Triple lines and Eckardt points on a cubic threefold

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September 15, 2023

Overview

Introduction

Projective cubic hypersurfaces are smooth hypersurfaces of degree three in the projective space $\mathbb{P}^n$.

- Elliptic curves.
  Example:
  $$x_0^3 + x_1^3 + x_2^3 = 0. \tag{1}$$
- Cubic surfaces.
  Example:
  $$x_0^3 + x_1^3 + x_2^3 + x_3^3 - (x_0 + x_1 + x_2 + x_3)^3 = 0. \tag{2}$$

Introduction

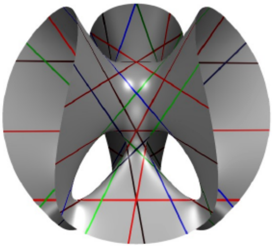

Figure 1: The 27 lines of the Clebsch's cubic (view Source).

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Introduction

- Cubic threefolds.

Example:

1. The Fermat cubic:

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0. \tag{3}$$

2. The Klein cubic:

$$x_0^2 x_1 + x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_4 + x_4^2 x_0 = 0. \tag{4}$$

Introduction

In what follows, we denote by X a smooth cubic threefold defined over $\mathbb{C}$. Let $\mathbb{G}(1,4)$ be the grassmannian of lines in $\mathbb{P}^4$, or equivalently the grassmannian $G(2,5)$ of 2-planes in a vector space $V \simeq \mathbb{C}^5$. Let $[\ell] \in \mathbb{G}(1,4)$ be a line $\ell \subset \mathbb{P}^4$ spanned by two points v_0 and v_1 in $\mathbb{P}^4$. Through the Plücker embedding

$$p : \mathbb{G}(1,4) \longrightarrow \mathbb{P}(\bigwedge^2 V), [\ell] \longmapsto v_0 \wedge v_1$$

the homogeneous coordinates on $\mathbb{P}(\bigwedge^2 V)$ are called Plücker coordinates. They correspond to the 2×2 minors of the 2×5 matrix whose rows are the points v_0 and v_1.

Introduction

Definition

The Fano surface $\mathrm{F}(X)$ of X is the set parametrizing the lines of $\mathbb{G}(1,4)$ which lie entirely in X.

We write

$$\mathrm{F}(X) = \{[\ell] \in \mathbb{G}(1,4) \mid \ell \subset X\}.$$

Remark: The Fano surface $\mathrm{F}(X)$ of a smooth cubic threefold X is smooth.

Lines on a cubic threefold

- Clemens and Griffiths in 1972.

Definition

Let ℓ be a line on X. Lines with $\mathcal{N}_{\ell/X} \simeq \mathcal{O}_\ell \bigoplus \mathcal{O}_\ell$ are called lines of the first type and those with $\mathcal{N}_{\ell/X} \simeq \mathcal{O}_\ell(1) \bigoplus \mathcal{O}_\ell(-1)$ are called lines of the second type.

An alternative description is given by the following lemma:

Lemma

The line $\ell \subset X$ is of the second type if and only if there exists a unique 2-plane $P \supset \ell$ in $\mathbb{P}^4$ tangent to X at every point of ℓ. If $\ell \subset X$ is a line of the first type, then there is no 2-plane tangent to X in all points of ℓ.

Lines on a cubic threefold

Some notations:

- $(x_0 : x_1 : x_2 : x_3 : x_4)$ the homogeneous coordinates on $\mathbb{P}^4$.
- $F(x_0, x_1, x_2, x_3, x_4) = 0$ the equation of $X \subset \mathbb{P}^4$.
- $p_{i,j}, 0 \leq i < j \leq 4$ the Plücker coordinates of $\mathbb{G}(1,4) \subset \mathbb{P}^9$.

