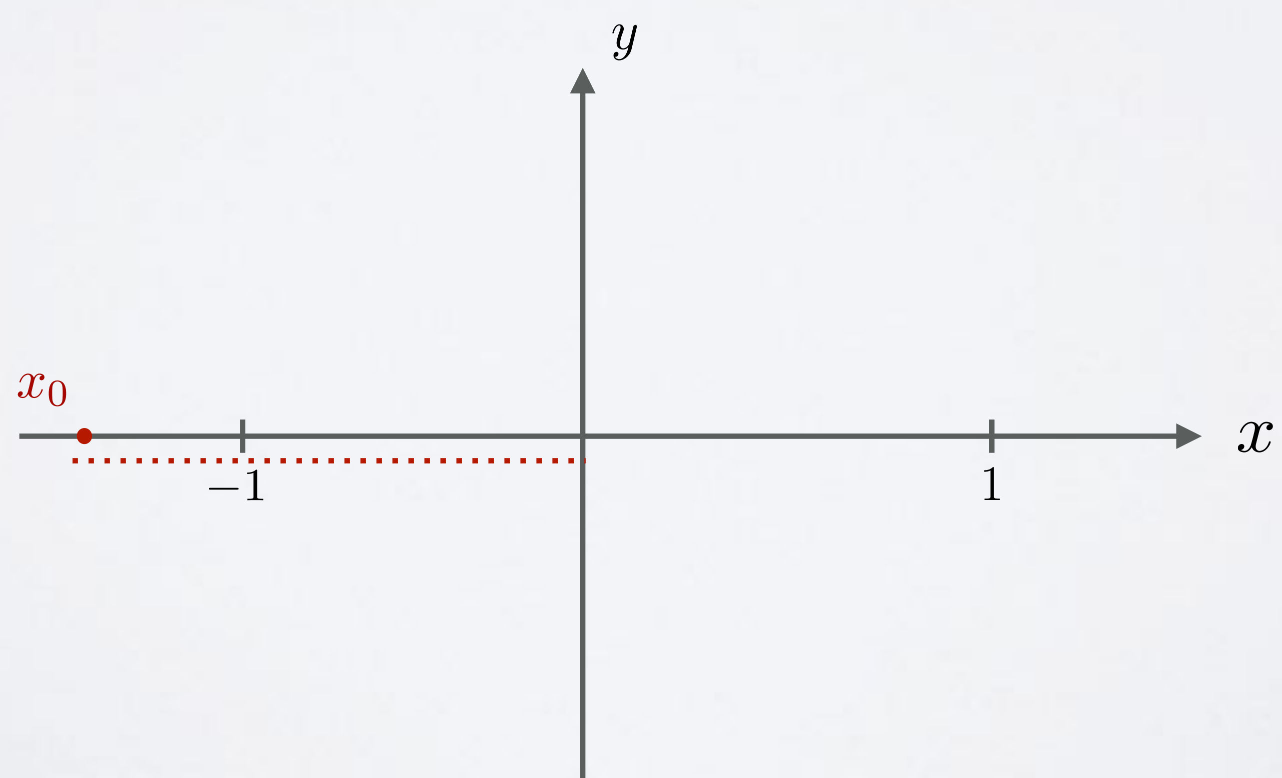

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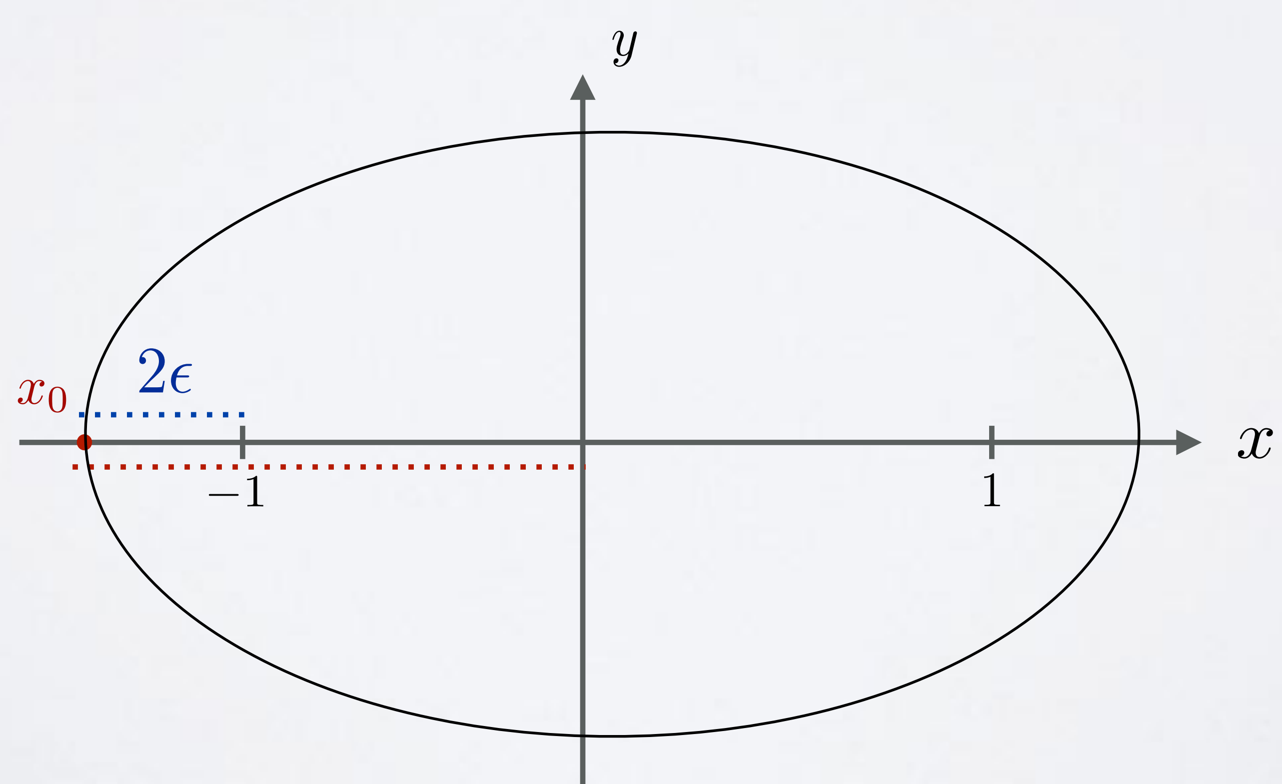
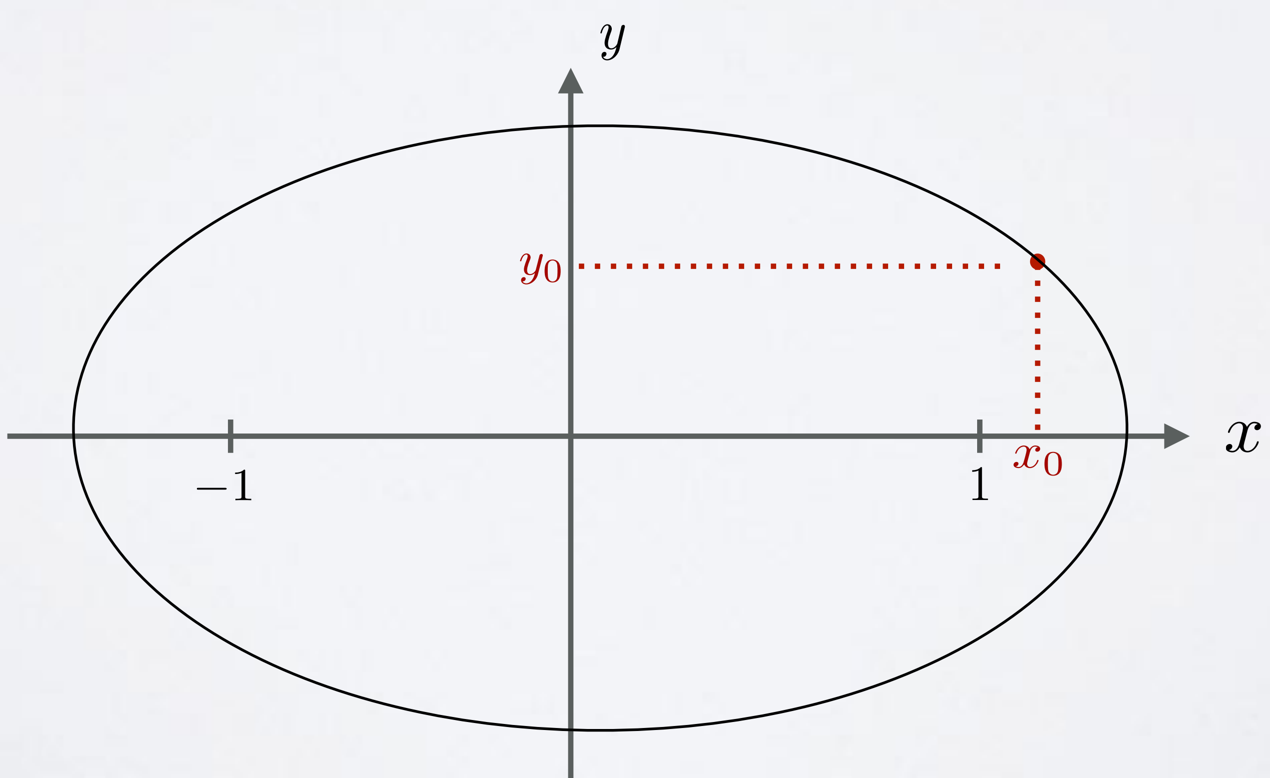


ANALYTICAL FUNCTIONS

- **Spectral approximation:** $\mathrm{Error} \sim \varrho^{-N}$
- **Decay rate** ϱ **:** a well definite distance between the singular points (x_0, y_0) of the function in the complex plane and the domain in the real axis $x \in [-1, 1]$
- **Conclusion :** complex singularities closer to real domain leads to slower exponential convergence
- **Remark :** discussion in terms of the Fourier basis is actually simpler (sorry!)



