Queen Mary
University of London

HIGH-ACCURACY NUMERICAL METHODS IN GENERAL RELATIVITY

Rodrigo Panosso Macedo
(Part I)

The 4th Afro-Franco-Brazilian meeting on Mathematics and Physics - July 2017

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Equations related to General Relativity

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- **Motivation:** two body problem

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Use com … **elated to**
(numerical a … **elativity**

. S. … **64)**.

ANNALS OF PHYSICS: **29**, 304–331 (1964)

The Two-Body Problem in Geometrodynamics

Susan G. Hahn

International Business Machines Corporation, New York, New York

and

Richard W. Lindquist

Adelphi University, Garden City, New York

The problem of two interacting masses is investigated within the framework of geometrodynamics. It is assumed that the space-time continuum is free of all real sources of mass or charge; particles are identified with multiply connected regions of empty space. Particular attention is focused on an asymptotically flat space containing a "handle" or "wormhole." When the two "mouths" of the wormhole are well separated, they seem to appear as two centers of gravitational attraction of equal mass. To simplify the problem, it is assumed that the metric is invariant under rotations about the axis of symmetry, and symmetric with respect to the time $t = 0$ of maximum separation of the two mouths. Analytic initial value data for this case have been obtained by Misner; these contain two arbitrary parameters, which are uniquely determined when the mass of the two mouths and their initial separation have been specified. We treat a particular case in which the ratio of mass to initial separation is approximately one-half. To determine a unique solution of the remaining (dynamic) field equations, the coordinate conditions $g_{0\alpha} = -\delta_{0\alpha}$ are imposed; then the set of second order equations is transformed into a quasi-linear first order system and the difference scheme of Friedrichs used to obtain a numerical solution. Its behavior agrees qualitatively with that of the one-body problem, and can be interpreted as a mutual attraction and pinching-off of the two mouths of the wormhole.

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- **Breakthrough:** F. Pretorius, Phys. Rev. Lett. 95, 121101 **(2005)**

PRL **95,** 121101 (2005) PHYSICAL REVIEW LETTERS week ending 16 SEPTEMBER 2005

Evolution of Binary Black-Hole Spacetimes

Frans Pretorius[1,2,*]

[1]*Theoretical Astrophysics, California Institute of Technology, Pasadena, California 91125, USA*
[2]*Department of Physics, University of Alberta, Edmonton, AB T6G 2J1 Canada*
(Received 6 July 2005; published 14 September 2005)

We describe early success in the evolution of binary black-hole spacetimes with a numerical code based on a generalization of harmonic coordinates. Indications are that with sufficient resolution this scheme is capable of evolving binary systems for enough time to extract information about the orbit, merger, and gravitational waves emitted during the event. As an example we show results from the evolution of a binary composed of two equal mass, nonspinning black holes, through a single plunge orbit, merger, and ringdown. The resultant black hole is estimated to be a Kerr black hole with angular momentum parameter $a \approx 0.70$. At present, lack of resolution far from the binary prevents an accurate estimate of the energy emitted, though a rough calculation suggests on the order of 5% of the initial rest mass of the system is radiated as gravitational waves during the final orbit and ringdown.

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- **Field:** solve numerically the *complete (non-linear) system* of Einstein's equation

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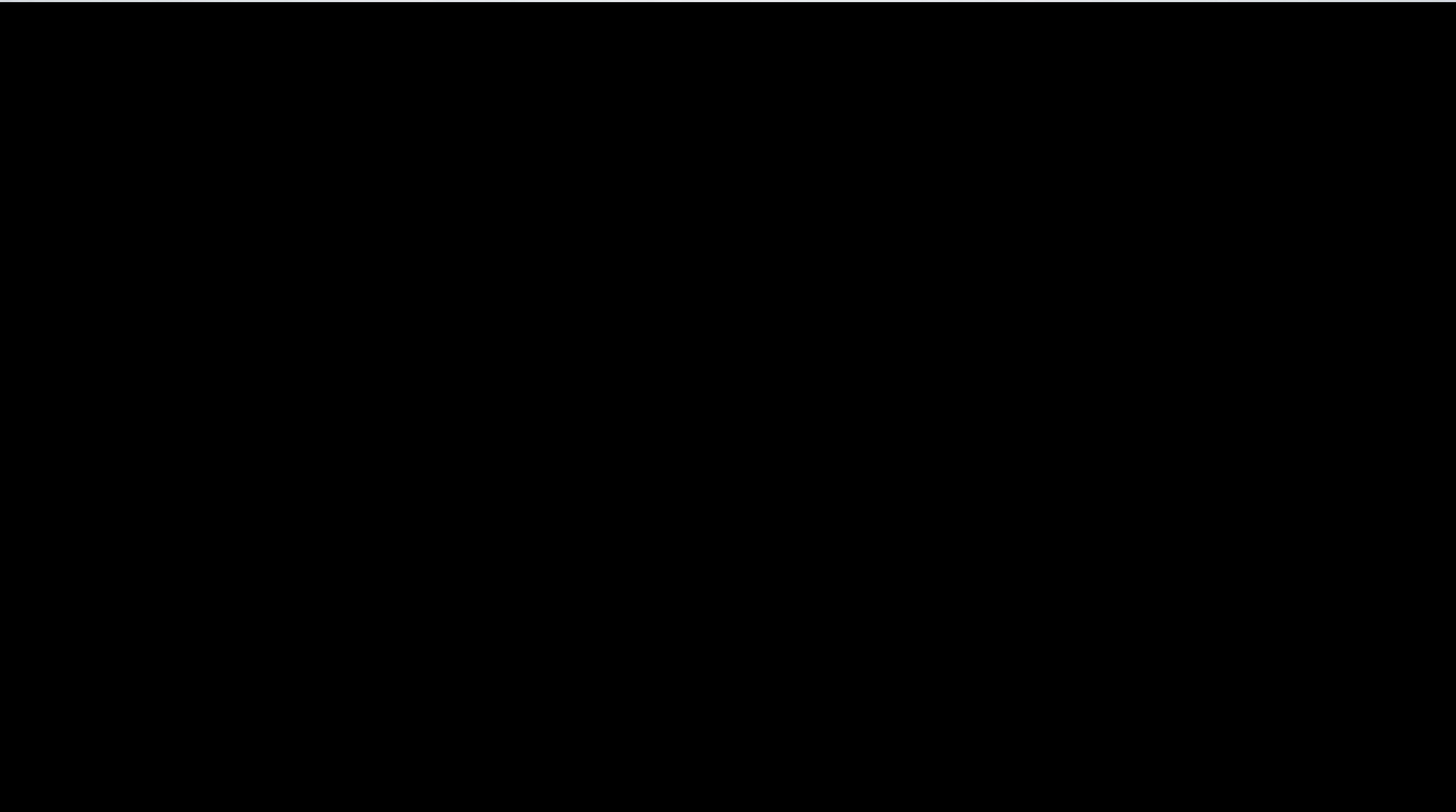
  - Anti de Sitter spacetime: AdS/CFT correspondence (numerical holography)

GRAVITATIONAL WAVES



$m_1 \approx 35 M_\odot$ $\quad$ $m_2 \approx 30 M_\odot$

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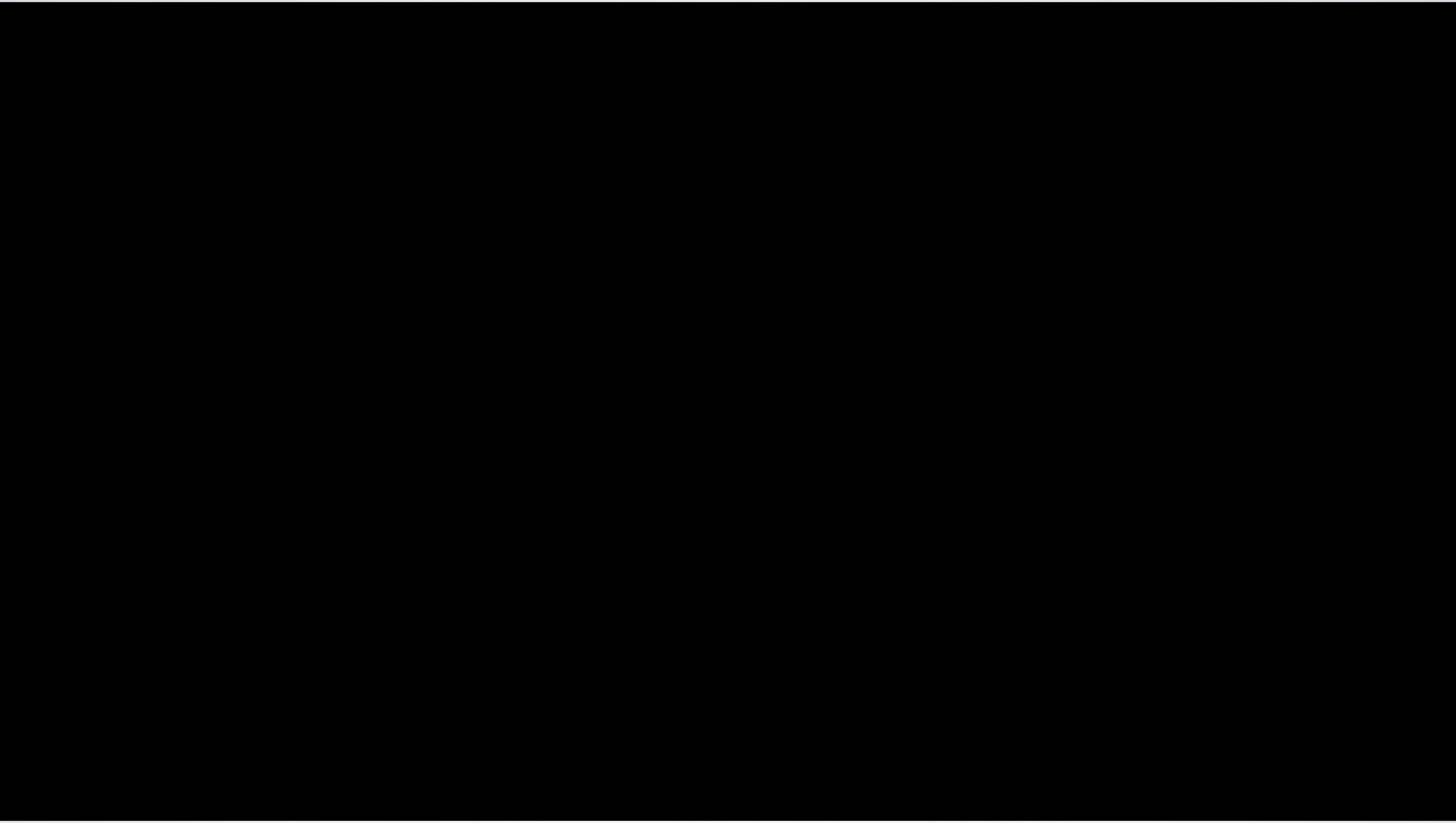

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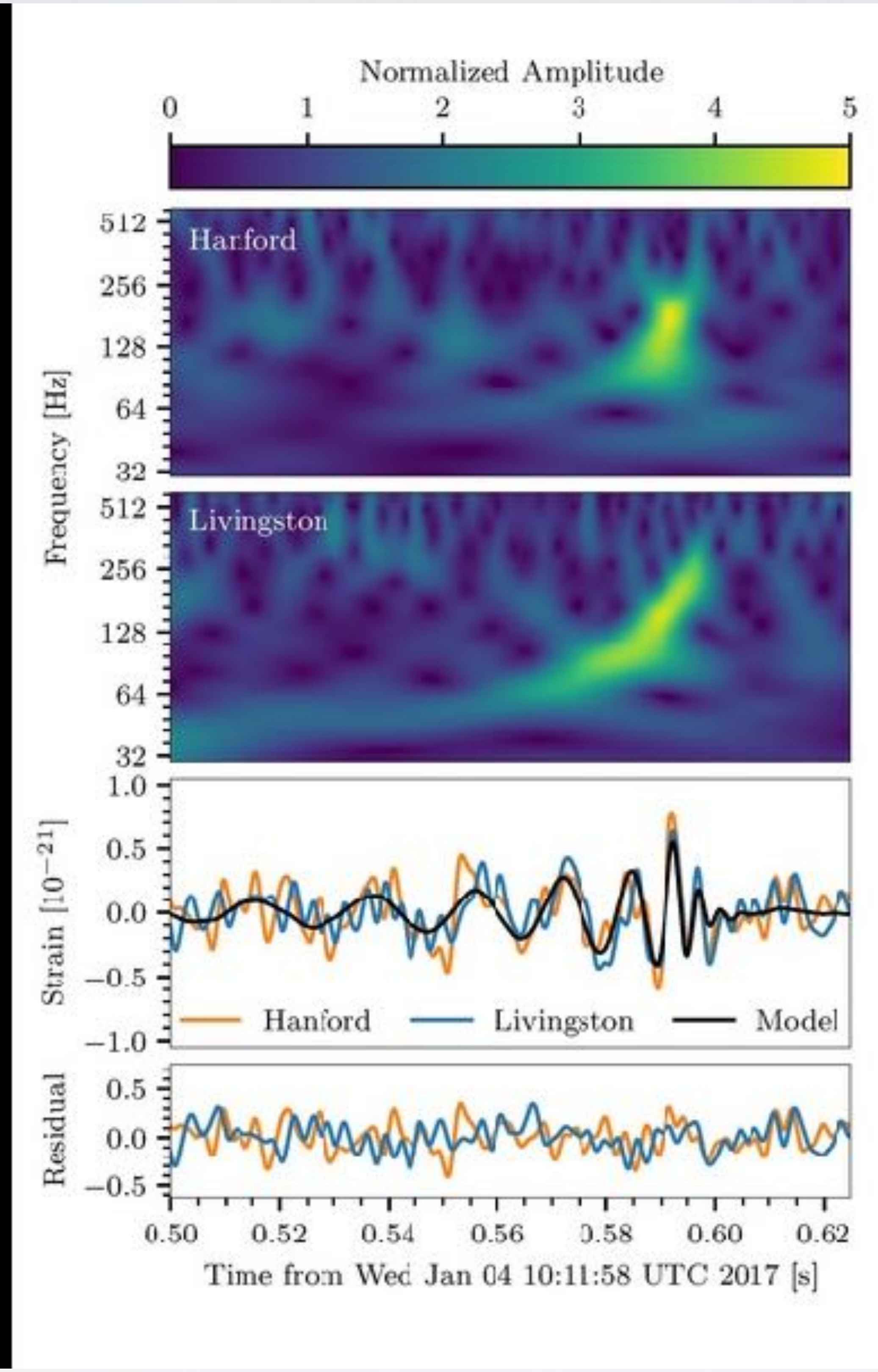
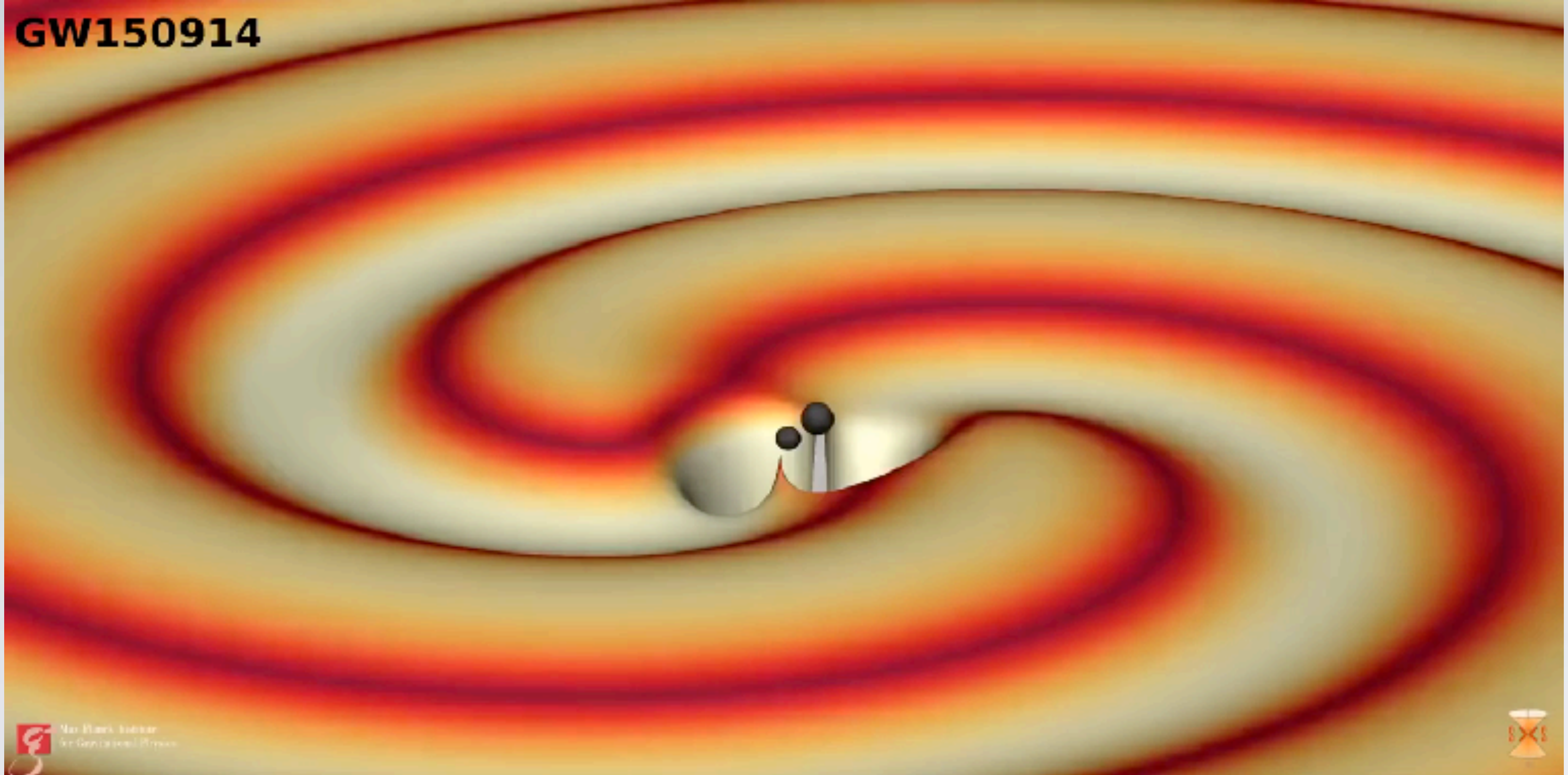


$m_1 \approx 35 M_\odot \quad m_2 \approx 30 M_\odot \quad m_{\text{final}} \approx 62 M_\odot \quad z \approx 0.093 (440 \text{Mpc})$