On the affine chart $p_{0,1} = 1$, the grassmannian $\mathbb{G}(1,4)$ is isomorphic to $\mathbb{C}^6$ through the embedding

$$\begin{array}{ccc} \mathbb{C}^6 & \longrightarrow & \mathbb{P}^9 \\ (p_{0,2}, p_{0,3}, p_{0,4}, p_{1,2}, p_{1,3}, p_{1,4}) & \longmapsto & (1, p_{0,2}, \ldots, p_{1,4}, p_{2,3}, p_{2,4}, p_{3,4}) \end{array}$$

given by the following relations:

$$\begin{aligned} p_{2,3} &= p_{0,2}p_{1,3} - p_{0,3}p_{1,2} \\ p_{2,4} &= p_{0,2}p_{1,4} - p_{0,4}p_{1,2} \\ p_{3,4} &= p_{0,3}p_{1,4} - p_{0,4}p_{1,3}. \end{aligned}$$

Lines on a cubic threefold

In fact, let $v_0 = (a_0 : a_1 : a_2 : a_3 : a_4)$ and $v_1 = (b_0 : b_1 : b_2 : b_3 : b_4)$. We have

$$\begin{pmatrix} a_0 & b_0 \\ a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ a_4 & b_4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ a_1' & b_1' \\ a_2' & b_2' \\ a_3' & b_3' \\ a_4' & b_4' \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a_2'' & b_2' \\ a_3'' & b_3' \\ a_4'' & b_4' \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -p_{1,2} & p_{0,2} \\ -p_{1,3} & p_{0,3} \\ -p_{1,4} & p_{0,4} \end{pmatrix}.$$

- On the affine chart $p_{0,1} = 1$, a point $[\ell] \in \mathbb{G}(1,4)$ corresponds thus to a line $\ell \subset \mathbb{P}^4$ spanned by $v_0 = (1 : 0 : -p_{1,2} : -p_{1,3} : -p_{1,4})$ and $v_1 = (0 : 1 : p_{0,2} : p_{0,3} : p_{0,4})$ in $\mathbb{P}^4$.
- We take $(p_{0,2}, p_{0,3}, p_{0,4}, p_{1,2}, p_{1,3}, p_{1,4})$ as the local coordinates of $\mathbb{G}(1,4)$ on the affine chart $p_{0,1} = 1$.

Lines on a cubic threefold

An arbitrary point $p \in \ell$ has coordinates

$$\begin{aligned} x_0 &= t_0 \\ x_1 &= t_1 \\ x_2 &= -p_{1,2}\,t_0 + p_{0,2}\,t_1 \\ x_3 &= -p_{1,3}\,t_0 + p_{0,3}\,t_1 \\ x_4 &= -p_{1,4}\,t_0 + p_{0,4}\,t_1 \end{aligned}$$

with $[t_0 : t_1] \in \mathbb{P}^1$. The line ℓ is on X if and only if

$$\begin{aligned} 0 &= F(t_0, t_1, -p_{1,2}\,t_0 + p_{0,2}\,t_1, -p_{1,3}\,t_0 + p_{0,3}\,t_1, -p_{1,4}\,t_0 + p_{0,4}\,t_1) \quad (5) \\ &= t_0^3\phi^{3,0}(\ell) + t_0^2 t_1 \phi^{2,1}(\ell) + t_0 t_1^2 \phi^{1,2}(\ell) + t_1^3 \phi^{0,3}(\ell) \quad (6) \end{aligned}$$

for all $[t_0 : t_1] \in \mathbb{P}^1$, where $\phi^{i,j}(\ell)$ are functions of the local Plücker coordinates.

Lines on a cubic threefold

This implies

$$\phi^{3,0}(\ell) = 0, \phi^{2,1}(\ell) = 0, \phi^{1,2}(\ell) = 0, \phi^{0,3}(\ell) = 0$$

which are the local equations of the Fano surface $\mathrm{F}(X)$ on the affine chart $p_{0,1} = 1$ of $\mathbb{G}(1,4)$.

Multiple lines on a cubic threefold

Let ℓ be a line on X and P a 2-plane containing it. We look at the intersection $P \cap X$. We write $P \cap X = \ell \cup C$, where C is a conic.

- If the conic degenerates, then we have $P \cap X = \ell \cup \ell^{'} \cup \ell^{''}$ where $\ell, \ell^{'}$ and $\ell^{''}$ are three distinct lines.
- If $P \cap X = 2\ell \cup \ell^{'}$ then we say the plane P is tangent to X at every point of ℓ. The line ℓ is called a multiple line and $\ell^{'}$ the residual line of ℓ. If $\ell = \ell^{'}$ then ℓ is called a triple line, otherwise we say that ℓ is a double line.