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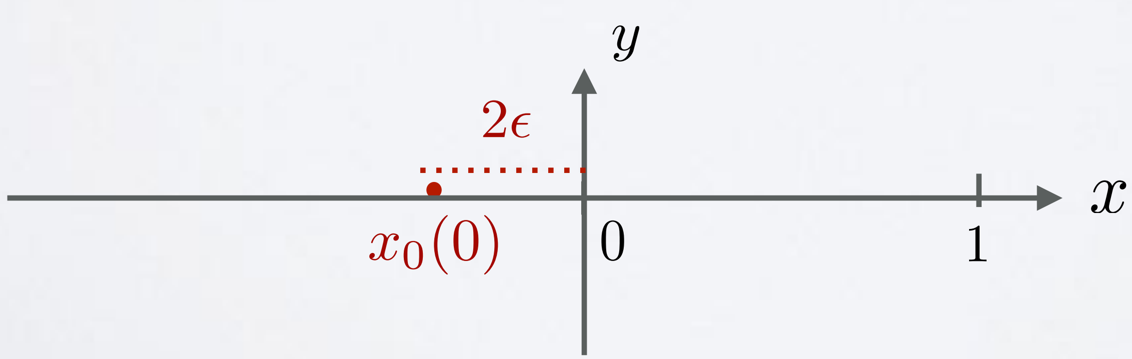

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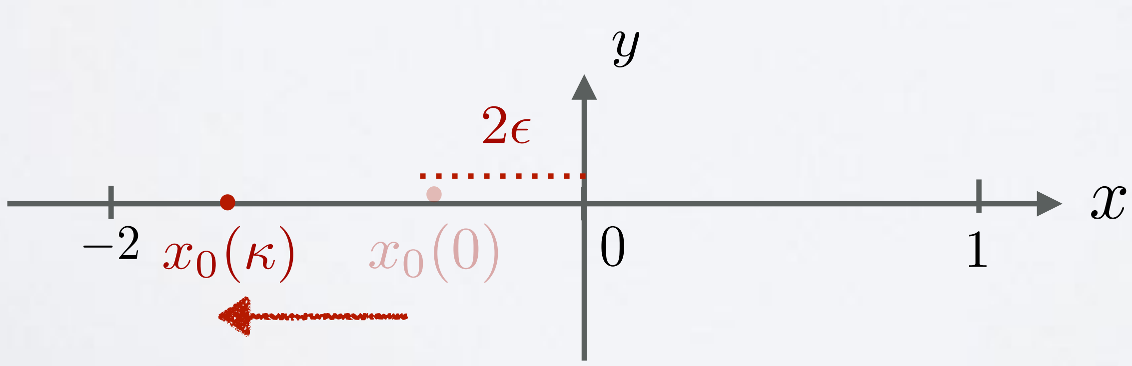
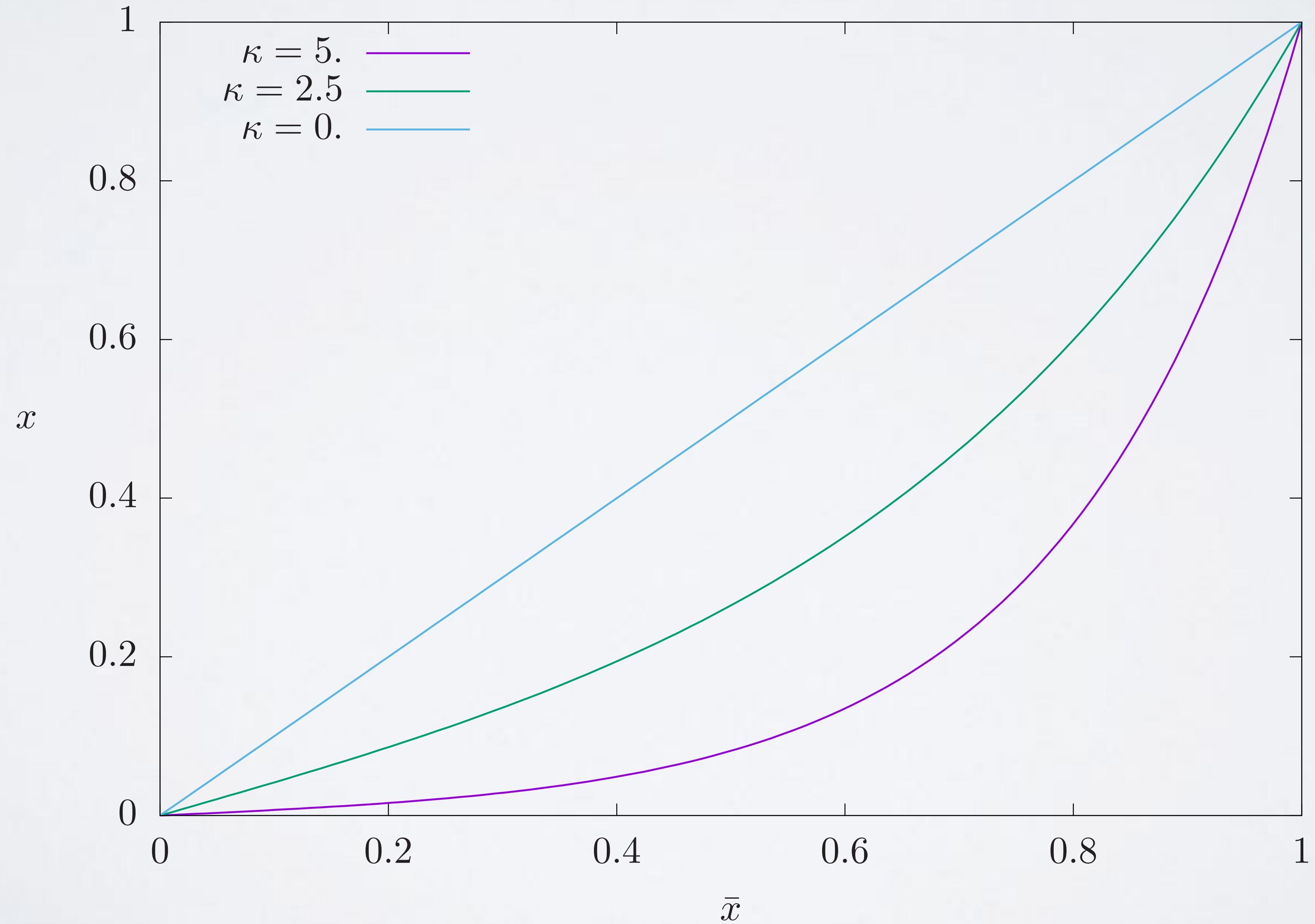
Singular point: $x_0(\kappa) = -\dfrac{\operatorname{asinh}(\epsilon \sinh(2\kappa))}{\kappa} \qquad \lim_{\kappa \to \infty} x_0(\kappa) = -2$