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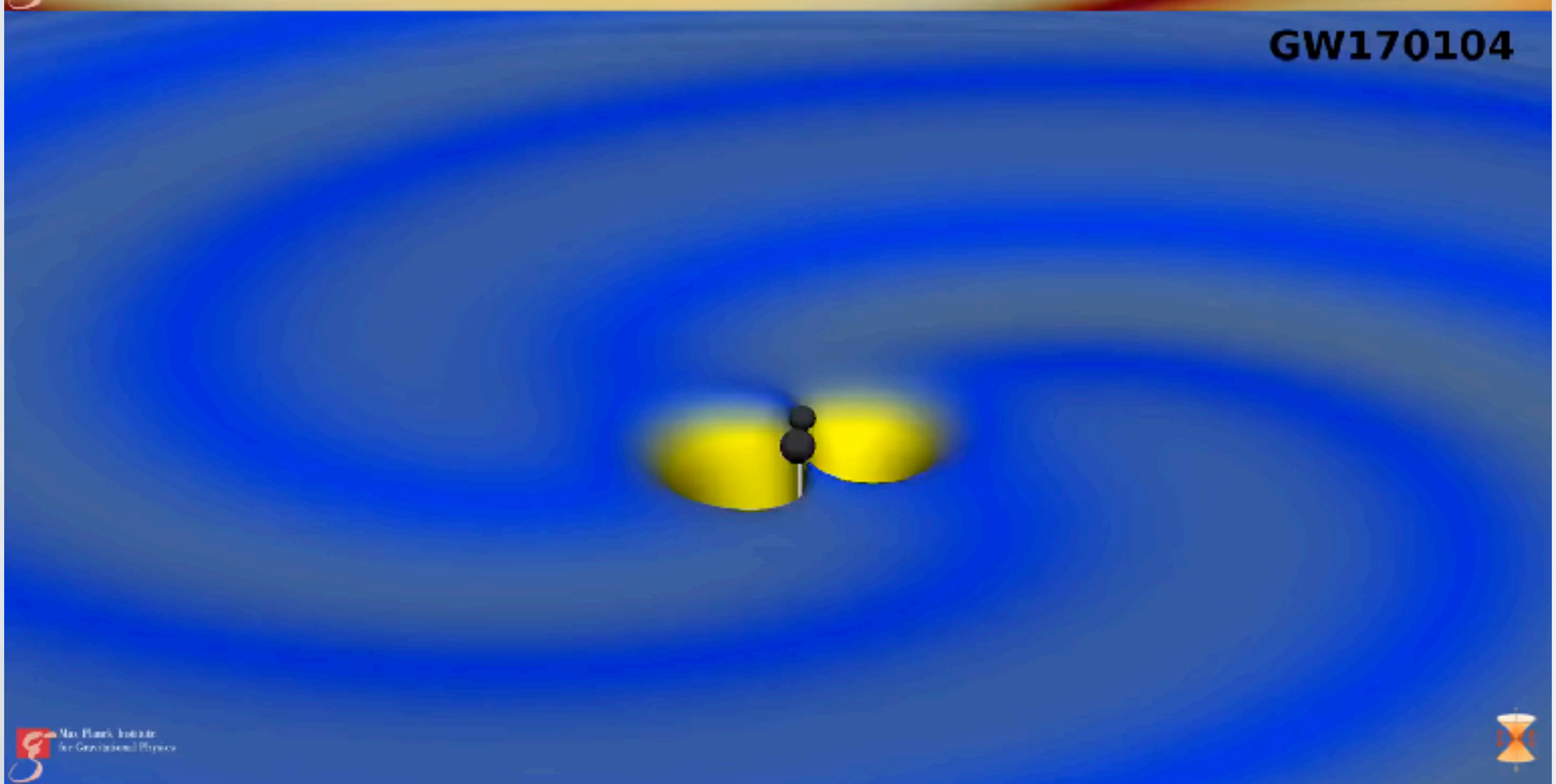

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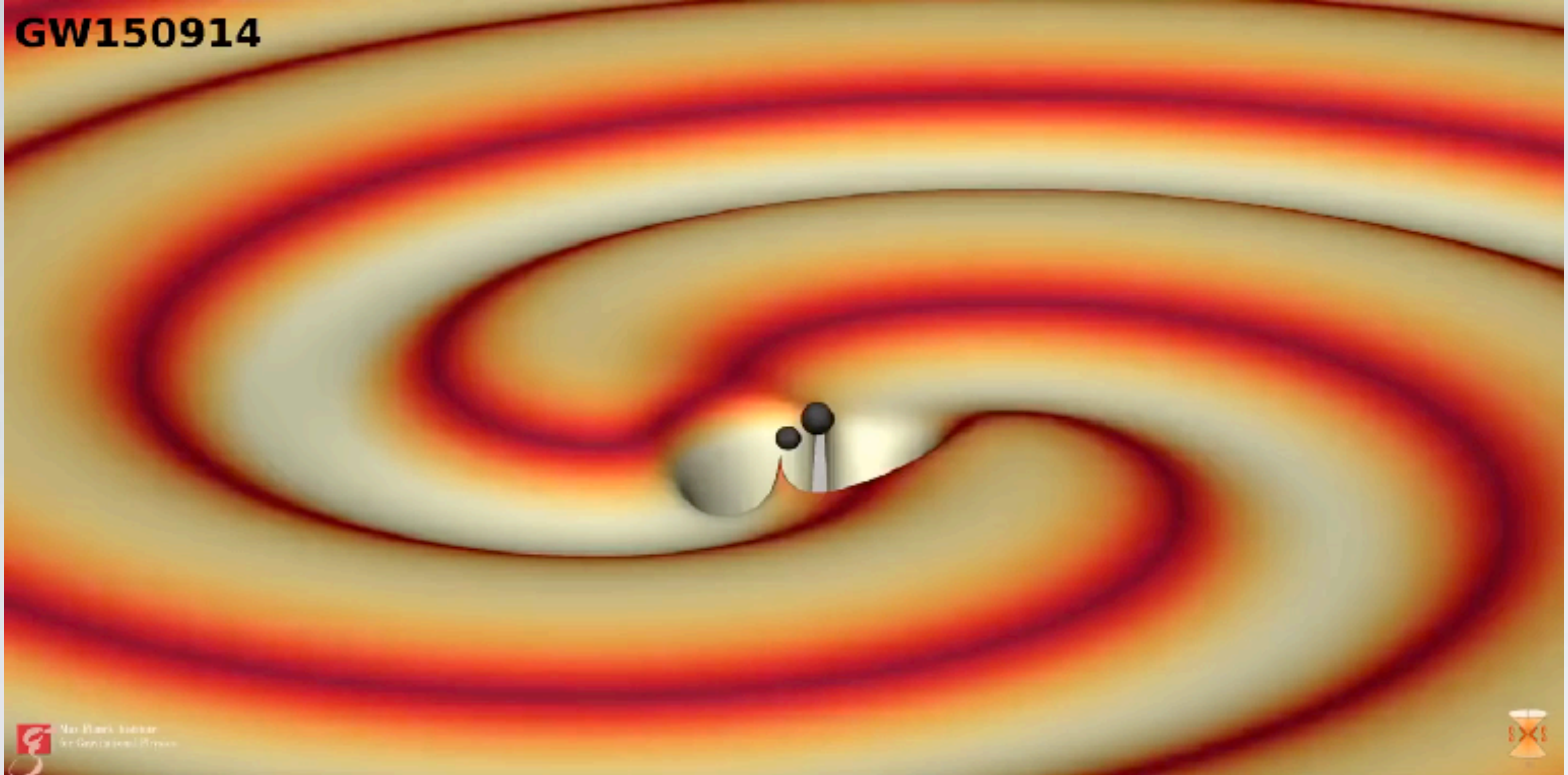

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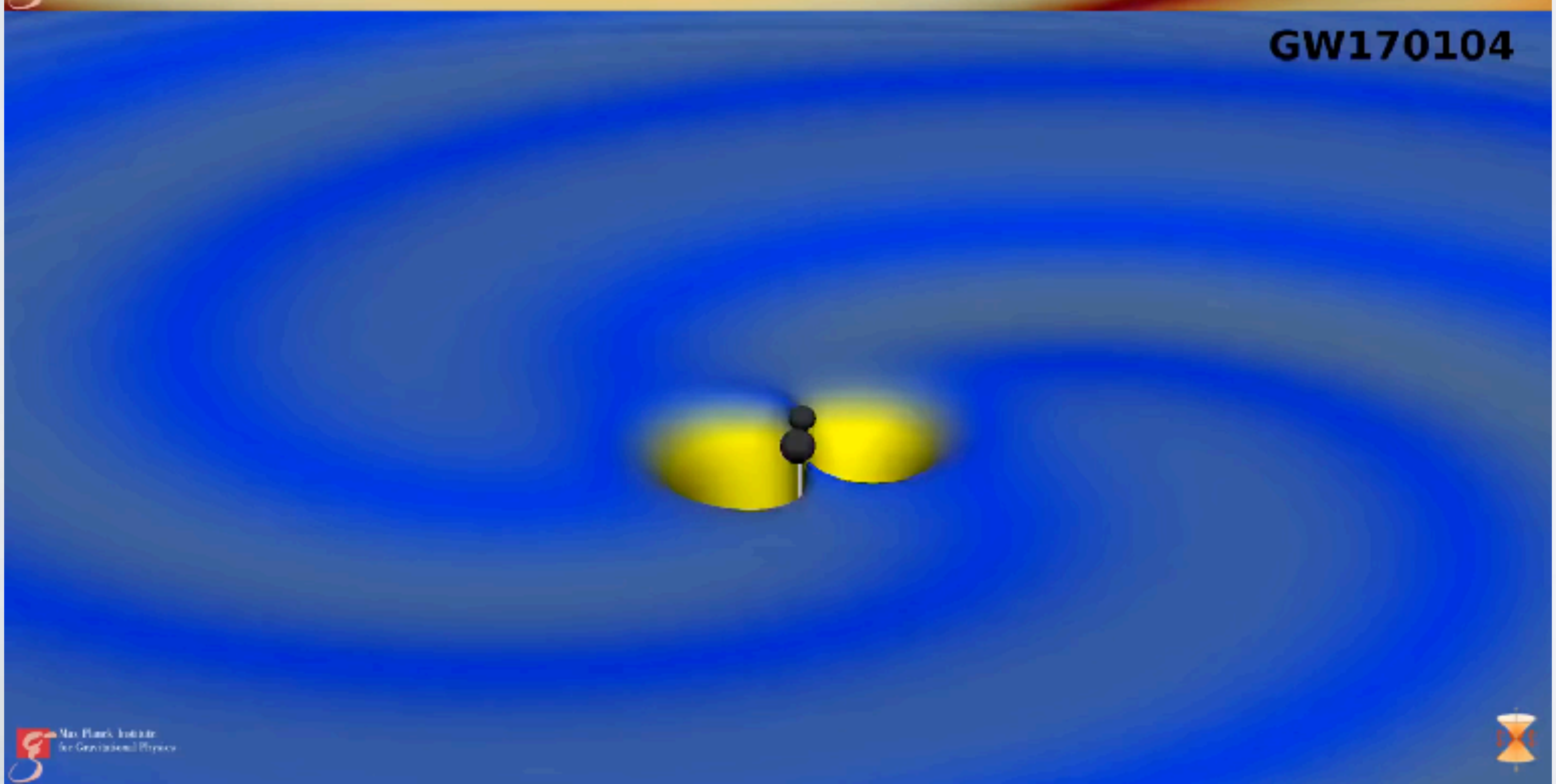

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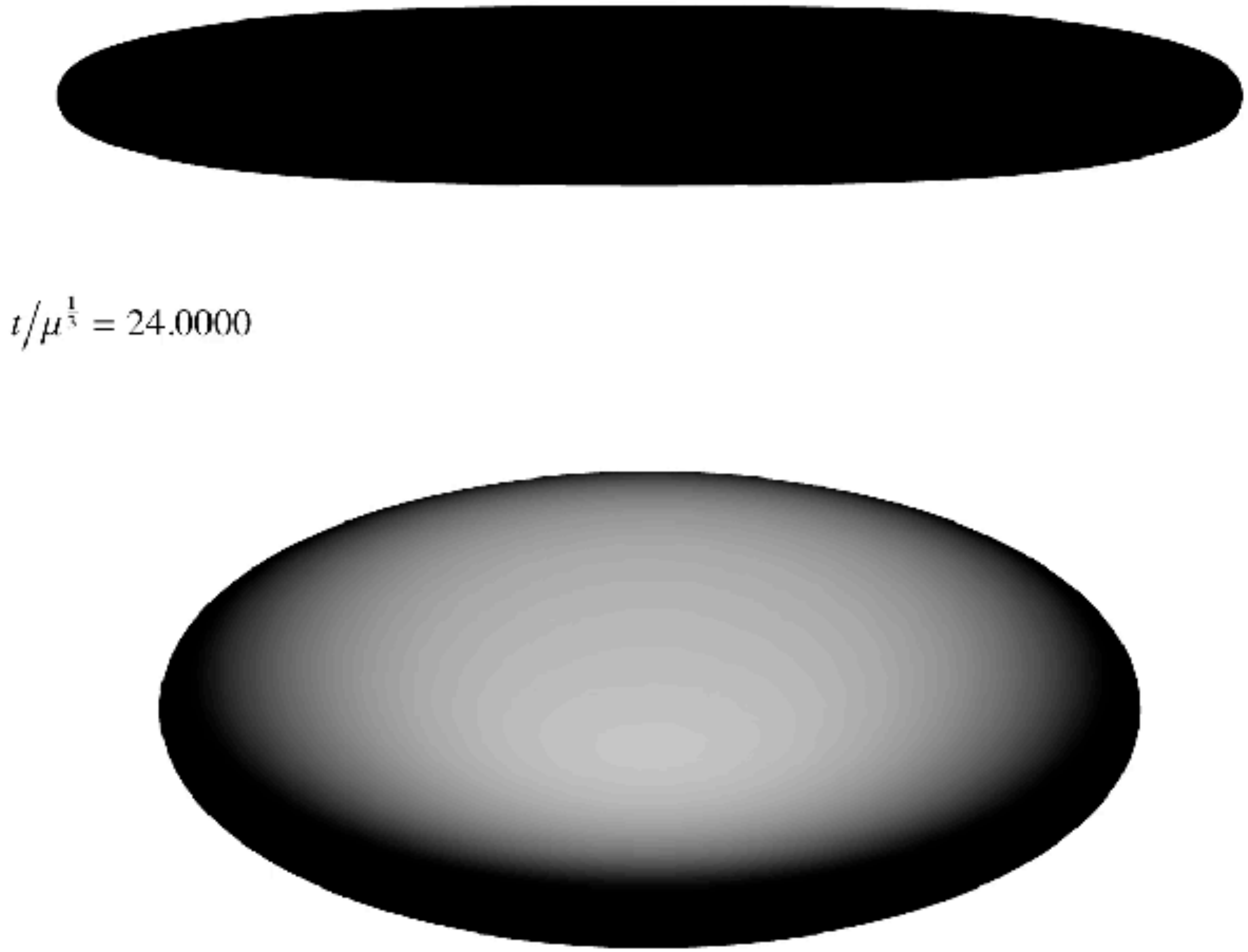
HIGHER DIMENSIONAL BLACK HOLES



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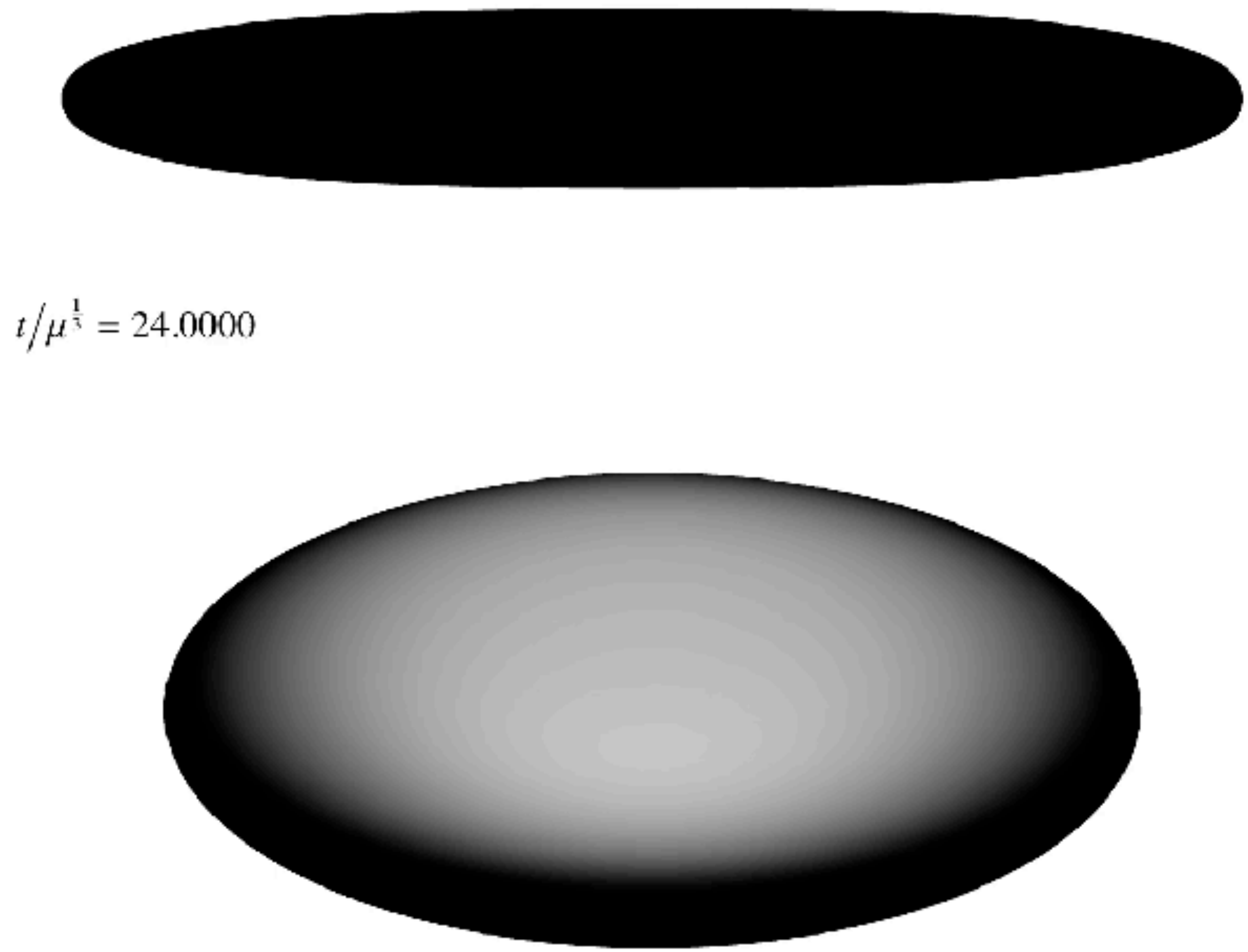

Numerical Relativity

Computer science:

- Codes development
- Super computers
- New technology (graphic card)
- Visualisation
- Data analysis (Einstein@Home)

Numerical Relativity

Mathematics

- Numerical analysis, algorithms
- Formulation of equations: 3+1, generalised harmonic gauge
- Non-linear PDE: well-posedness, existence
- Elliptic and strongly hyperbolic systems
- Lorentzian Geometry: causal structure, trapped surfaces
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- (Binary) Black hole dynamics
- Neutron Stars
- Gravitational Waves
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- New geometries (higher dimensional spacetimes, AdS)