Multiple lines on a cubic threefold

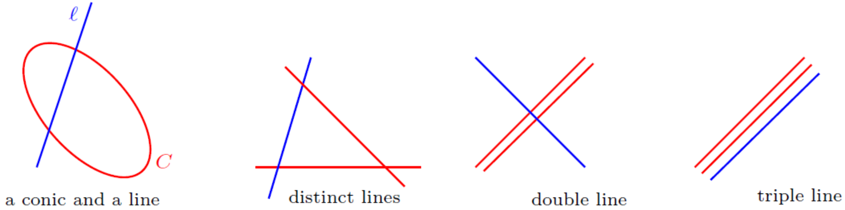

Figure 2: From conic degeneration to triple line.

Multiple lines on a cubic threefold

- In 1972 Murre proved that the multiple lines on a cubic threefold are exactly the lines of the second type.
- Until now almost nothing is known about triple lines on a cubic threefold except that the set of triple lines in a cubic threefold is finite, which was proved by Clemens and Griffiths in 1972.

Eckardt points

Definition

A point $p \in X$ is called an Eckardt point if it is a point of multiplicity 3 for the intersection $X \cap \mathrm{T}_p X$, where $\mathrm{T}_p X$ denotes the projective tangent space of X at p.

This is equivalent to saying that the intersection $X \cap \mathrm{T}_p X \subset \mathrm{T}_p X$ is a cone with vertex p over an elliptic curve E_p.

Proposition

An Eckardt point $p \in X$ is a point such that there are infinitely many lines through p contained in X.

Eckardt points

Let $p = (1:0:0:0:0) \in X$. Then the equation of X may be written

$$F(x_0,\dots,x_4) = x_0^2 l(x_1,\dots,x_4) + x_0 Q(x_1,\dots,x_4) + C(x_1,\dots,x_4) \quad (7)$$

where $l(x_1,\dots,x_4)$, $Q(x_1,\dots,x_4)$ and $C(x_1,\dots,x_4)$ are homogeneous polynomials of degree one, two and three respectively. Since the tangent space $T_p(X)$ is defined by the hyperplane $l(x_1,x_2,x_3,x_4) = 0$ then the intersection $X \cap T_pX$ can take the form

$$F(x_0,\dots,x_4) = x_0 Q(x_1,\dots,x_4) + C(x_1,\dots,x_4). \quad (8)$$

For p an Eckardt point one has thus $Q(x_1,\dots,x_4) = 0$, and the equation of X becomes

$$F(x_0,\dots,x_4) = x_0^2 l(x_1,\dots,x_4) + C(x_1,\dots,x_4). \quad (9)$$

Eckardt points

The intersection $X \cap T_pX$ is then a cone with vertex p over the elliptic curve

$$E_p = \{C(x_1, \dots, x_4) = 0, l(x_1, \dots, x_4) = 0, x_0 = 0\}. \qquad (10)$$

Some features on Eckardt points on a cubic threefold:

- Clemens and Griffiths proved in 1972 that a smooth cubic threefold can contain at most finitely many Eckardt points, and in fact at most 30 according to Canonero, Catalisano and Serpico in 1997.
- The Fermat cubic is the unique cubic threefold that contains 30 Eckardt points.
- A generic cubic threefold does not contain Eckardt points.

Eckardt points

- Each Eckardt point $p \in X$ determines an elliptic curve E_p on its Fano surface, which is the base of the cone $X \cap \mathrm{T}_p X$ (Tjurin in 1971).
- Lines going through Eckardt points are of the second type.
- In 2009 Roulleau proved that the number of elliptic curves on a Fano surface $\mathrm{F}(X)$ is at most 30. He also proved that the Fano surface of the Fermat cubic is the unique one that contains 30 elliptic curves.

Curves on the Fano surface

Proposition (Murre)

The set

$$\mathrm{M}(X) = \{[\ell] \in \mathrm{F}(X), \exists P \simeq \mathbb{P}^2 | P \cap X = 2\ell \cup \ell'\}$$

of lines of the second type is an algebraic curve on the Fano surface $\mathrm{F}(X)$.

It is defined by the equations

$$m(\ell) = 0, \phi^{3,0}(\ell) = 0, \phi^{2,1}(\ell) = 0, \phi^{1,2}(\ell) = 0, \phi^{0,3}(\ell) = 0$$

where $m(\ell) = 0$ is the local equation of $\mathrm{M}(X)$ in $\mathrm{F}(X)$ on the affine chart $p_{0,1} = 1$ of $\mathbb{G}(1,4) \subset \mathbb{P}^9$.