SINH TRANSFORMATION

- **Analytical Mesh Refinement :** from the numerical perspective, the transformation concentrate more grid points in the needed region

$$x = \frac{\sinh(\kappa \bar{x})}{\sinh(\kappa)}$$



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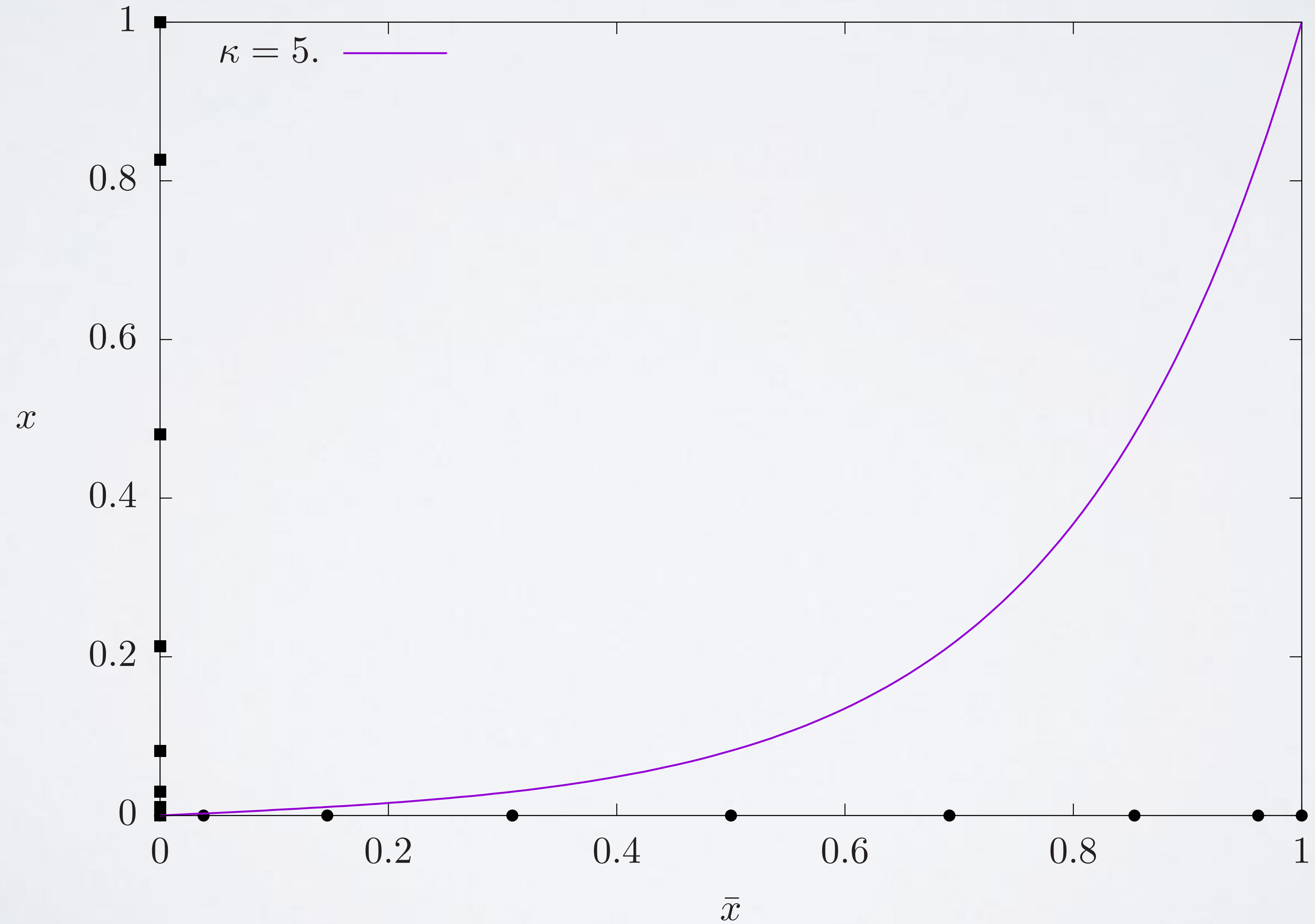
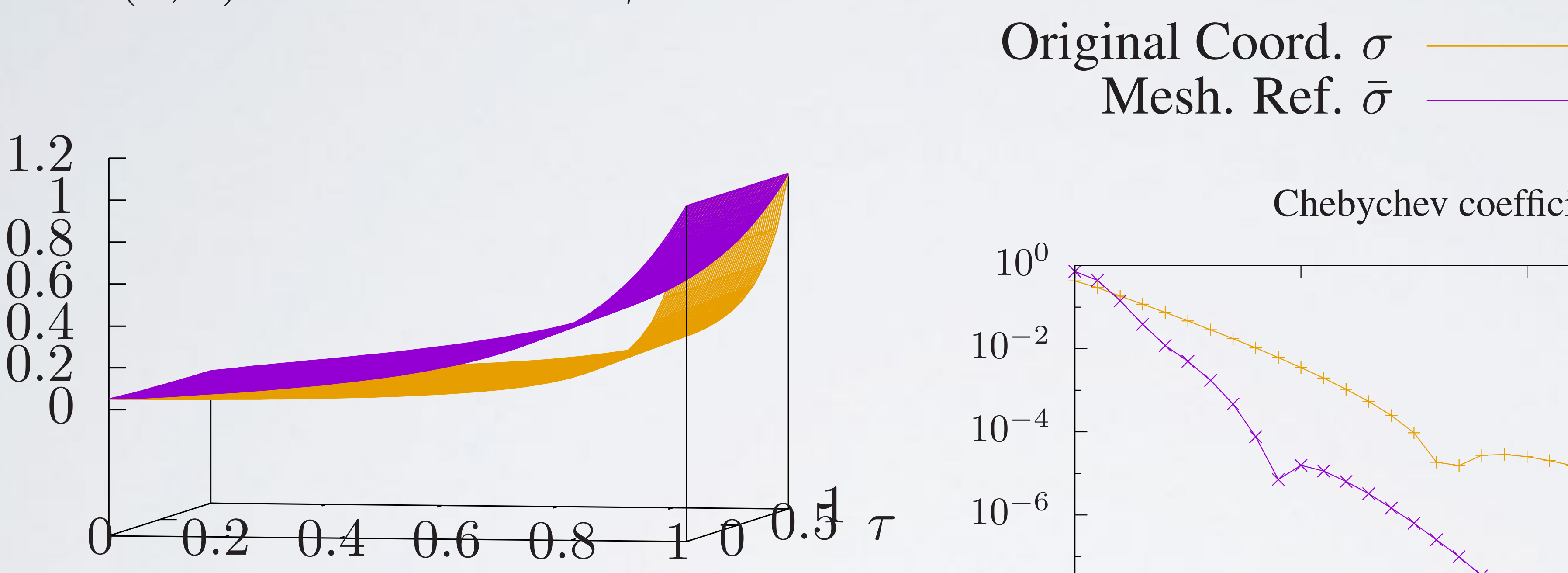