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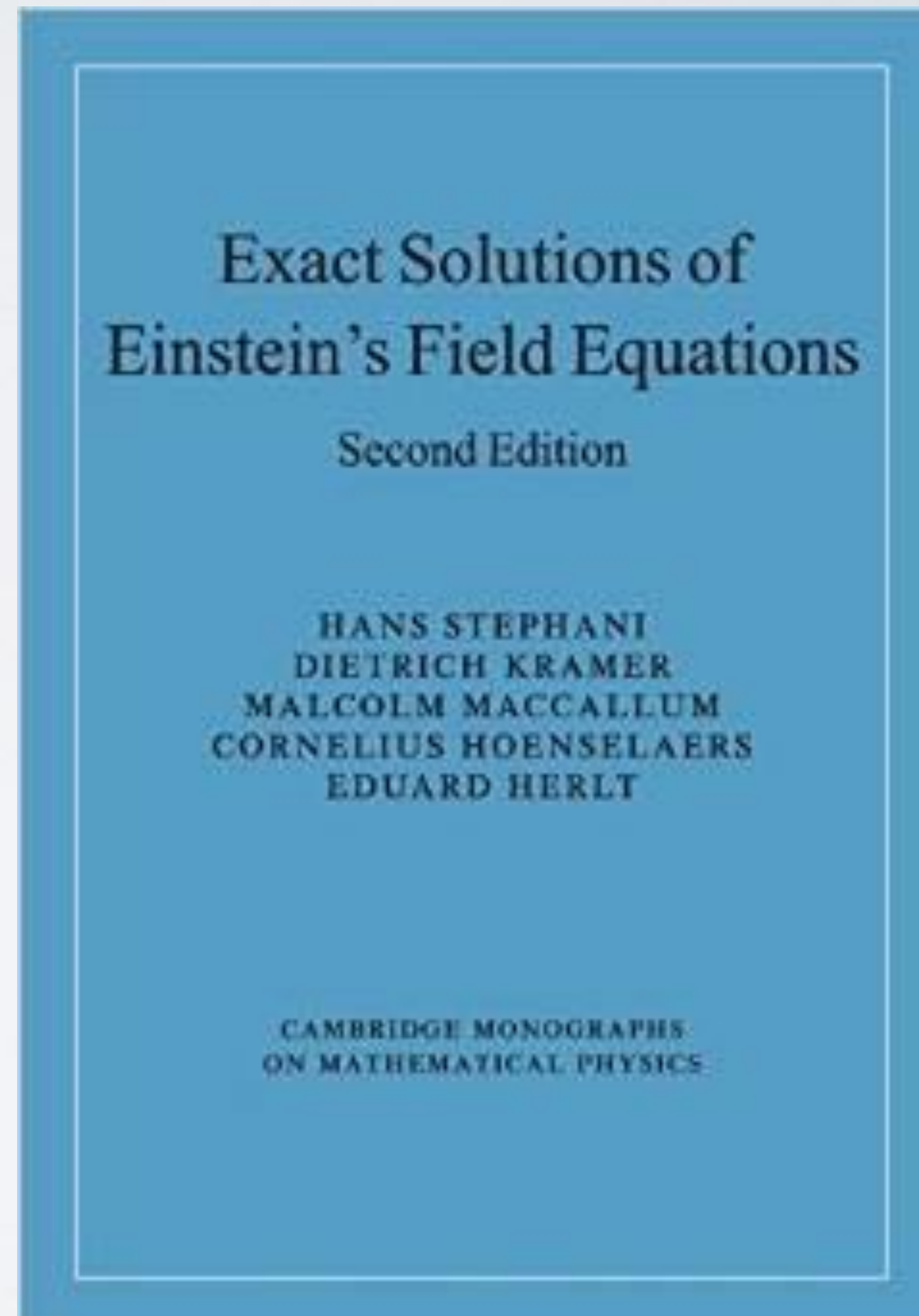
HISTORICAL BACKGROUND

- Department for Theoretical Physics (Uni Jena): H. Stephani, D. Kramer
- School of Mathematical Sciences (Queen Mary): M. MacCalulum

Long tradition on

Exact Solutions of
Einstein's Field Equations



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Generation techniques (P. Letelier)

then imply that σ satisfies the same equation (17.13) as ψ does for $A = 0$, i.e.

$$\sigma^{,a}{}_{;a} = 0 = (W\sigma_{,z})_{,z} - (W\sigma_{,t})_{,t} \qquad (10.110)$$

(Letelier 1975, Ray 1976). Taking now a vacuum solution of the form (10.109) and adding a function Ω to M,

$$\tilde{M} = M_{vac} + \Omega, \qquad (10.111)$$

one sees that the resulting Ricci tensor $\tilde{R}_{ab}$ is exactly of the form

$$\tilde{R}_{ab} = 2\sigma_{,a}\sigma_{,b}, \qquad (10.112)$$

provided the function Ω satisfies the equation

$$\Omega_{,a} = 2W\sigma^{,b}(2W_{,b}\sigma_{,a} - \sigma_{,b}W_{,a})/(W^{,n}W_{,n}). \qquad (10.113)$$

Note that only the metric function W enters this equation (M drops out).

Theorem 10.2 *If the metric (10.109) – with $W^{,a}W_{,a} \neq 0$ – satisfies the vacuum field equations $R_{ab} = 0$, then the perfect fluid field equations for a stiff fluid $\tilde{R}_{ab} = 2\sigma_{,a}\sigma_{,b}$ are satisfied by the metric $d\tilde{s}^2$ which differs from (17.4) by the substitution $\tilde{M} = M + \Omega$, where σ and Ω are solutions of (10.110) and (10.113), respectively* (Letelier and Tabensky (1975) for $A = 0 = \Psi$, Wainwright *et al.* (1979), Belinskii (1979)).

For $W^{,a}W_{,a} = 0$, $W \neq$ const, by an obvious generalization one can generate from a vacuum solution with $W = W(z+t)$ a pure radiation solution with $\sigma = \sigma(z+t)$ by adding an $\Omega = \omega_0(z-t) + \omega_1(z+t)$ with $\omega_1' = W(\sigma')^2/2W'$.

The technique described by the above theorem can be – and has been – applied to many solutions: the class of vacuum metrics (17.4) belongs to those metrics with an orthogonally transitive Abelian G_2 on S_2 where soliton techniques can be used to generate vacuum solutions, see above and Chapter 34. Moreover, since the metric function W does not change under these generation techniques, the same functions σ and Ω can be used for all vacuum solutions obtained from the same vacuum seed. A different way of looking at the class of solutions covered by Theorem 10.2 is to start from a stiff fluid solution, perform a transformation to a vacuum solution, apply a soliton generation technique and then go back to the stiff fluid: one then may speak of a solution describing solitons travelling in the background of a (particular) stiff fluid of higher symmetry. Of course, since σ and Ω need not be changed, it suffices to immediately transform only the vacuum part of the metric.

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The Neugebauer-Meinel rotating disk solution

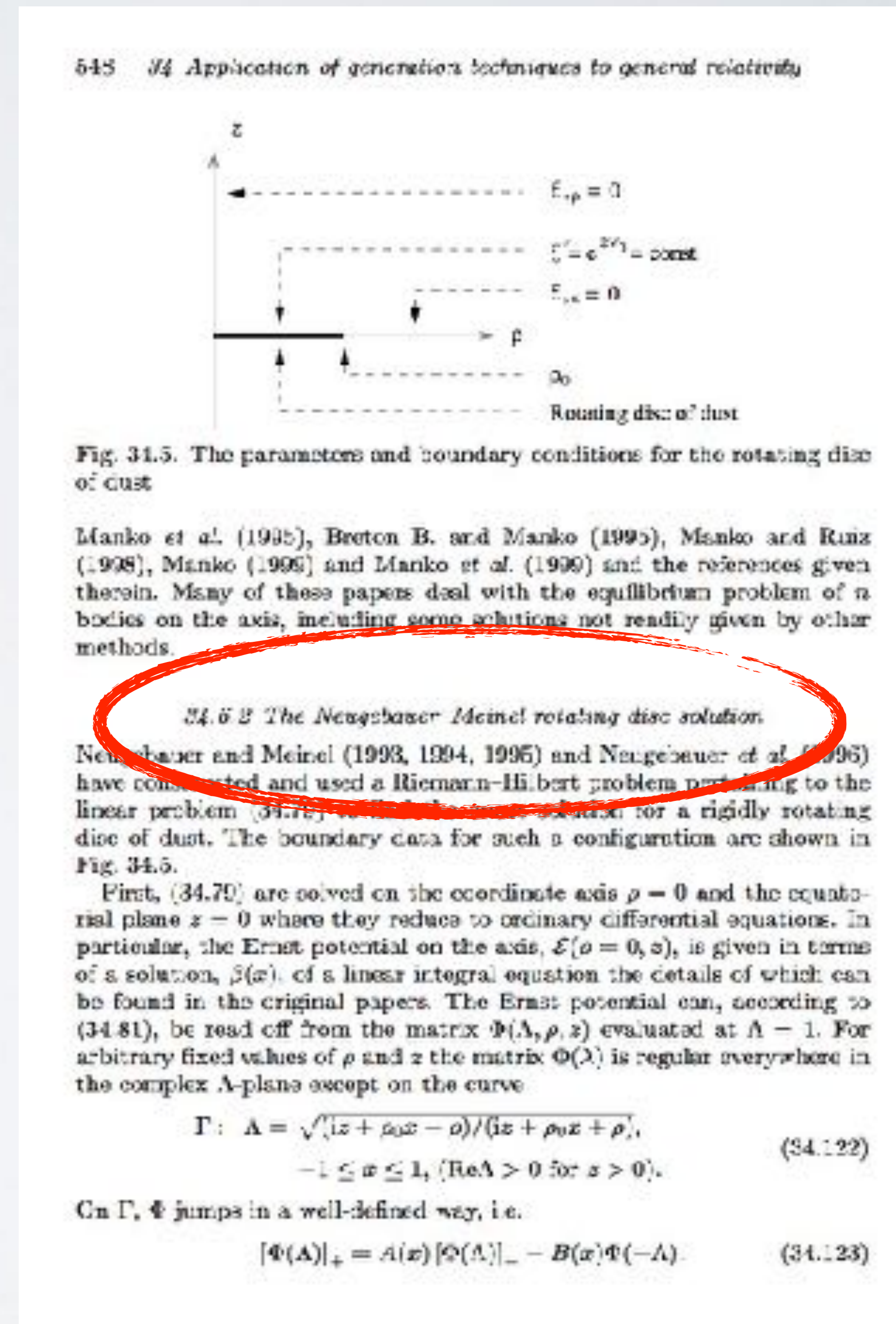

Fig. 34.5. The parameters and boundary conditions for the rotating disc of dust

Manko *et al.* (1995), Breton B. and Manko (1995), Manko and Ruiz (1998), Manko (1998) and Manko *et al.* (1999) and the references given therein. Many of these papers deal with the equilibrium problem of n bodies on the axis, including some solutions not readily given by other methods.

34.6.2 The Neugebauer Meinel rotating disc solution

Ne[illegible]bauer and Meinel (1993, 1994, 1995) and Neugebauer *et al.* ([illegible]996) have con[illegible]ted and used a Riemann–Hilbert problem pe[illegible]ing to the linear problem [illegible] for a rigidly rotating disc of dust. The boundary data for such a configuration are shown in Fig. 34.5.

First, (34.79) are solved on the coordinate axis $\rho = 0$ and the equatorial plane $z = 0$ where they reduce to ordinary differential equations. In particular, the Ernst potential on the axis, $\mathcal{E}(\rho = 0, z)$, is given in terms of a solution, $\beta(x)$, of a linear integral equation the details of which can be found in the original papers. The Ernst potential can, according to (34.81), be read off from the matrix $\Phi(\Lambda, \rho, z)$ evaluated at $\Lambda = 1$. For arbitrary fixed values of ρ and z the matrix $\Phi(\lambda)$ is regular everywhere in the complex Λ-plane except on the curve

$$\Gamma: \quad \Lambda = \sqrt{(iz + \rho_0 x - \rho)/(iz + \rho_0 x + \rho)}, \qquad -1 \le x \le 1, \; (\mathrm{Re}\Lambda > 0 \text{ for } z > 0). \tag{34.122}$$

On Γ, Φ jumps in a well-defined way, i.e.

$$[\Phi(\Lambda)]_+ = A(x)\,[\Phi(\Lambda)]_- - B(x)\Phi(-\Lambda). \tag{34.123}$$

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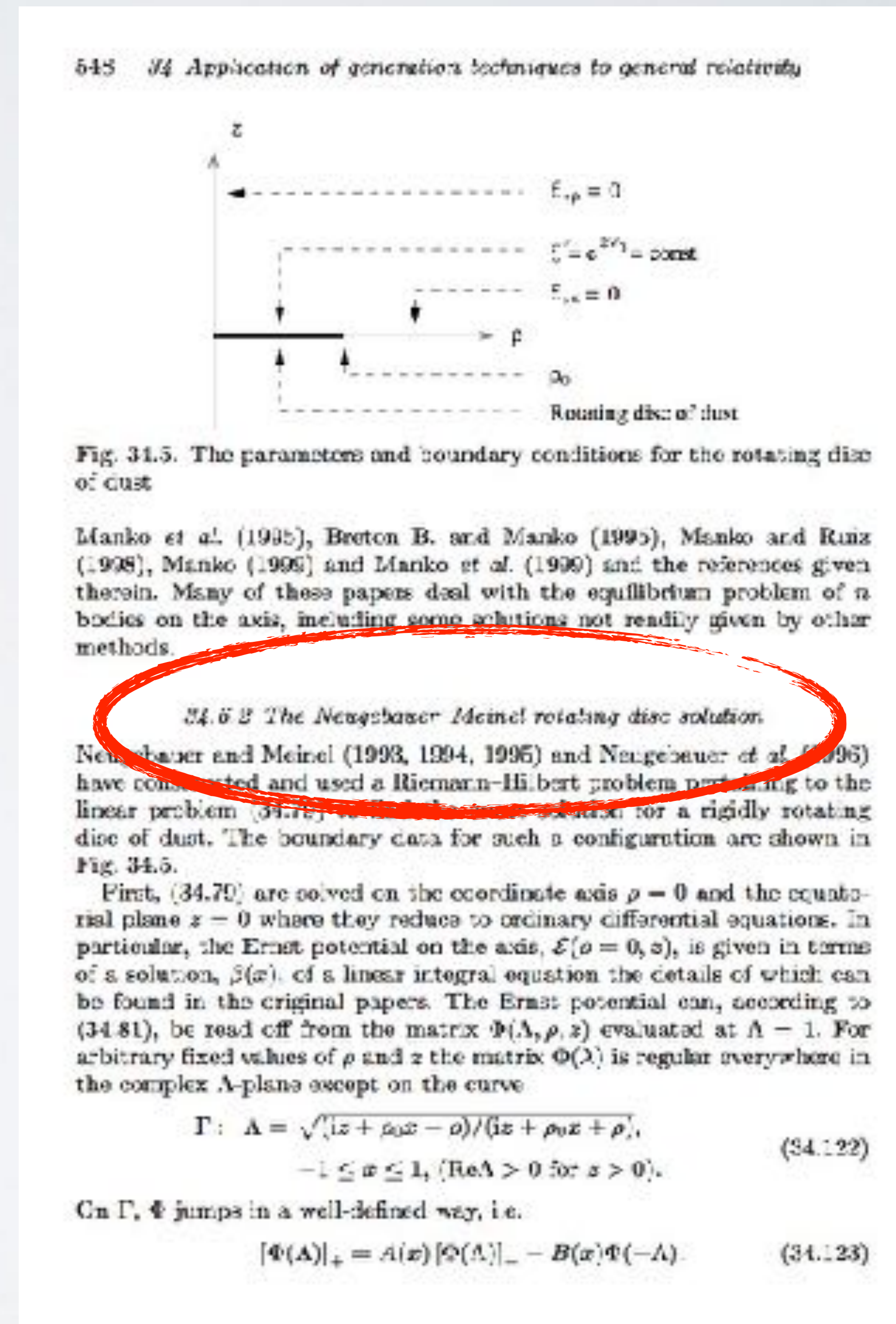
Exact Solutions are only found under very special assumptions and simplifications



548 *34 Application of generation techniques to general relativity*

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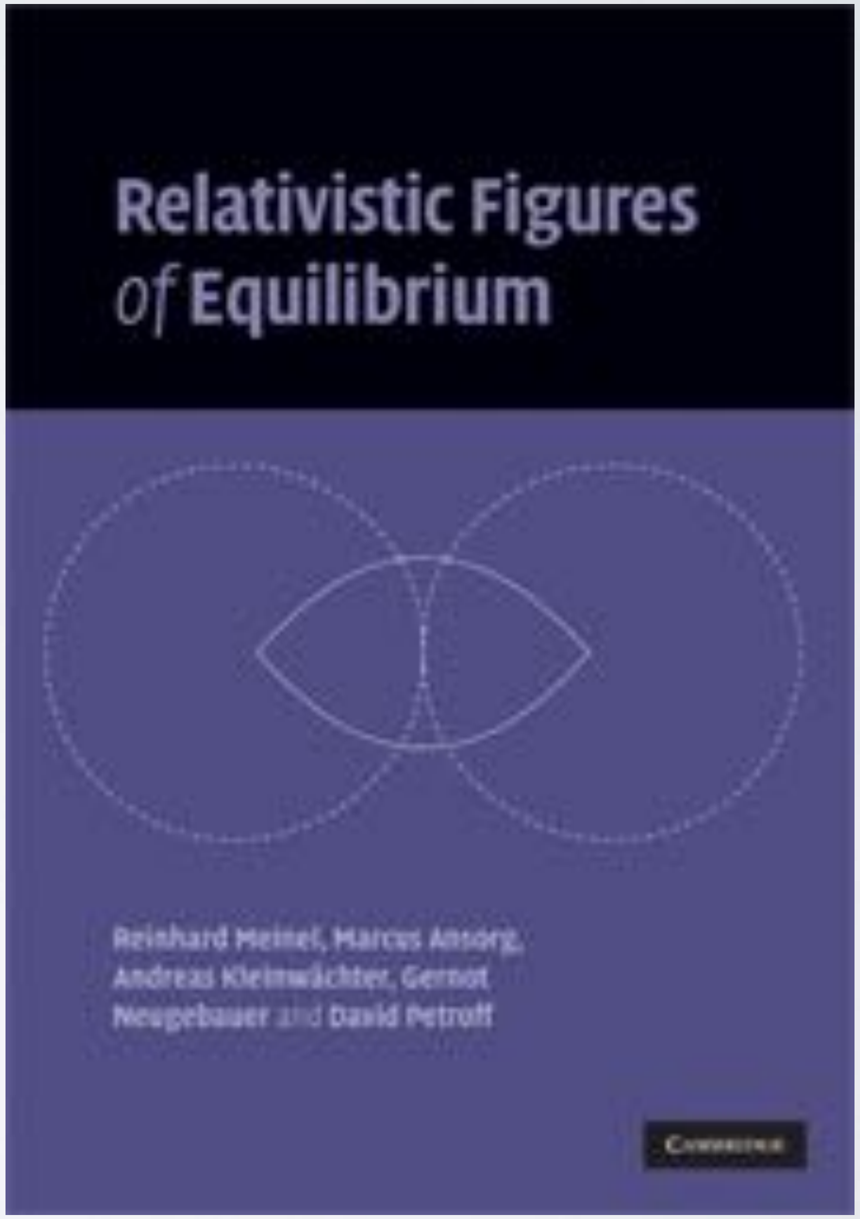
Shift from exact to numerical solution



(★1970; ✝2016)