Curves on the Fano surface

Proposition

The set

$$\mathrm{R}(X) = \{\ell^{'} \in \mathrm{F}(X), \exists P \simeq \mathbb{P}^2 | P \cap X = 2\ell \cup \ell^{'}\}$$

of residual lines is a curve on the the Fano surface $\mathrm{F}(X)$.

Lahoz, Naranjo and Rojas proved that for a generic smooth cubic threefold X, the curve $\mathrm{R}(X)$ is irreducible and singular.

Triple lines and the curve of lines of the second type

Question

What do triple lines represent in the geometry of the curve $\mathrm{M}(X)$ of lines of the second type?

- Murre in 1972.

Proposition

The curve $\mathrm{M}(X)$ is nonsingular.

Later on, people who cited his work precised that:

- The curve $\mathrm{M}(X)$ is nonsingular for a generic cubic threefold X.
- What are the singular points of $\mathrm{M}(X)$ when it is not smooth?

Triple lines and the curve of lines of the second type

Theorem

The triple lines on a cubic threefold are exactly the singular points of the curve $\mathrm{M}(X)$.

Proposition

The Fermat cubic threefold contains 135 triple lines.

Triple lines and the curve of lines of the second type

Sketch of the proof

Consider the Fermat cubic in $\mathbb{P}^4$ defined by $F_4 = \{x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0\}$. In the affine chart $p_{0,1} = 1$ the set $\mathrm{M}(F_4)$ is a non smooth curve given by the following equations:

$$\begin{aligned}
p_{1,2}^3 + p_{1,3}^3 + p_{1,4}^3 - 1 &= 0 \\
p_{0,2}p_{1,2}^2 + p_{0,3}p_{1,3}^2 + p_{0,4}p_{1,4}^2 &= 0 \\
p_{0,2}^2 p_{1,2} + p_{0,3}^2 p_{1,3} + p_{0,4}^2 p_{1,4} &= 0 \\
p_{0,2}^3 + p_{0,3}^3 + p_{0,4}^3 + 1 &= 0 \\
(p_{0,4}p_{1,3} - p_{0,3}p_{1,4})(p_{0,4}p_{1,2} - p_{0,2}p_{1,4})(p_{0,3}p_{1,2} - p_{0,2}p_{1,3}) &= 0.
\end{aligned}$$

Triple lines and the curve of lines of the second type

Sketch of the proof

Consider the quadrics

$$Q_1 = p_{0,4}p_{1,3} - p_{0,3}p_{1,4}, \quad Q_2 = p_{0,4}p_{1,2} - p_{0,2}p_{1,4}, \quad Q_3 = p_{0,3}p_{1,2} - p_{0,2}p_{1,3}.$$

The intersection of the Fano surface $\mathrm{F}(F_4)$ with Q_1, Q_2 and Q_3, denoted by $\mathrm{M}_1(F_4), \mathrm{M}_2(F_4)$ and $\mathrm{M}_3(F_4)$ respectively, are smooth curves that correspond to the irreducible components of $\mathrm{M}(F_4)$. Looking at the intersection points of $\mathrm{M}_1(F_4)$ and $\mathrm{M}_2(F_4)$, $\mathrm{M}_1(F_4)$ and $\mathrm{M}_3(F_4)$, $\mathrm{M}_2(F_4)$ and $\mathrm{M}_3(F_4)$, we get 18 triple lines at each intersection. Also, the intersection of $\mathrm{M}_1(F_4), \mathrm{M}_2(F_4)$ and $\mathrm{M}_3(F_4)$ is empty. There are thus 54 triple lines in the affine chart $p_{0,1} = 1$.