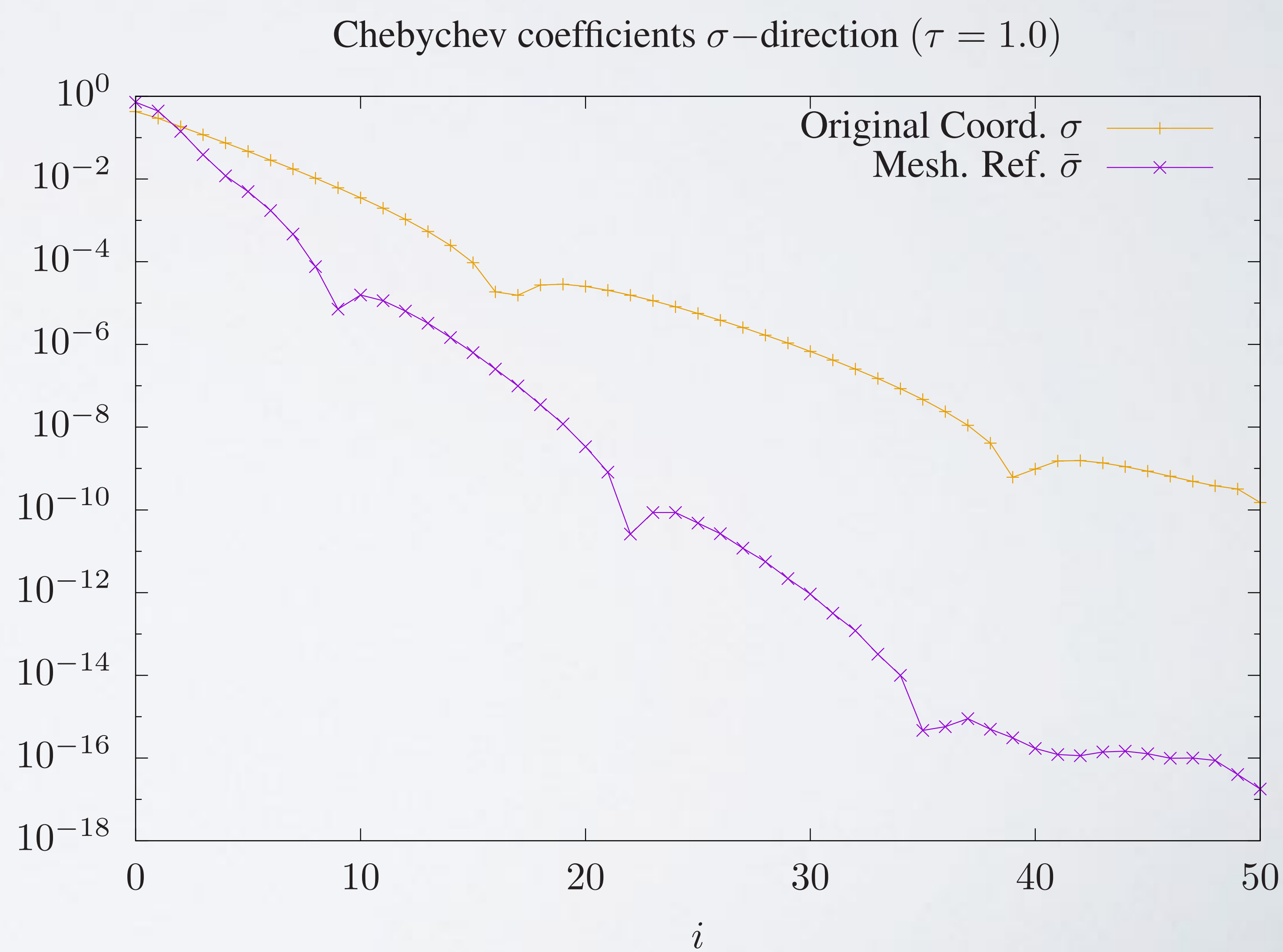
ELLIPTIC EQUATION

- **Disk of charged dust in General Relativity**

Y.C.Liu, RPM, M. Breithaupt, S. Palenta, R. Meinel. PRD 94 104035 (2016)