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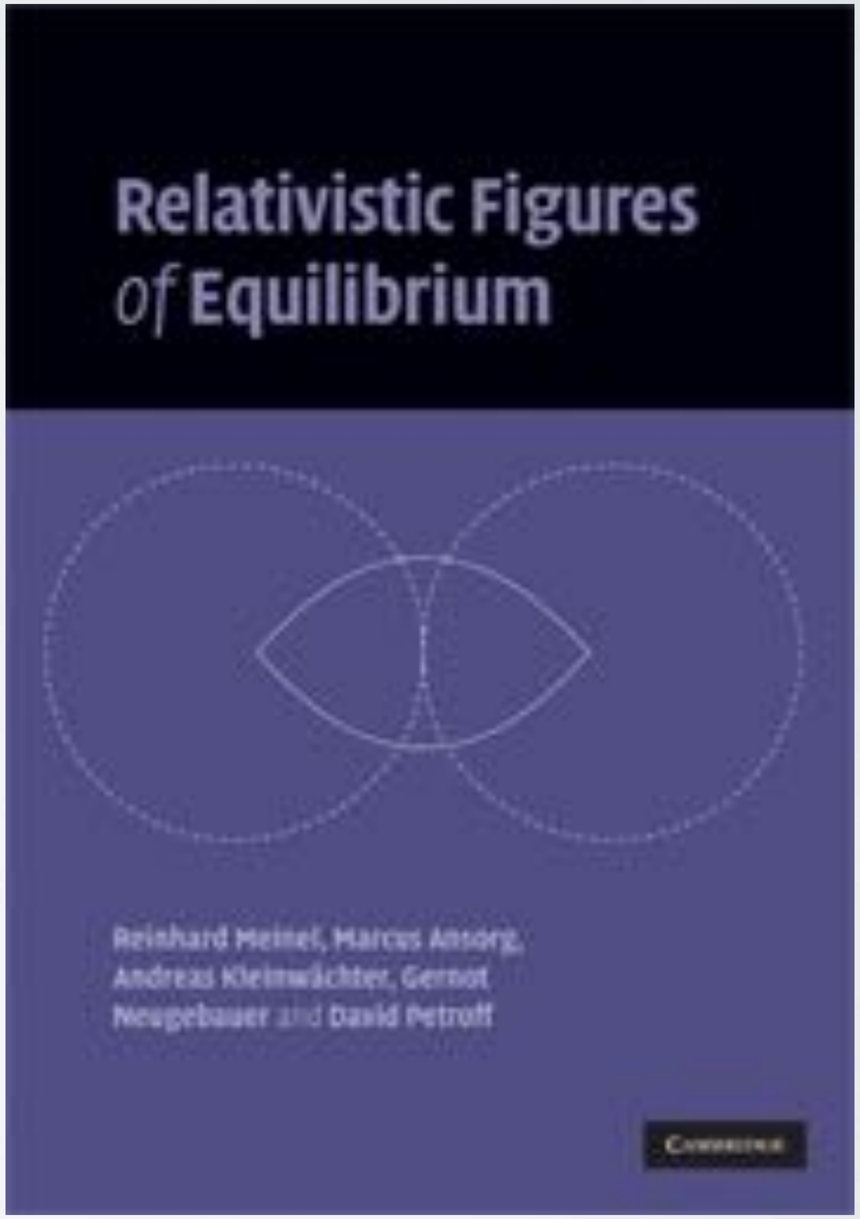
If numerical, then as exact as possible

(★1970; †2016)



HIGH ACCURACY METHODS

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Spectral Methods

Spectral methods for numerical relativity, G. Granclemént and J. Novak
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- ◉ Numerically expensive (there are faster algorithms)
- ◉ Relies on regularity properties of solution (be careful with discontinuities)

OUTLINE PART 1

- Introduction to spectral methods
- Collocation points
- Accuracy

- Derivatives
- Example 1: eigenvalue problems (Quasi-normal modes)

SPECTRAL METHODS

- Spectral expansion of a real valued function $f(x), \ x \in [a, b]$

$$f(x) = \sum_{i=0}^{N} c_i^{(N)} \phi_i(x) + R^{(N)}(x)$$

N : expansion order

$c_i^{(N)}$: spectral coefficients

$R^{(N)}(x)$: Residual Term

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- What are the functions $\phi_i(x)$?
- How to calculate the coefficients $c_i^{(N)}$?
- Does the series converge? At which rate?

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$f(x)$ is periodic

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- Orthogonal basis functions $\{\phi_i(x)\}$: solutions of eigenvalue problem of Sturm-Lioville-Theory

Trigonometric, Legendre, Laguerre, Chebyschev, Hermite, Bessel,…

- The spectral basis function $\phi_i(x)$ are chosen accordingly to the underlying properties of the original function $f(x)$

$f(x)$ is defined on sphere

SPECTRAL METHODS

- Spectral expansion of a real valued function $f(x), \;\; x \in [a, b]$

$$f(x) = \sum_{i=0}^{N} c_i^{(N)} \phi_i(x) + R^{(N)}(x)$$

Basis Functions

N : expansion order

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SPECTRAL METHODS

MORAL PRINCIPLE 1:

(J.P. Boyd: Chebyshev and Fourier Spectral Methods)

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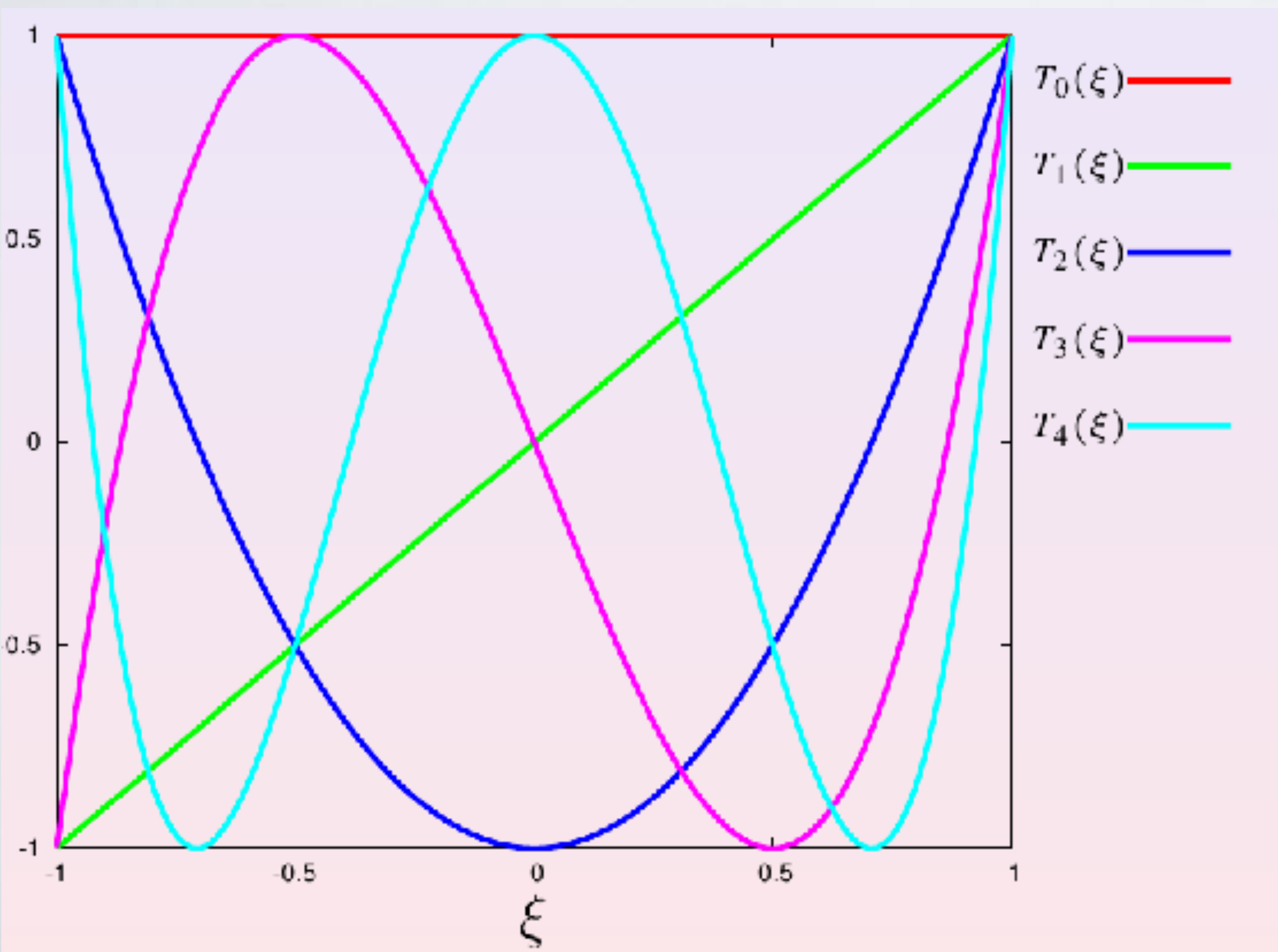
CHEBYSHEV POLYNOMIALS

- Basis functions: $\phi_i(x) = T_i(\xi(x))$. Linear map $\xi : [a,b] \to [-1,1]$

- Chebyshev Polynomials of first kind $T_i(\xi) = \cos(i \arccos \xi)$



if $\xi = \cos(\theta)$, $\theta \in [0, \pi]$

$$T_i(\theta) = \cos(i\theta)$$

- Orthogonality

$$\int_{-1}^{1} \frac{T_i(\xi) T_k(\xi)}{\sqrt{1-\xi^2}} d\xi = \mathrm{N}_i \delta_{ij}$$

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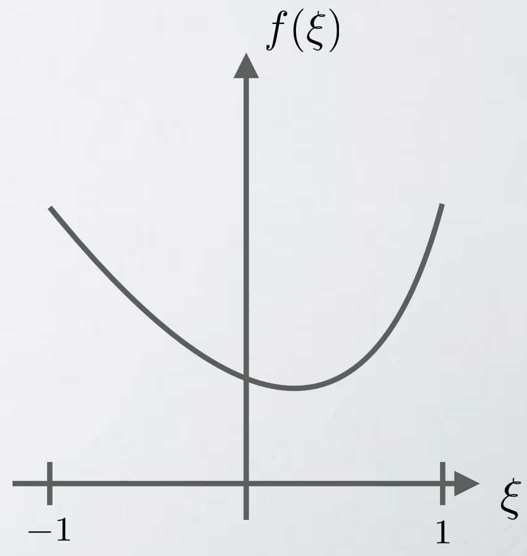
Spectral Coefficients

- Different approaches leads to different methods: Galerkin Method, Tau-method, **Collocation method**

COLLOCATION METHOD

- Spectral Representation on domain $[-1, 1]$

$$f(\xi) = \sum_{i=0}^{N} c_i^{(N)} T_i(\xi) + R^{(N)}(\xi)$$



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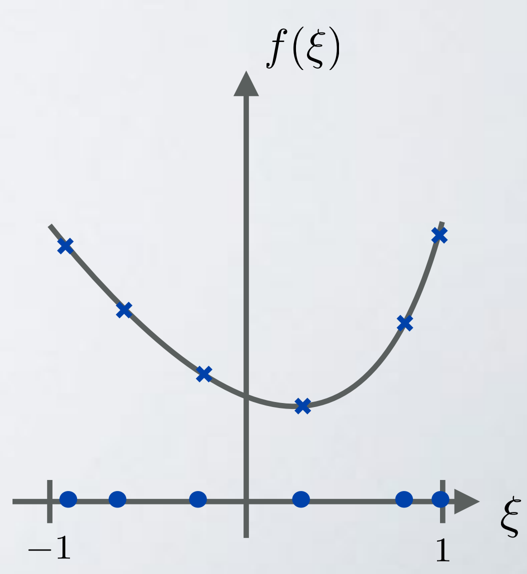
$$\{\xi_k\} \quad k = 0 \ldots N$$



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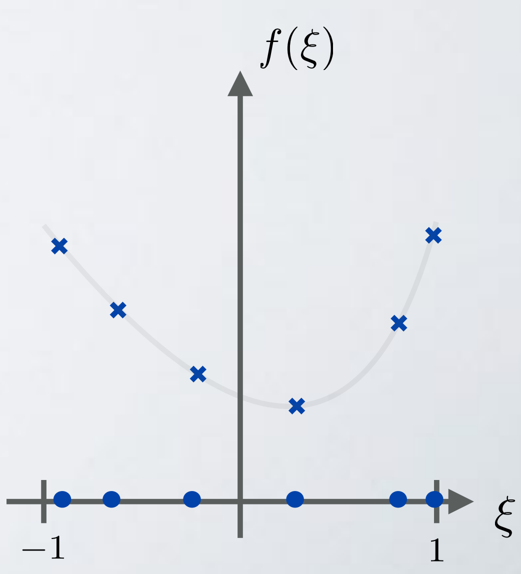

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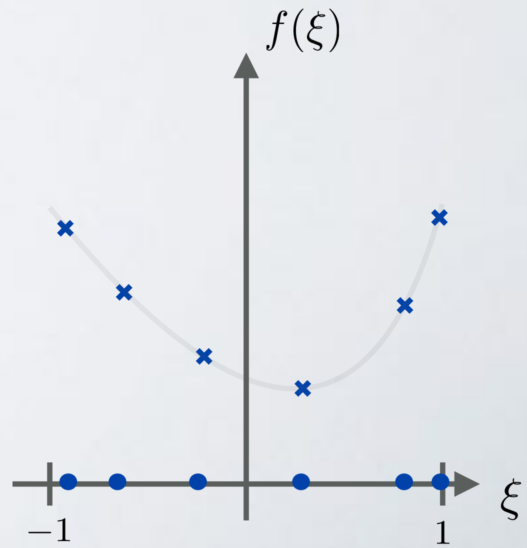

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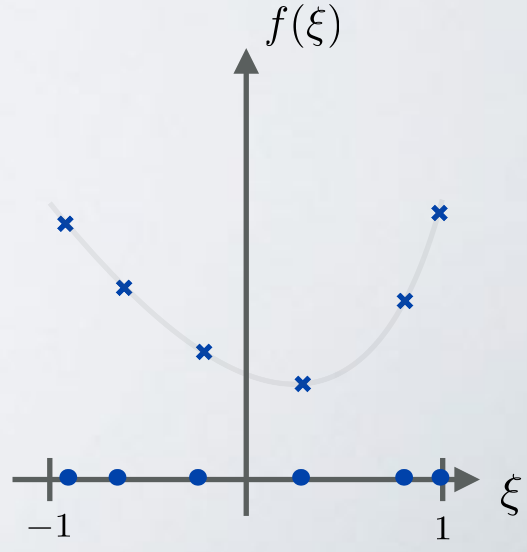

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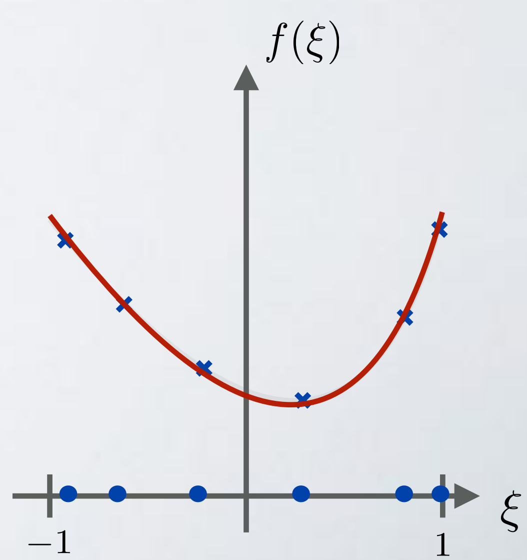

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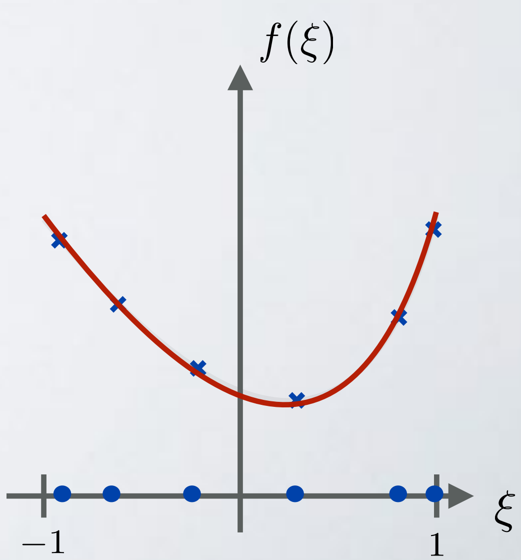
- Truncation error

$$||R^{(N)}(\xi)|| = ||f(\xi) - \tilde{f}(\xi)||$$



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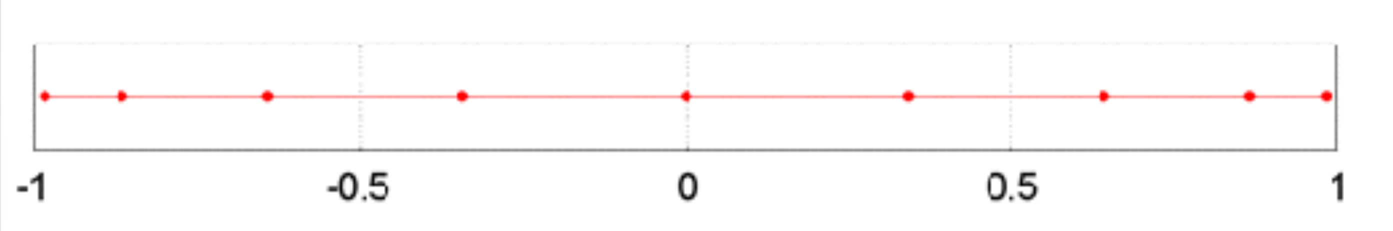

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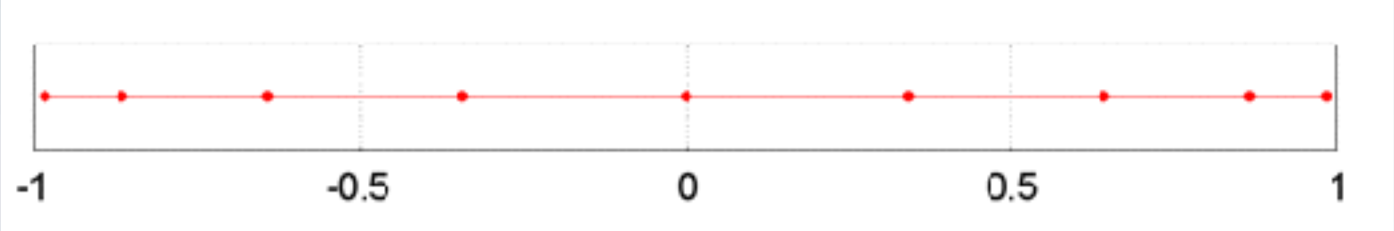

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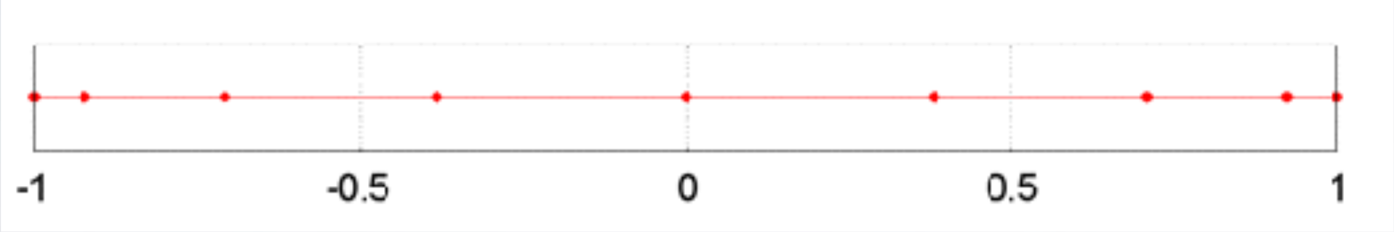