Triple lines and the curve of lines of the second type

Sketch of the proof

stratum 1	$p_{0,1} = 1$	54 triple lines
stratum 2	$p_{0,1} = 0$ and $p_{0,2} = 1$	36 triple lines
stratum 3	$p_{0,1} = 0, p_{0,2} = 0$ and $p_{0,3} = 1$	18 triple lines
stratum 4	$p_{0,1} = 0, \dots, p_{0,3} = 0$ and $p_{0,4} = 1$	no triple line
stratum 5	$p_{0,1} = 0, \dots, p_{0,4} = 0$ and $p_{1,2} = 1$	18 triple lines
stratum 6	$p_{0,1} = 0, \dots, p_{1,2} = 0$ and $p_{1,3} = 1$	9 triple lines
stratum 7	$p_{0,1} = 0, \dots, p_{1,3} = 0$ and $p_{1,4} = 1$	no triple line
stratum 8	$p_{0,1} = 0, \dots, p_{1,4} = 0$ and $p_{2,3} = 1$	
stratum 9	$p_{0,1} = 0, \dots, p_{2,3} = 0$ and $p_{2,4} = 1$	
stratum 10	$p_{0,1} = 0, \dots, p_{2,4} = 0$ and $p_{3,4} = 1$	

Triple lines and the curve of lines of the second type

Remark: It is clear that the curve $\mathrm{M}(F_4)$ is reducible. Roulleau proved that the Fano surface of the Fermat cubic contains 30 elliptic curves that intersect in 135 points. These intersection points are exactly the triple lines on the Fermat cubic. We wish to point out that the elliptic curves of the Fano surface are exactly the irreducible components of $\mathrm{M}(F_4)$.

Triple lines and Eckardt points

Theorem (Tjurin)

The triple lines on X correspond to the inflection points of the elliptic curve E_p which is the base of the cone $X \cap T_pX \subset T_pX$.

Corollary

If a smooth complex cubic threefold contains Eckardt points then it contains triple lines.

Triple lines and Eckardt points

Example:

1. The Fermat cubic threefold has 30 Eckardt points and contains 135 triple lines.
2. The Klein cubic defined by $x_0^2x_1 + x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_0 = 0$ contains neither Eckardt points nor triple lines.
3. The cubic threefold defined by $x_0^2x_2 + x_2^2x_4 + x_1^2x_3 + x_3^2x_0 + x_4^3 = 0$ has one Eckardt point with coordinates $(0:1:0:0:0)$ and contains 9 triple lines.

Triple lines and Eckardt points

Remark: It follows that there are exactly nine triple lines going through an Eckardt point on a smooth cubic threefold.

Question

Does every triple line come from an Eckardt point?

- In 1997, Canonero, Serpico and Catalisano stated, without giving much details, that the smooth cubic threefold defined by

$$F_2 = \{x_0^2 x_4 + x_1^2 x_3 + x_3^3 + x_3^2 x_4 + x_3 x_4^2 - x_4^3 + x_2^3 = 0\}$$

  has only two Eckardt points with coordinates $(1:0:0:0:0)$ and $(0:1:0:0:0)$.
- The cubic F_2 has 39 triple lines.

Triple lines and Eckardt points

We recall that if a smooth complex cubic threefold contains Eckardt points then it contains triple lines.

Question

Is the reverse true?

Definition

The polar quadric of a point $p = (p_0 : \ldots : p_4) \in \mathbb{P}^4$ with respect to X is the hypersurface defined by $\displaystyle\sum_{i=0}^{4} p_i \frac{\partial f}{\partial x_i} = 0$.

Triple lines and Eckardt points

Proposition

A point $p \in X$ is an Eckardt point if and only if the polar quadric $\triangle_p(X)$ is of rank at most two.

Using the previous proposition, we propose the following method for computing all Eckardt points on a cubic threefold.

Triple lines and Eckardt points

Method for computing Eckardt points on a cubic threefold

Let $X \subset \mathbb{P}^4$ be a smooth complex cubic threefold, $p = (p_0 : \ldots : p_4) \in X$ a point and $\mathcal{B}$ the matrix associated with the polar quadric $\triangle_p(X)$. Eckardt points on X are given by the vanishing locus of all 3×3 minors of $\mathcal{B}$. In order to count each point only once, we use a stratification of $\mathbb{P}^4$ described as follows: the first stratum is the affine chart $p_0 = 1$ and the i-th one is defined by $p_0 = 0, \ldots, p_{i-2} = 0, p_{i-1} = 1$ for $i = 1, \ldots, 4$.