$\nu(\sigma, \tau) \quad @ \quad \epsilon = 0.5 \quad \gamma = 0.97$





Analytical Mesh
Refinement parameter

$\kappa \sim \ln(1-\gamma)$

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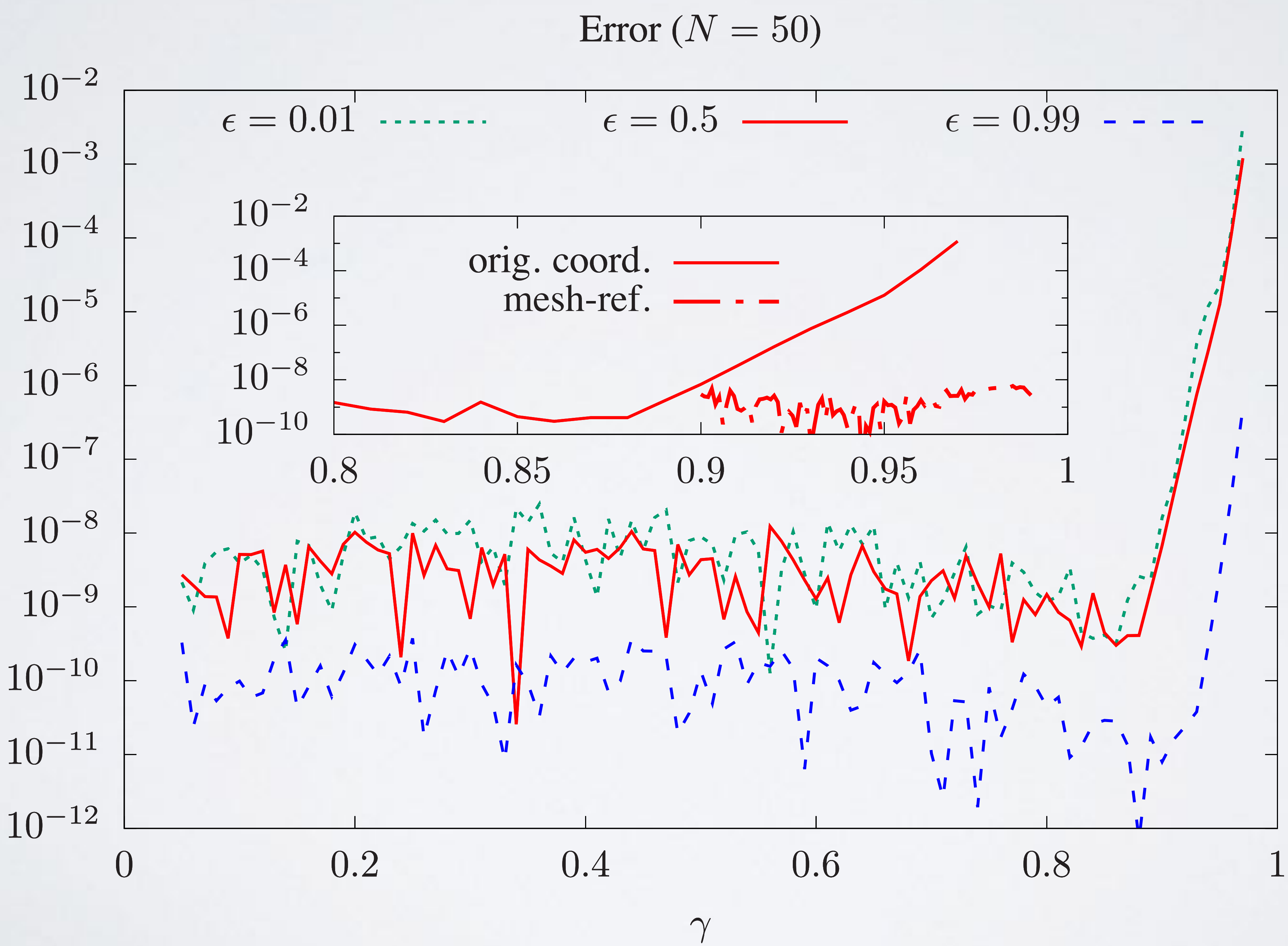