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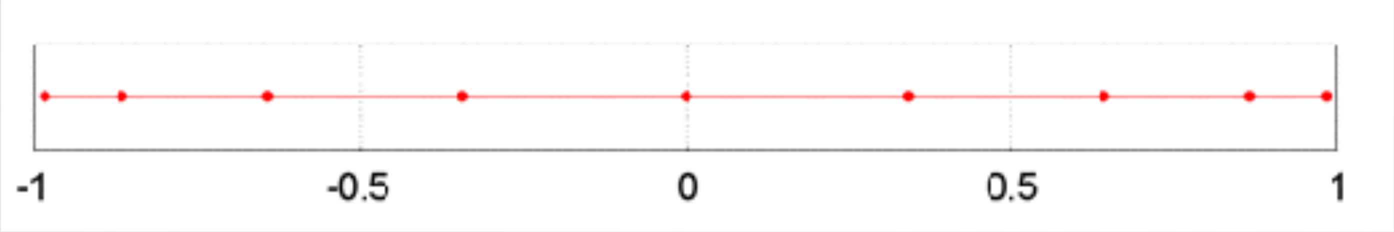

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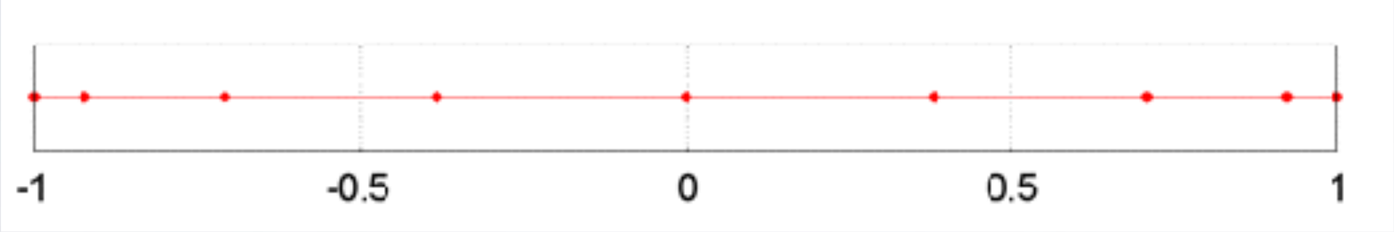

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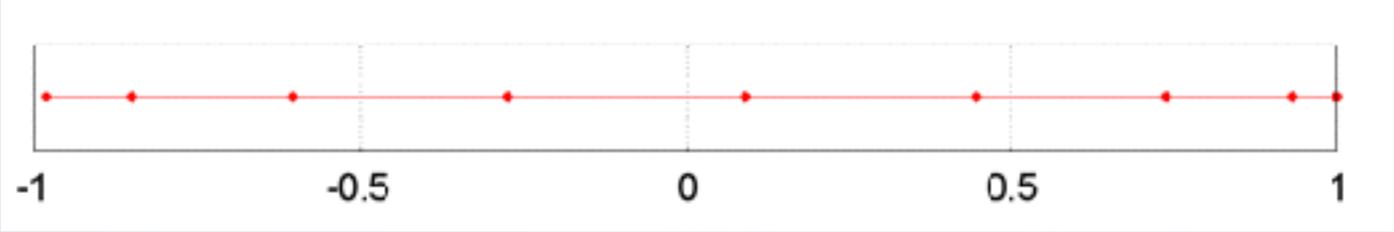

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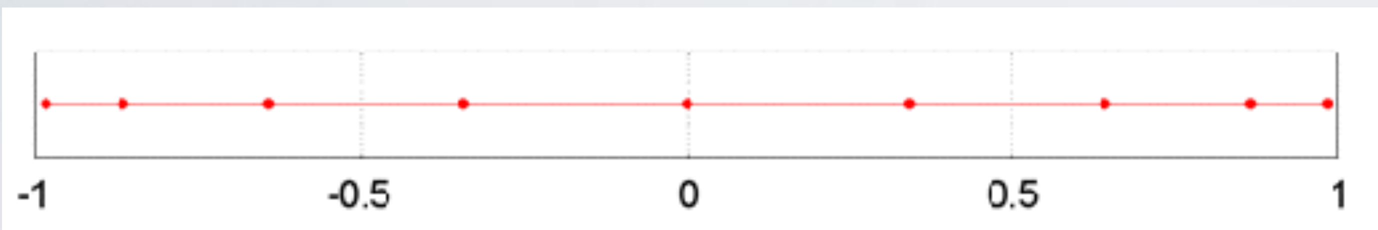

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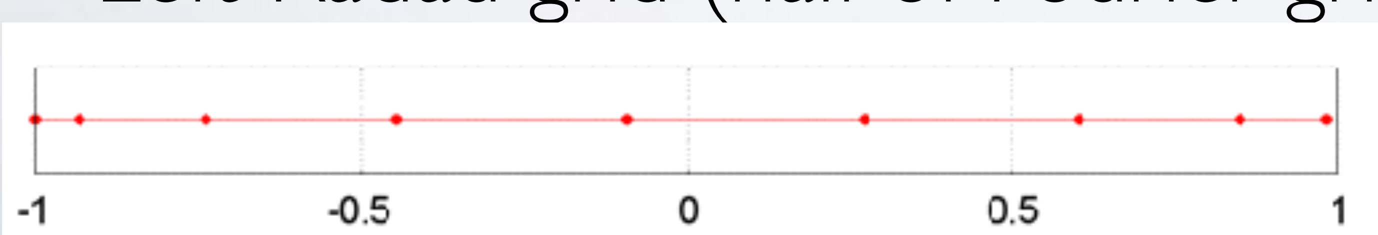

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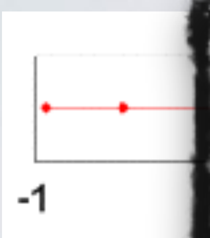

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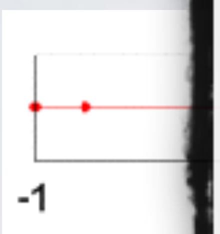

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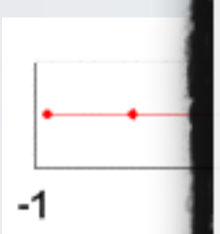

Discrete orthogonality

- With respect to grid points

$$\sum_{i}^{N} T_{i}(\xi_j) T_{i}(\xi_k) = \tilde{N}_j \delta_{jk}$$

- With respect to degree

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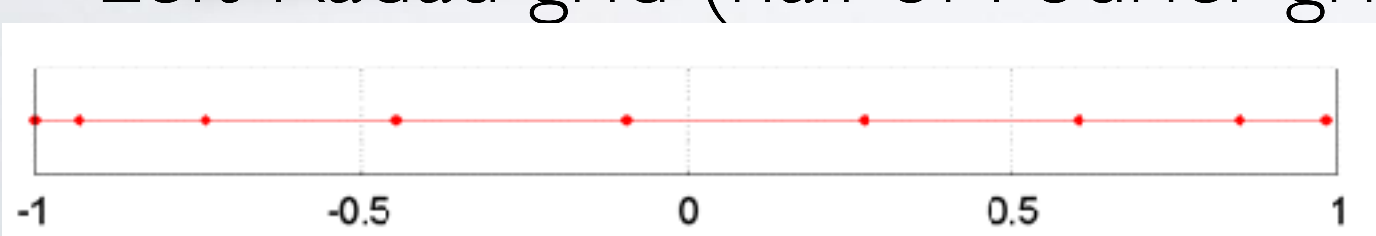

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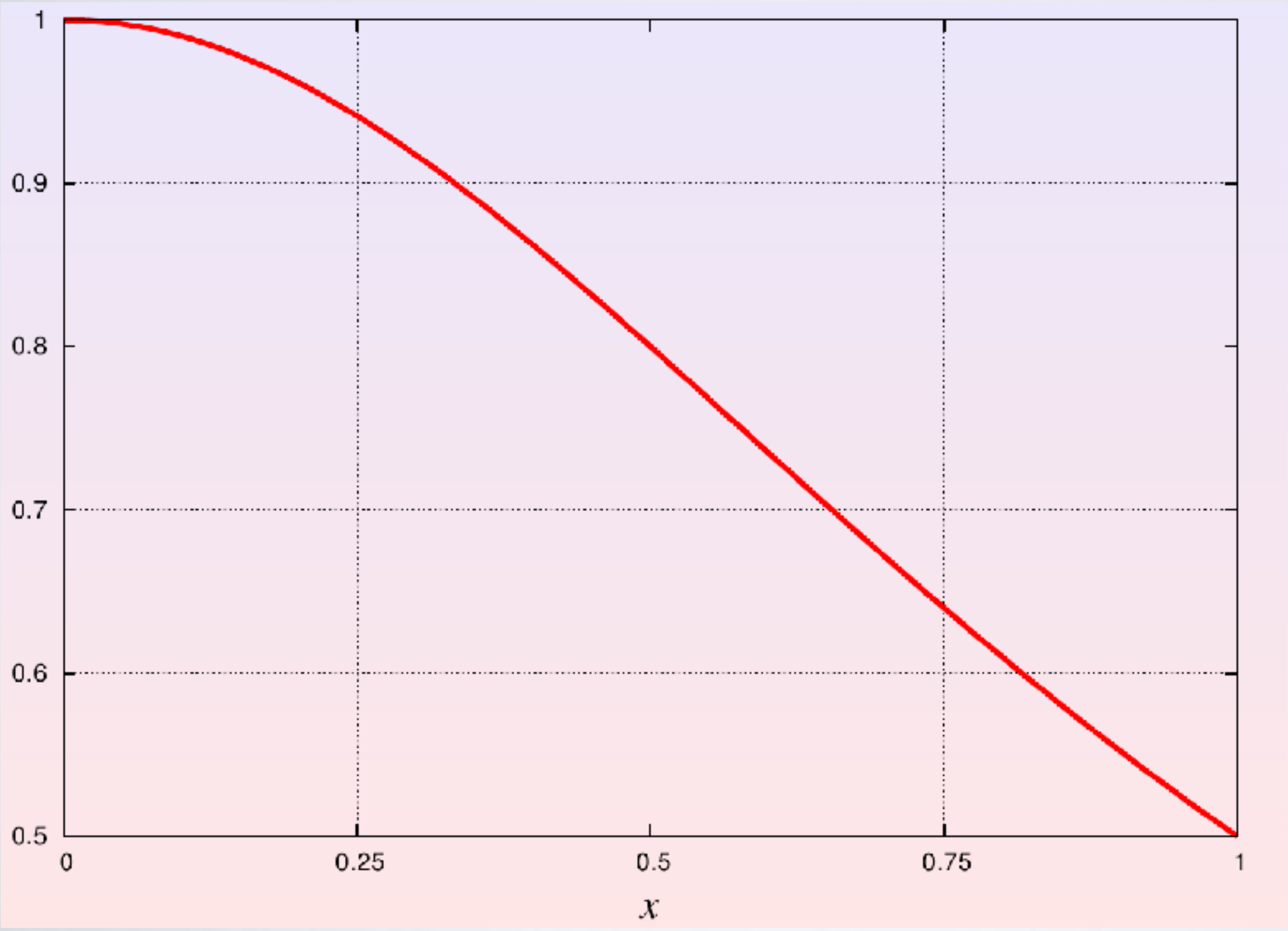
EXAMPLE: ACCURACY

- Analytical Function: $\mathcal{C}^\omega$

(Taylor expansion in the neighbourhood of any $x \in [0,1]$)

$$f_1(x) = \frac{1}{1+x^2}$$



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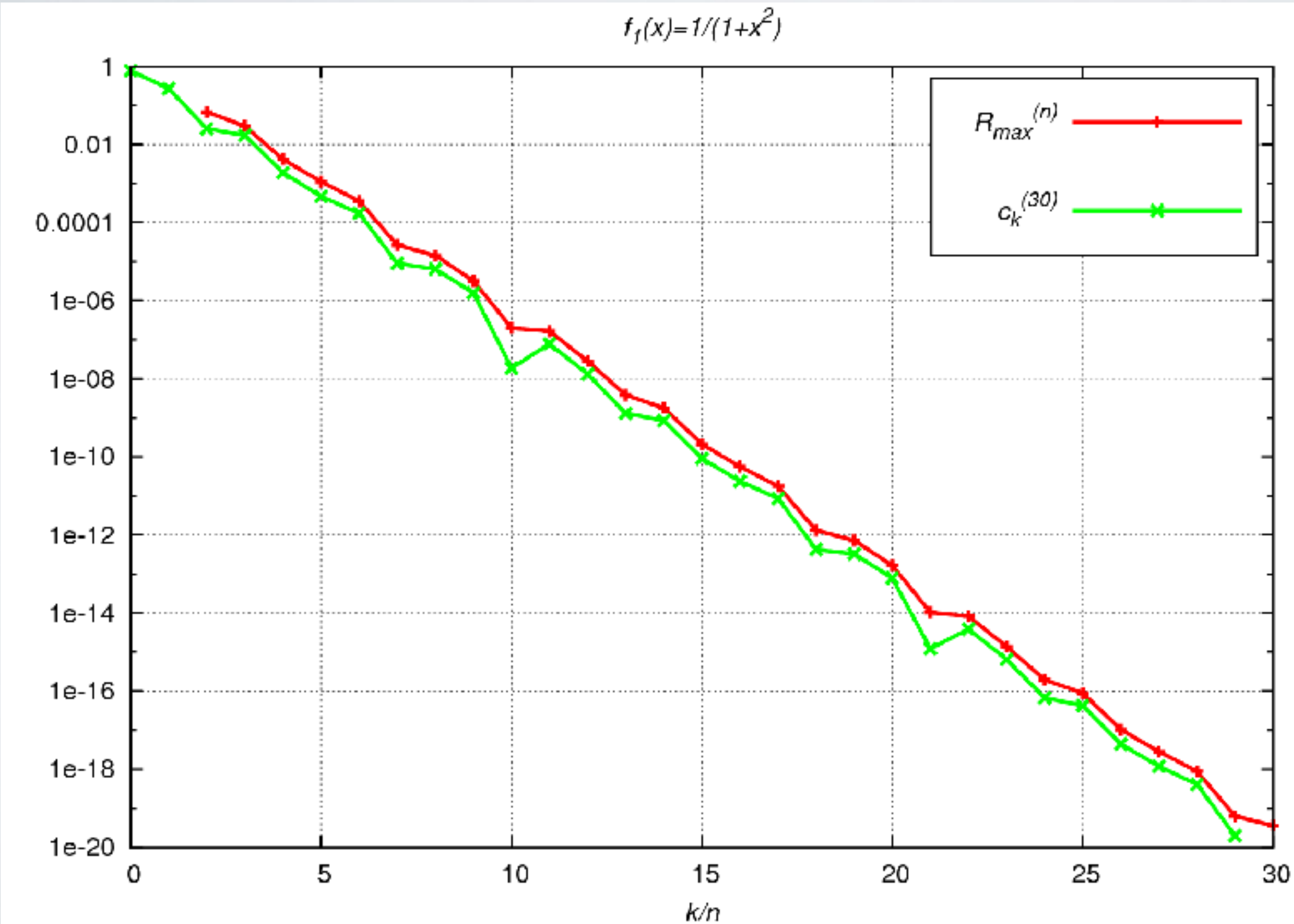

$$\text{Error} = \alpha e^{-\beta N}$$

Exponential Decay

Coefficients behaviour mirrors error

EXAMPLE: ACCURACY

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$$f_2(x) = \frac{\epsilon}{\epsilon + x}$$

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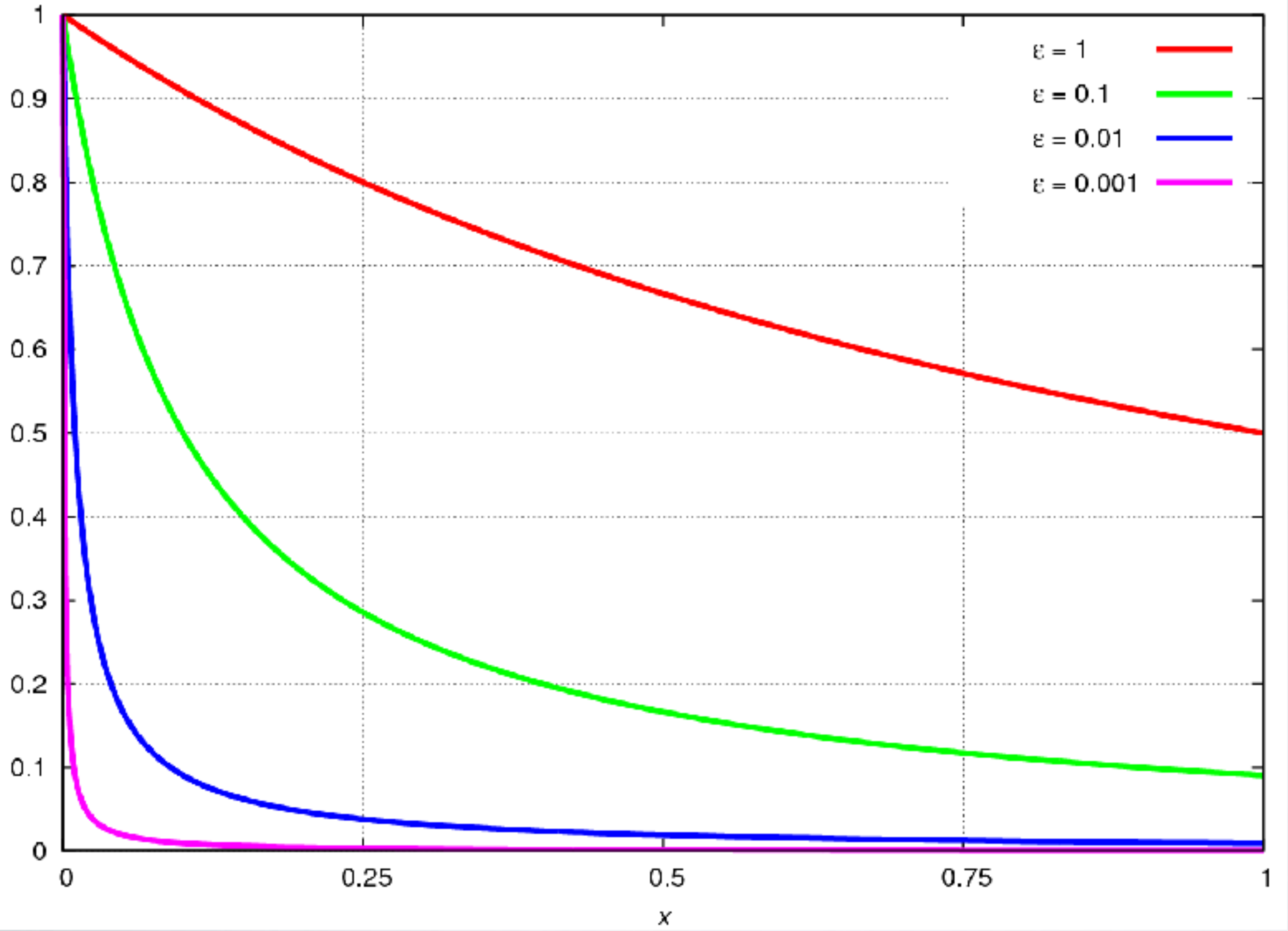