$\mathcal{C}^\ell([0,1])$ VERSUS $\mathcal{C}^\infty([0,1])$

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$$x(z) = e^{1-1/z}$$

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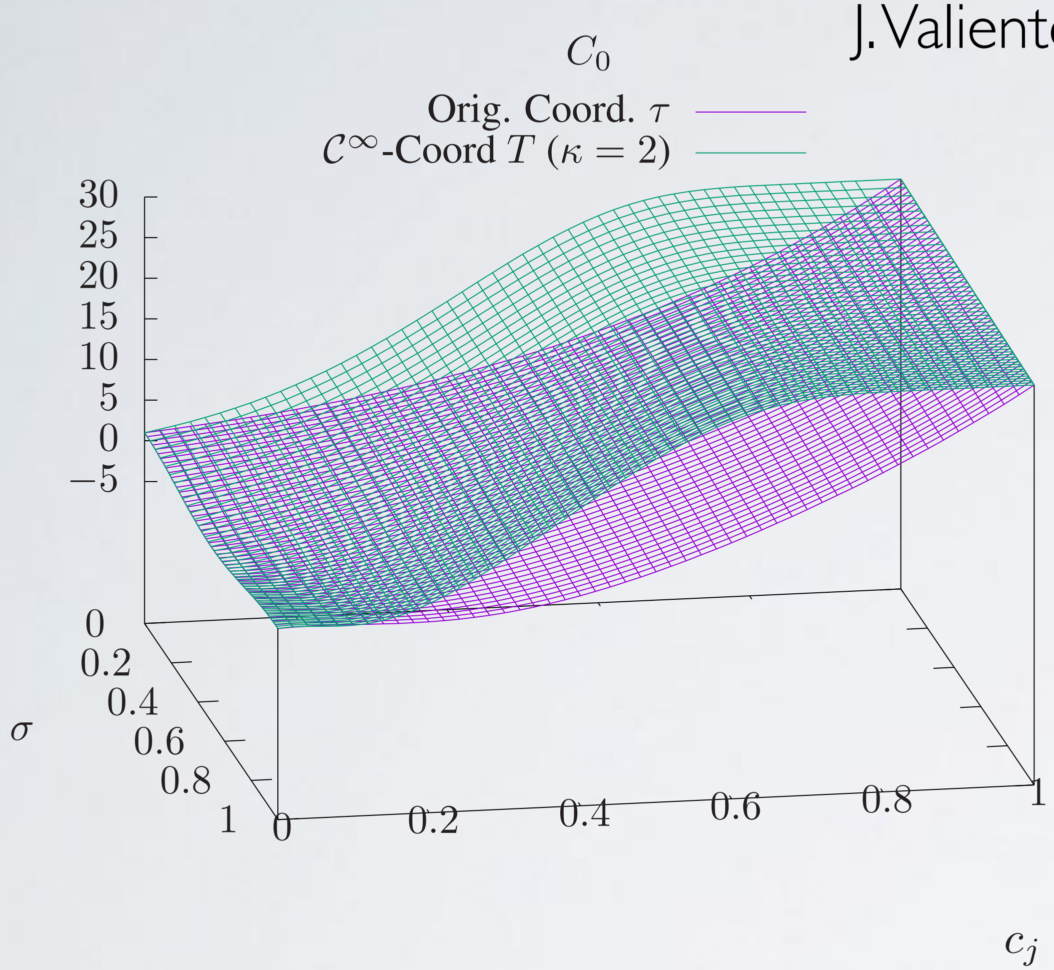
- **Consequence:** $F(z) = f\left(x(z)\right) \sim \dfrac{e^{k\left(1-\frac{1}{z}\right)}}{z}(z-1)$

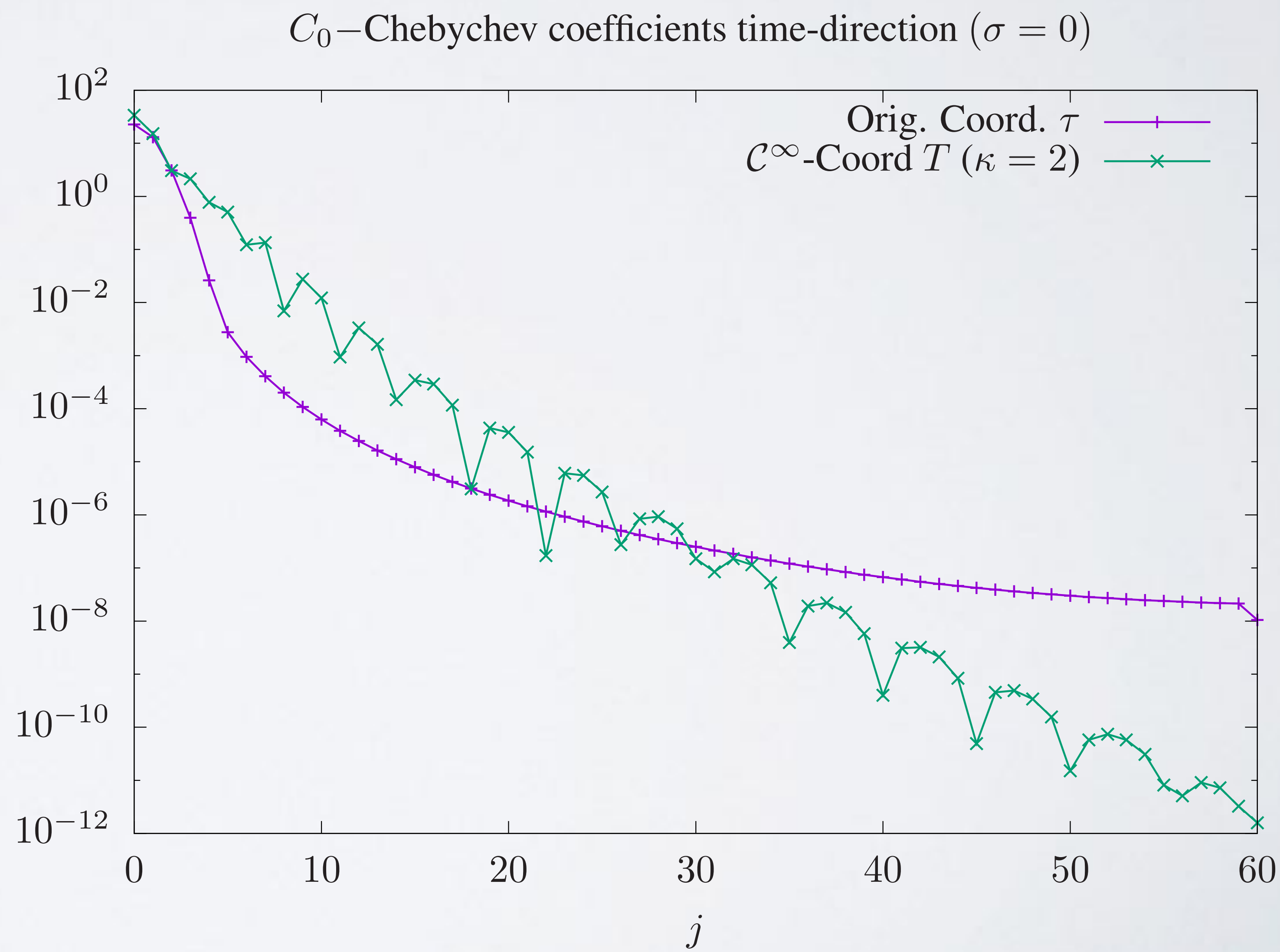
$$\lim_{z\to 0} \frac{d^j}{dz^j} F(z) = 0 \quad \text{i.e.} \quad F \in \mathcal{C}^\infty([0,1])$$

HYPERBOLIC EQUATION

Gravitational degrees of freedom near spatial infinity (linear regime on Minkowski background)

J. Valiente, RPM. In preparation (2017)





$$\tau = 1 - \exp\left[\kappa\left(1 - \frac{1}{1-T}\right)\right]$$

CONCLUSION

Spectral Methods

- Provide numerical solutions with high accuracy
- Treatment of unknown functions and unknown parameters in equal stage
- Allows one to explore regions in the parameter space where the solution is problematic
- Explore mathematical properties of the solution
- But can be slower then other algorithms
- Well stablished method for elliptic equations (General Relativity: stationary space times and initial data problem)
- Time dependent problems: spectral methods in the spatial direction together with some other time integrator (Runge-Kutta)
- Development of fully spectral codes: spectral methods applied to both space and time directions