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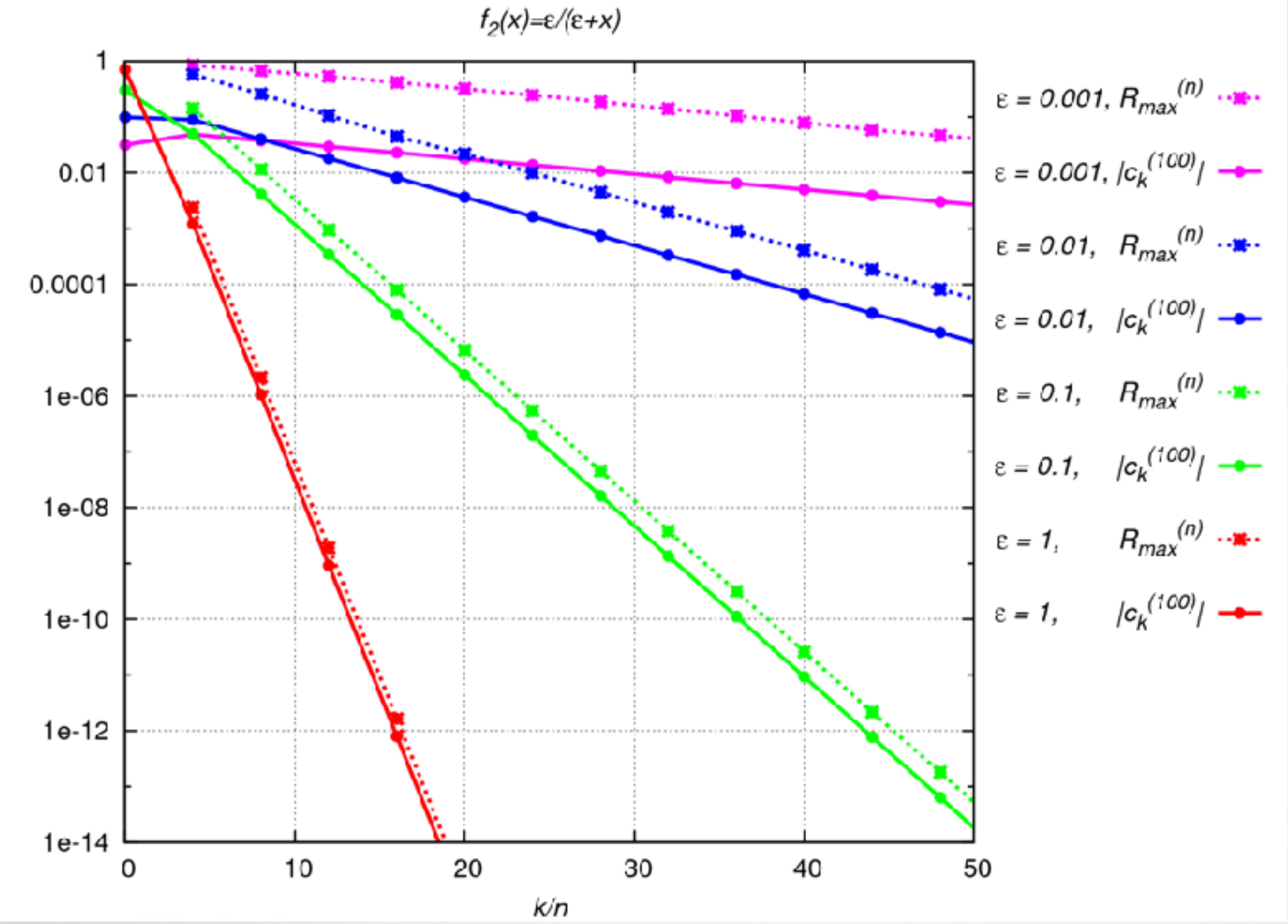

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(slower slope)

Coefficients behaviour mirrors error

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- Continuous ℓ-differentiable Functions: $\mathcal{C}^\ell$

$$g_k(x) = x^k \ln x$$
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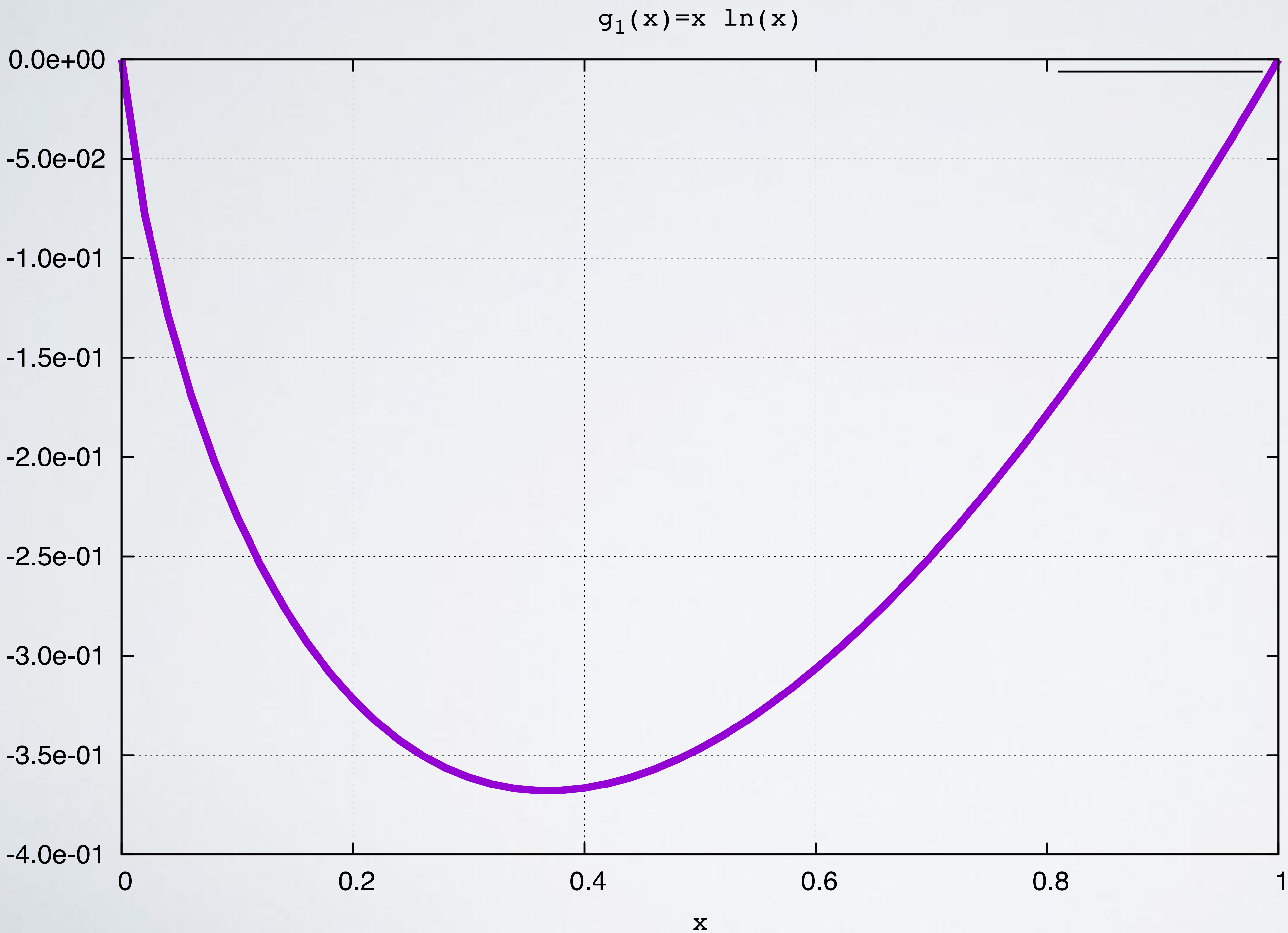

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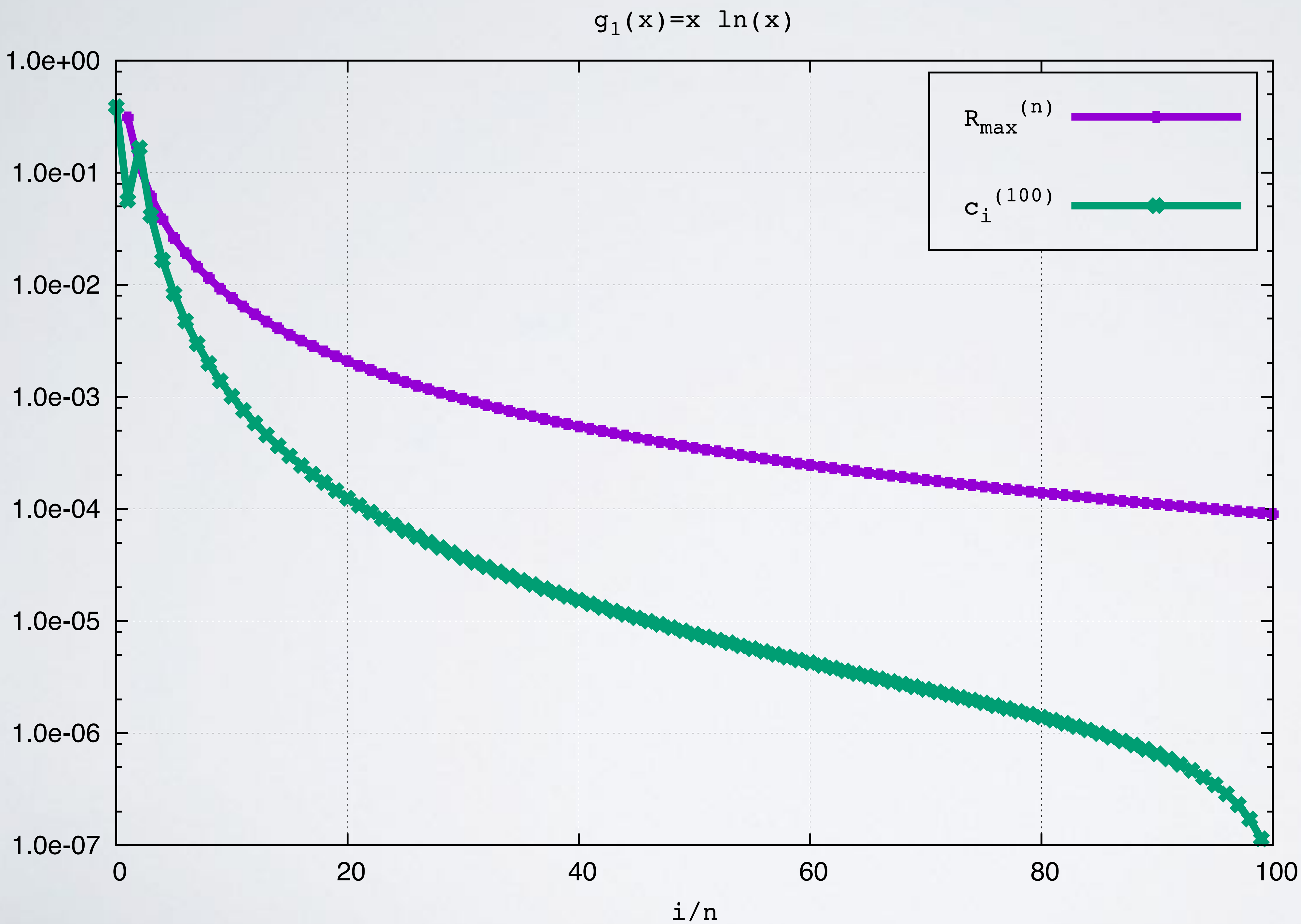

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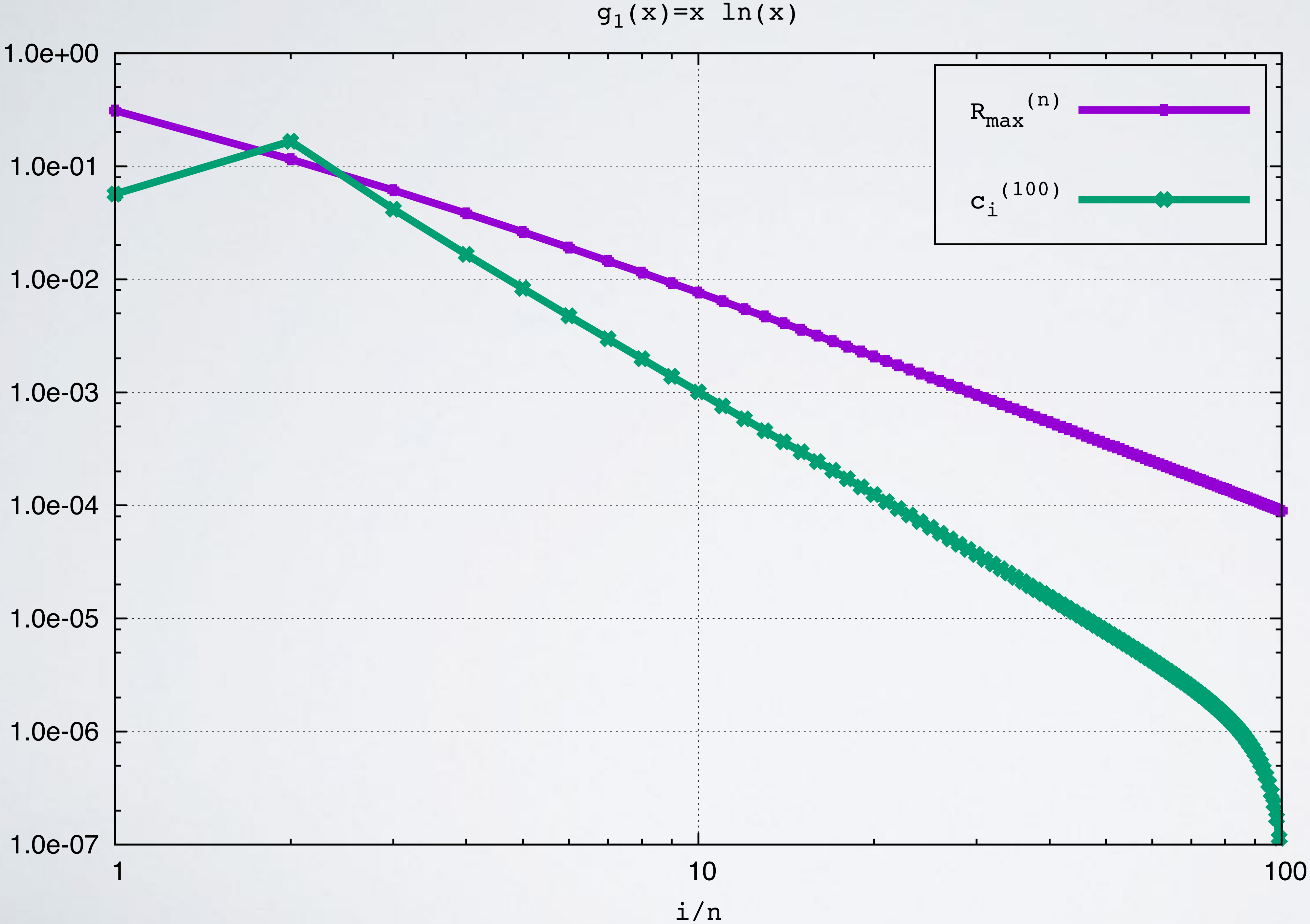

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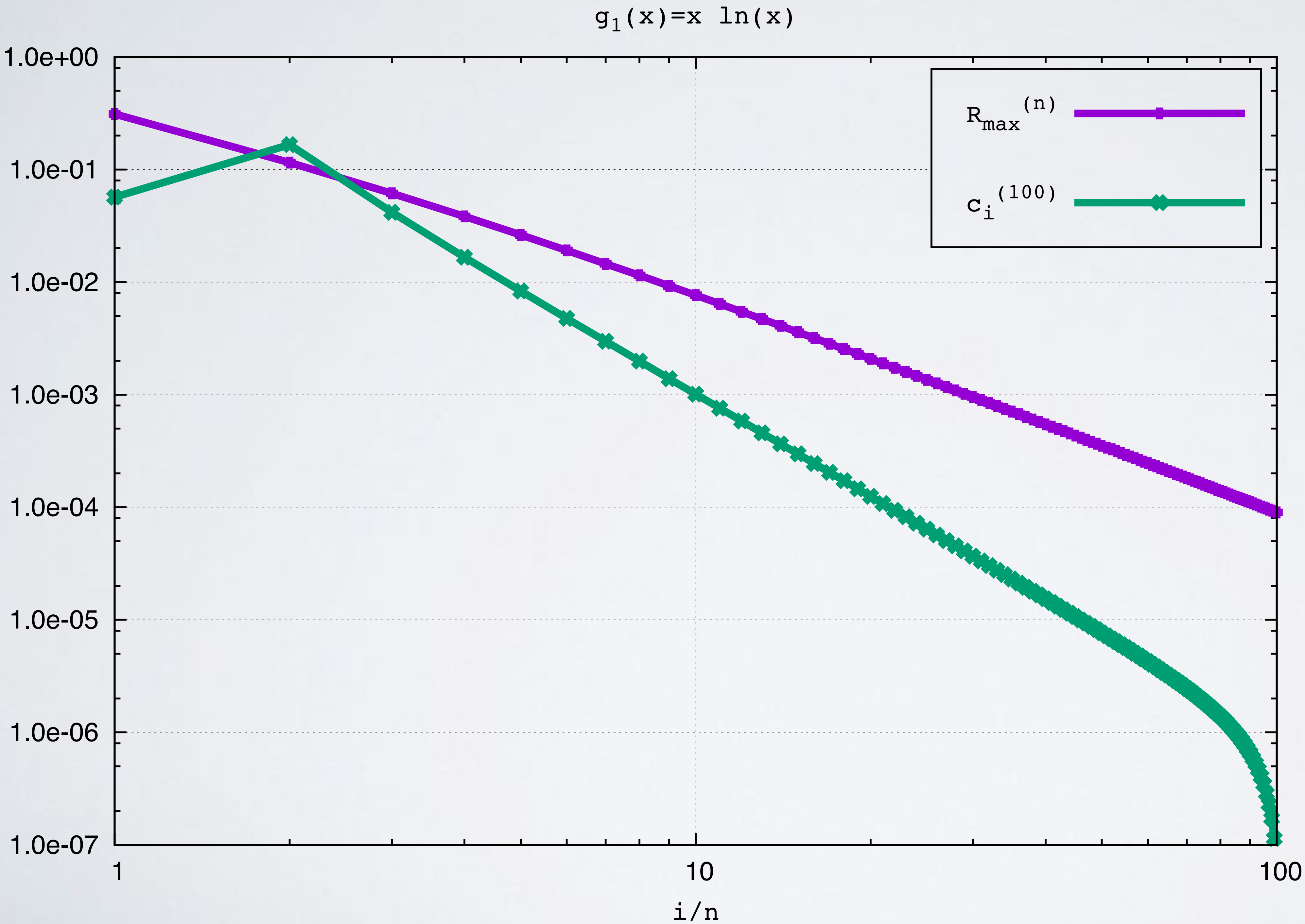

$$\text{Error} = AN^{-p}$$
$$p = 2k$$

Algebraic Decay

$$c_i = \tilde{A} i^{-\tilde{p}}$$
$$\tilde{p} = 2k + 1$$

Coefficients behaviour similar to error

SUMMARY: ACCURACY

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  ➡Chebyshev coefficients **must** decay exponentially

DERIVATIVES

- Interpolate at any point ξ : $$\tilde{f}(\xi) = \sum_{i=0}^{N} c_i^{(N)} T_i(\xi)$$
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are calculated exactly using properties of the Chebyshev Polynomials

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- The derivative reads: $$\frac{d}{d\xi}\tilde{f}(\xi) = \sum_{i=0}^{N} c_i^{(N)} \frac{d}{d\xi} T_i(\xi)$$

- In particular, the derivative of a Chebyshev Polynomial can be expressed in terms of the Chebyshev basis

$$\frac{d}{d\xi} T_i(\xi) = \sum_{j=0}^{N} d_{ij}^{(N)} T_j(\xi)$$

$$\downarrow$$

$$\frac{d}{d\xi}\tilde{f}(\xi) = \sum_{j=0}^{N} \underbrace{\sum_{i=0}^{N} c_i^{(N)} d_{ij}^{(N)}}_{c'_j} T_j(\xi)$$

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Derivative Coefficients algorithm

- Given resolution N and Chebyshev Coefficients $c_i^{(N)}$
- Recurrence relation for derivative coefficients $c'^{(N)}_i$
- Start with $c'^{(N)}_{N+1} = c'^{(N)}_N = 0$
- Run backward for $i = N \cdots 1$

$$c'_{i-1} = 2kc_i + c'_{k+1}$$

DISCRETE DERIVATIVES

- Consider the derivative at the grid points $\{\xi_k\}$

$$\frac{d}{d\xi}\tilde{f}(\xi_k) = \sum_{j=0}^{N}\sum_{i=0}^{N} c_i^{(N)} d_{ij}^{(N)} T_j(\xi_k)$$

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- Consider relation to find coefficients $\vec{c} = \hat{T}^{-1}\vec{f}$

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$$f'_k = \sum_{\ell=0}^{N} D_{k\ell} f_\ell \Rightarrow \vec{f}' = \hat{D}\vec{f}$$

- Spectral differentiation matrices: $\hat{D}$

DISCRETE DERIVATIVES

- Continuous derivative operator: ∂_μ
- Discrete derivative operator: $\hat{D}_\mu$

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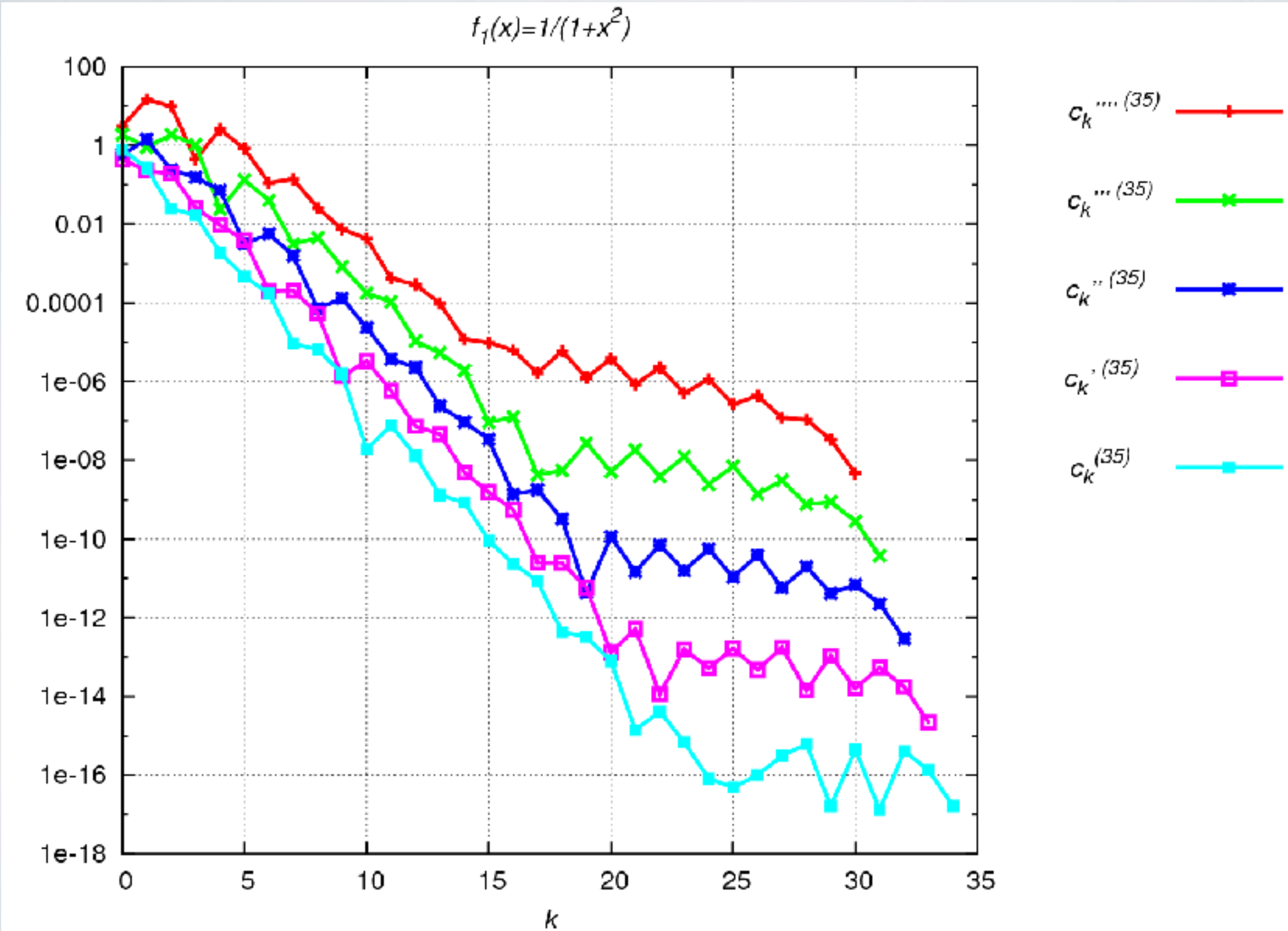
➡ Contains a lot of information (see example eigenvalue problem)

DISCRETE DERIVATIVES



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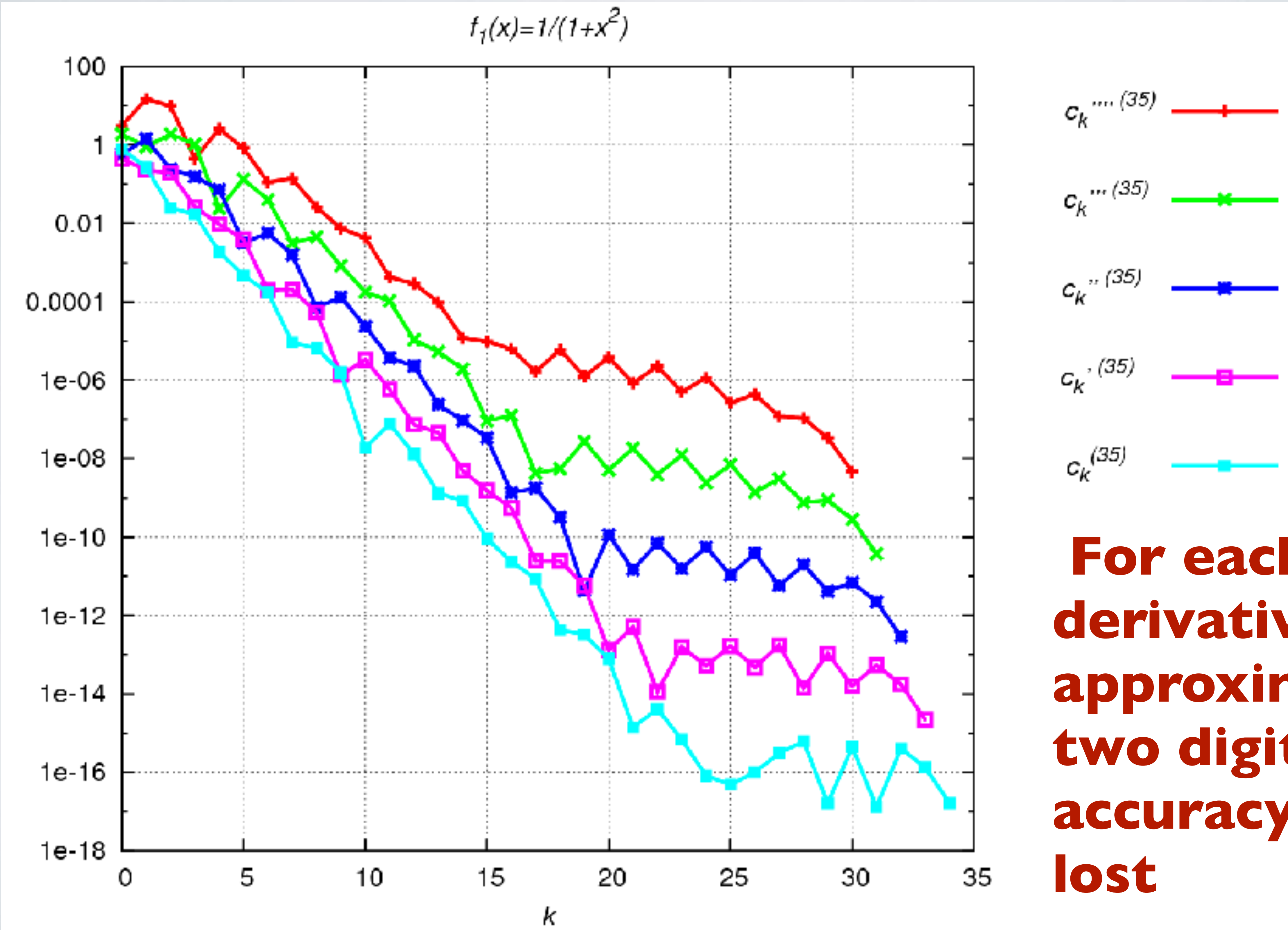

For each derivative, approximately two digits of accuracy are lost

EXAMPLE 1: QNM

- **Perturbation Theory: typical problem**

- Stationary (black-hole) spacetime as background
- Consider fields propagating on background
- Linearise or consider a linear theory
- Wave equation
- Take boundary conditions into account
- Fourier (or Laplace) transformation
- Non-trivial solutions to the homogenous equation (eigenvalue problem)

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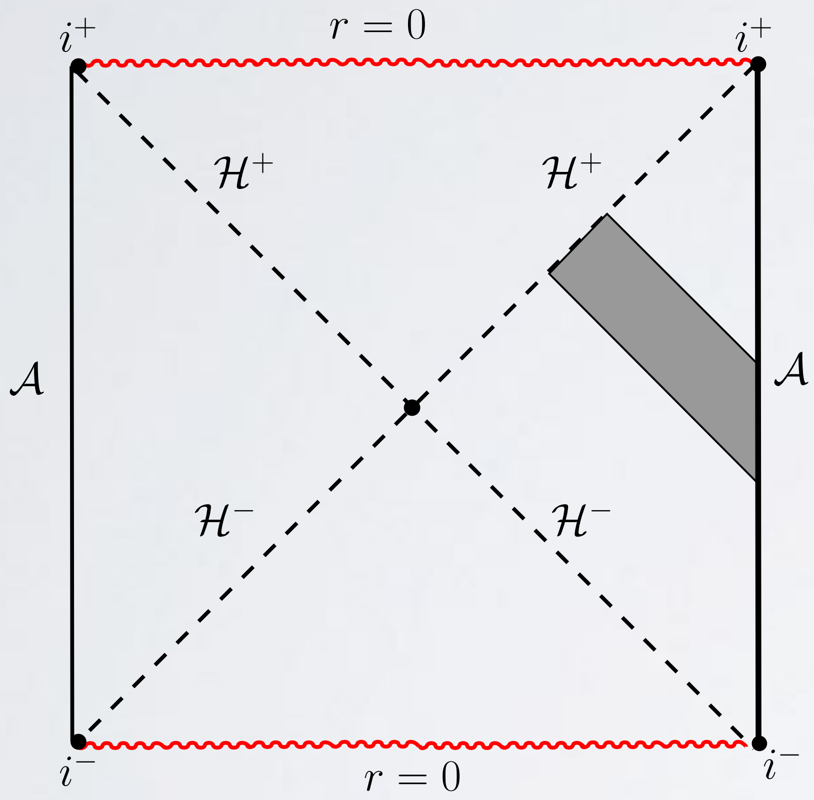
M. AMMON, S.GRIENINGER, A.J. ALBA, RPM, L. MELGAR, JHEP 09 (2016) 131

- **Black brane background spacetime:** Schwarzschild-AdS in 5D

$$ds^2 = \frac{1}{\rho^2}\left(-f(\rho)dv^2 - 2\,dvd\rho + dx^2 + dy^2 + dz^2\right) \qquad f(\rho) = 1 - \rho^4$$



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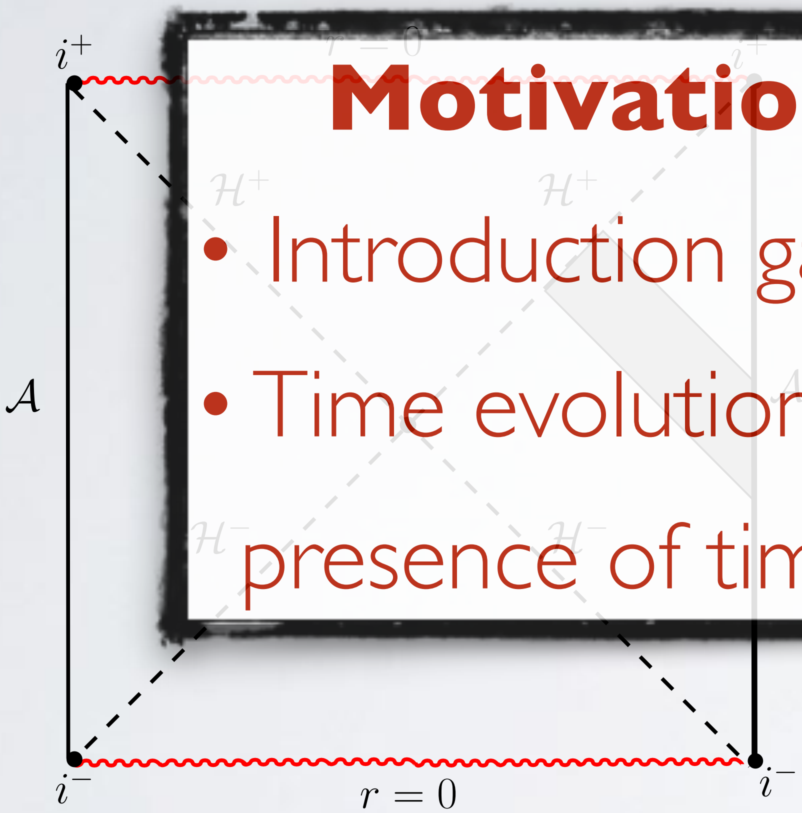

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Motivation AdS/CFT Correspondence

- Introduction gauge/gravity duality (O. Miskovic)
- Time evolution of the chiral magnetic effect in the presence of time dependent axial charges

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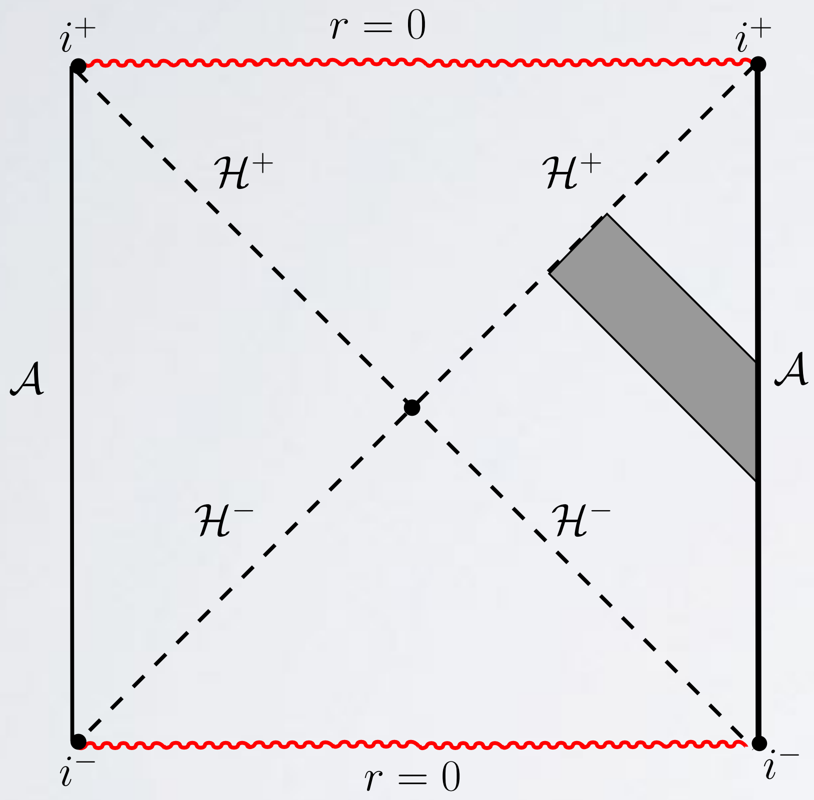

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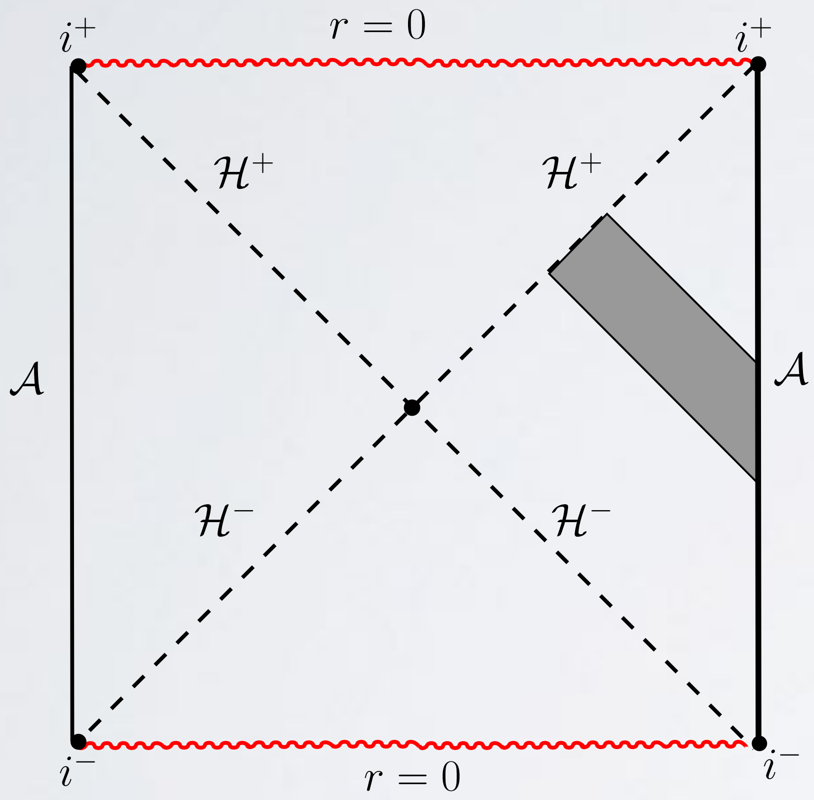

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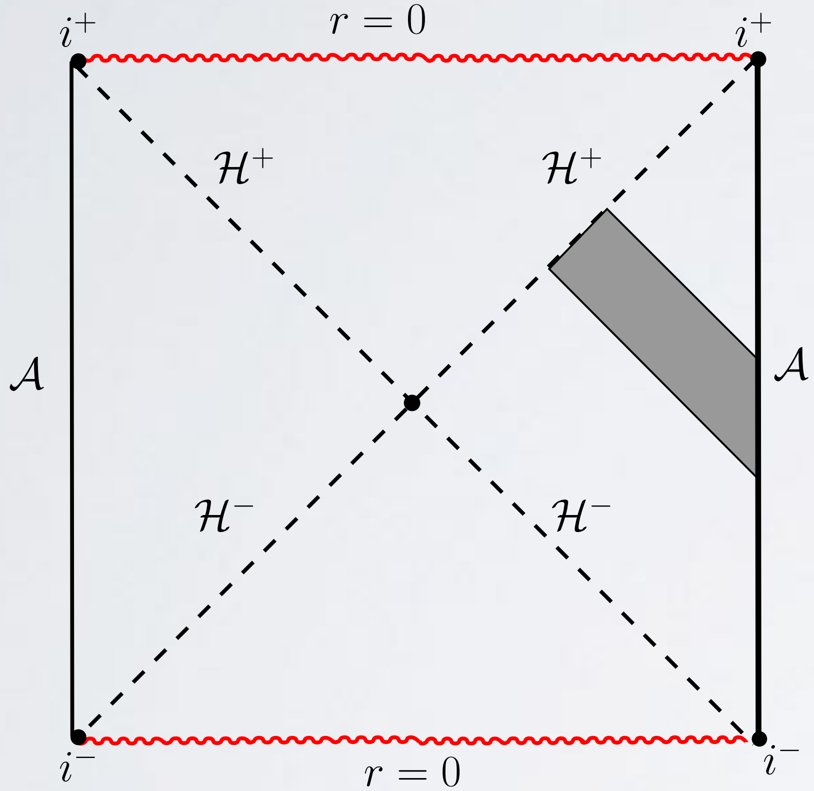

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time-derivative

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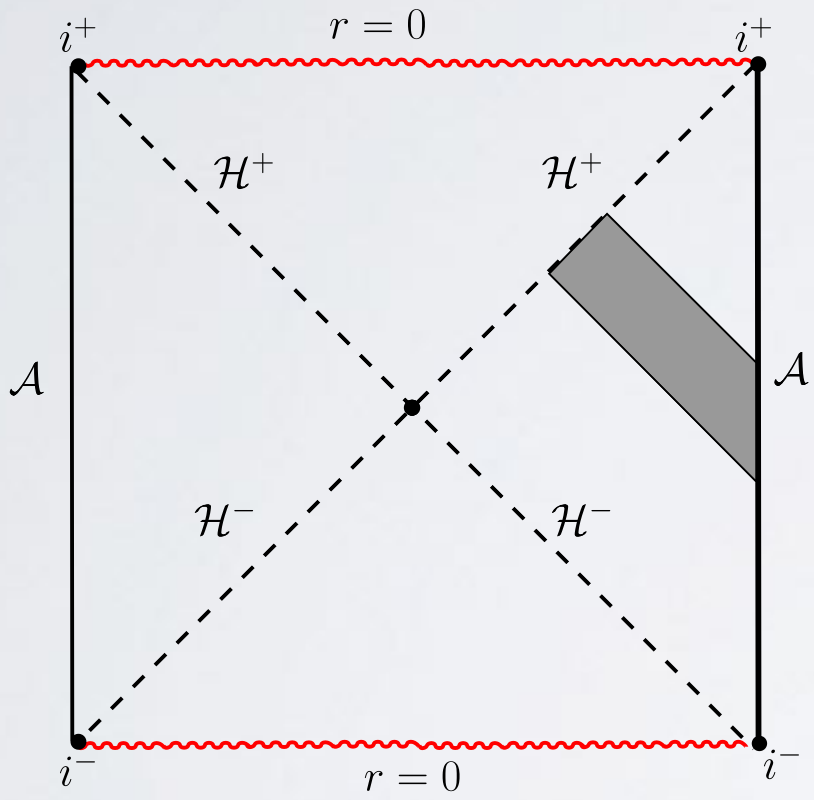

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time-derivative second order

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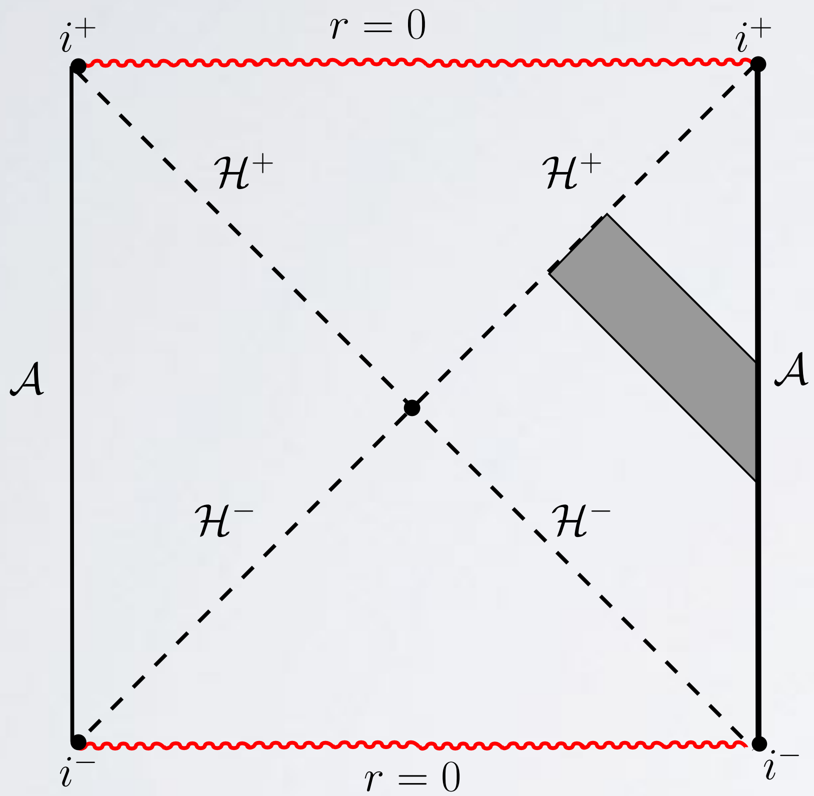

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- **Laplace transformation (homogenous equation):**

$$\left[-\rho\,(1-\rho^4)\,\frac{\partial^2}{\partial\rho^2} - \left(3 - 7\rho^4\right)\,\frac{\partial}{\partial\rho} + (8+\lambda^2)\,\rho^3\right]\bar{U}(\rho) + s\left[2\,\rho\,\frac{\partial}{\partial\rho} + 3\,\right]\bar{U}(\rho) = 0.$$

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2. Ask your favourite mathematical software to solve the (generalised) eigenvalue problem

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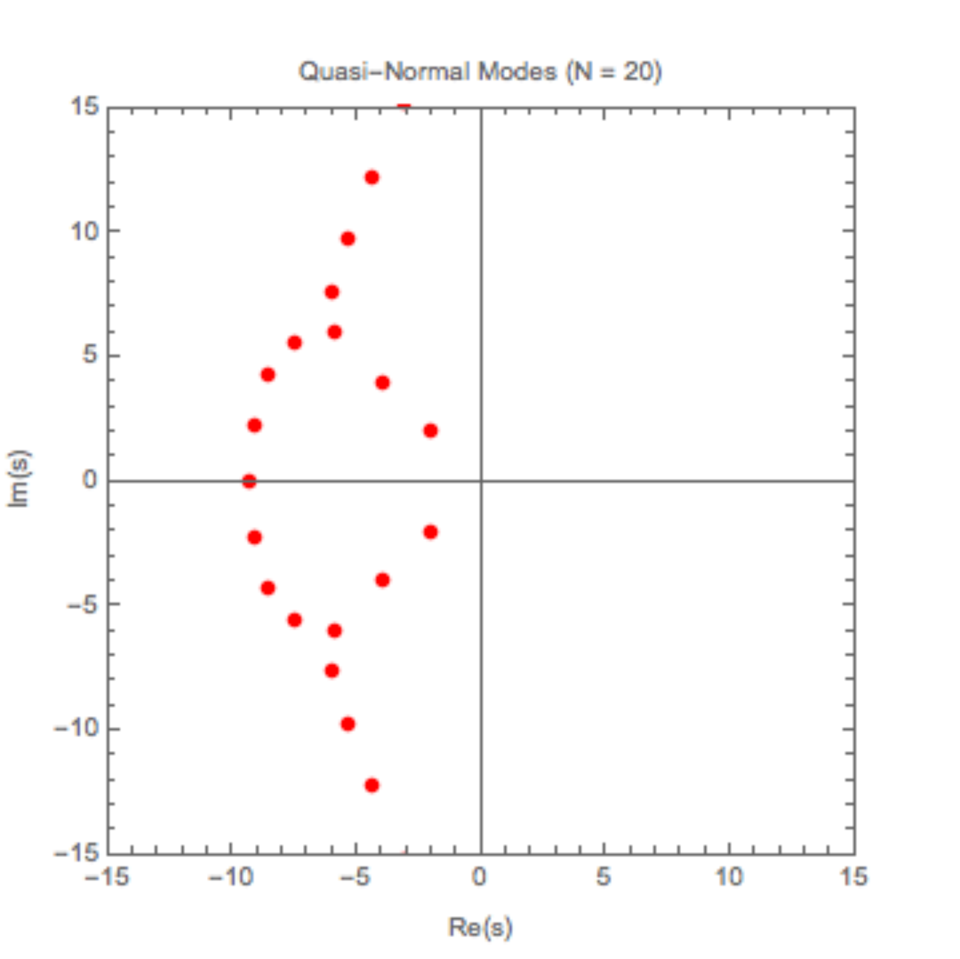
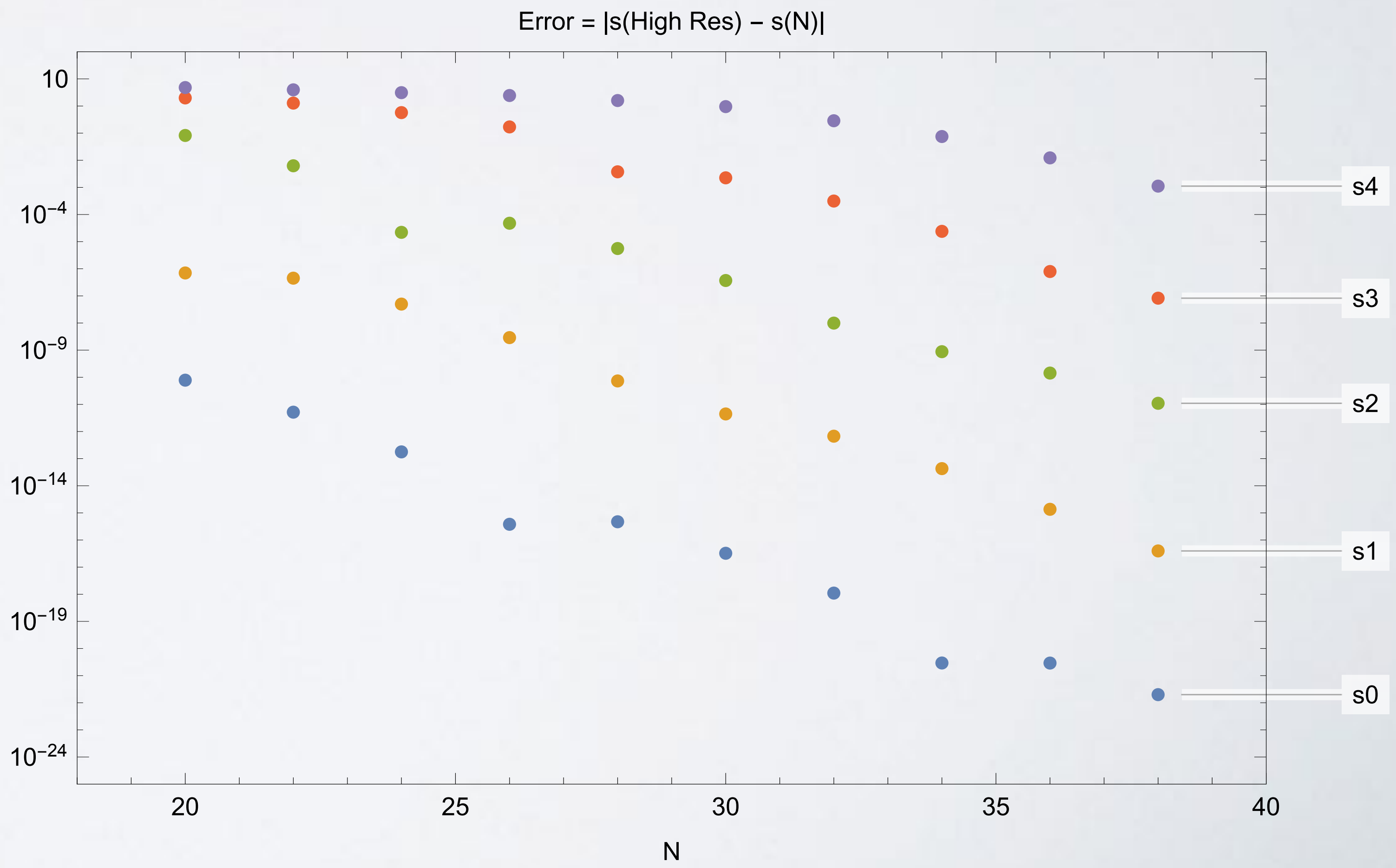
N N





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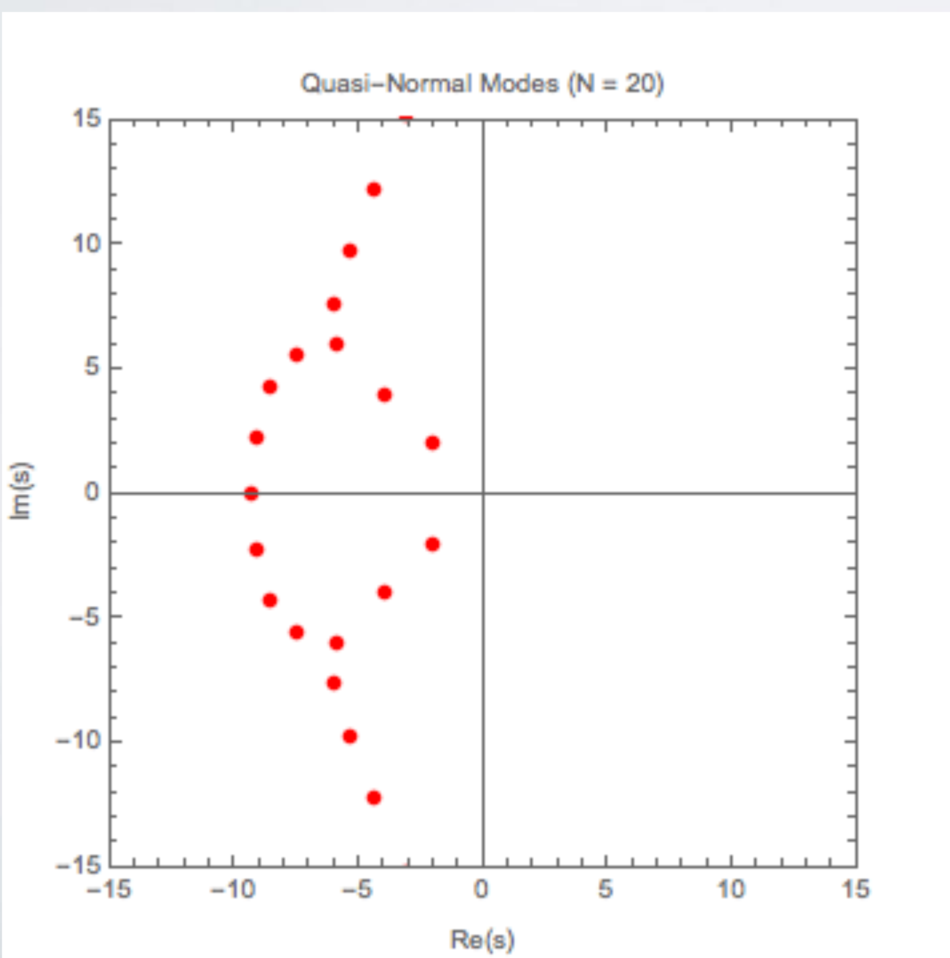
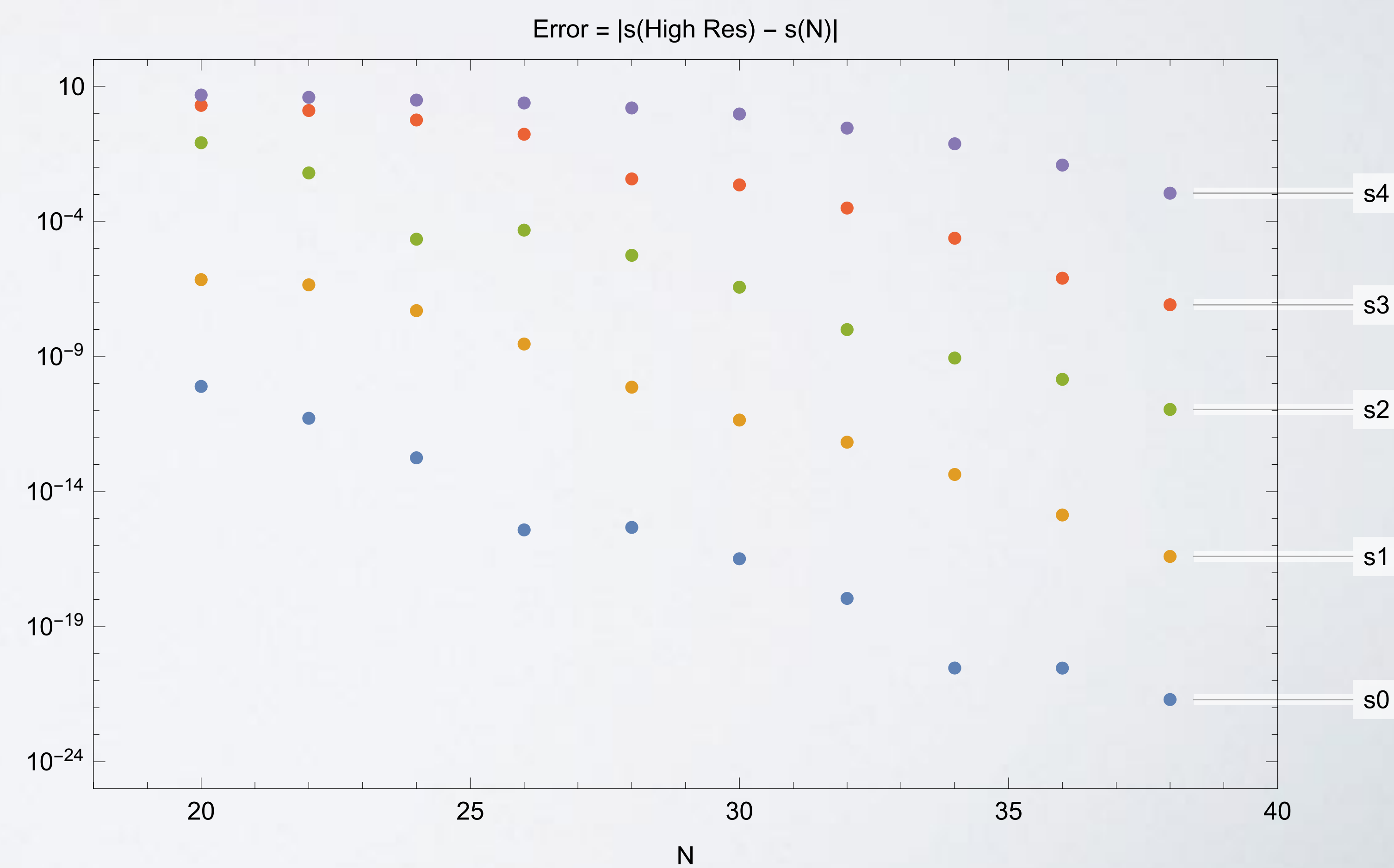
  2. However, most of them are rubbish
  3. One must study the convergence to see which values are stable and converge to a fixed value as we increase the resolution





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