Triple lines and Eckardt points

Example:

1. Consider the Fermat cubic $F_4 = \{x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0\} \subset \mathbb{P}^4$. Denote by $\mathcal{B}_1$ the matrix associated with the polar quadric $\triangle_p F_4$.
   - On the affine chart $p_0 = 1$ the vanishing locus of all 3×3 minors of $\mathcal{B}_1$ is defined by the following equations :

     $p_2^4 + p_2 = 0,\ p_3^4 + p_3 = 0,\ p_4^4 + p_4 = 0,\ p_1^3 + p_2^3 + p_3^3 + p_4^3 + 1 = 0,\ p_1 p_2 = 0,\ p_1 p_3 = 0,\ p_2 p_3 = 0,\ p_1 p_4 = 0,\ p_2 p_4 = 0,\ p_3 p_4 = 0.$

     If $p_i \neq 0$ then $p_j = 0$ for $i \neq j$ and $p_i^3 = -1$. We get 12 Eckardt points with coordinates $(1 : 0 : \ldots : p_i : \ldots : 0)$ with $p_i^3 = -1$.

Triple lines and Eckardt points

- In the stratum $p_0 = 0, p_1 = 1$ the vanishing locus of all 3×3 minors of $\mathcal{B}_1$ is defined by the following equations :

$$p_3^4 + p_3 = 0,\ p_4^4 + p_4 = 0,\ p_2^3 + p_3^3 + p_4^3 + 1 = 0,\ p_2 p_3 = 0,\ p_2 p_4 = 0,\ p_3 p_4 = 0.$$

If $p_i \neq 0$ then $p_j = 0$ for $i \neq j$ and $p_i^3 = -1$. We get 9 Eckardt points with coordinates $(0 : 1 : \ldots : p_i : \ldots : 0)$ with $p_i^3 = -1$.

Triple lines and Eckardt points

- In the stratum $p_0 = 0, p_1 = 0, p_2 = 1$ the vanishing locus of all 3×3 minors of $\mathcal{B}_1$ is defined by the following equations :
$$p_4^4 + p_4 = 0,\ \ p_3^3 + p_4^3 + 1 = 0,\ \ p_3 p_4 = 0.$$
We get three Eckardt points with coordinates $(0:0:1:\xi:0)$ and three others with coordinates $(0:0:1:0:\xi)$, with $\xi^3 = -1$.
- In the last stratum we get 3 Eckardt points with coordinates $(0:0:0:1:\xi)$ with $\xi^3 = -1$.

Triple lines and Eckardt points

2. The Klein cubic contains no Eckardt point because the vanishing locus of all 3×3 minors of the matrix associated with its polar quadric is empty in all strata.
3. Denote by $\mathcal{B}_2$ the matrix associated with the polar quadric $\triangle_p X_2$. On the affine chart $p_0 = 1$ the vanishing locus of all 3×3 minors of $\mathcal{B}_2$ is defined by $p_1 = 0, p_2 = 0, p_3 = 0, p_4 = 0$, and in the stratum $p_0 = 0, p_1 = 1$ it is defined by $p_2 = 0, p_3 = 0, p_4 = 0$ while it is empty in the other strata. This proves that X_2 has no Eckardt points besides $(1 : 0 : 0 : 0 : 0)$ and $(0 : 1 : 0 : 0 : 0)$.

Triple lines and Eckardt points

- Let ℓ be a line of the second type on X given by

$$x_2 = 0, x_3 = 0, x_4 = 0.$$

Following Clemens and Griffiths, the equation of X may take the form:

$$\begin{aligned} f(x_0, \dots, x_4) &= x_0^2 x_2 + x_1^2 x_3 + x_0 q_0(x_2, x_3, x_4) + x_1 q_1(x_2, x_3, x_4) \\ &+ P(x_2, x_3, x_4) \end{aligned}$$

Triple lines and Eckardt points

where $q_0(x_2, x_3, x_4) = \sum_{2 \leq j \leq k \leq 4} b_{0jk}$ and $q_1(x_2, x_3, x_4) = \sum_{2 \leq j \leq k \leq 4} b_{1jk}$ are homogeneous polynomials of degree two and $P(x_2, x_3, x_4)$ is a homogeneous polynomial of degree three. Assume ℓ is a triple line so that the plane given by $x_2 = 0, x_3 = 0$ is the plane tangent to X in all points of ℓ. Then the equation of X may be written

$$f(x_0, \ldots, x_4) = x_0^2 x_2 + x_1^2 x_3 + x_0 q_0(x_2, x_3, x_4) + x_1 q_1(x_2, x_3, x_4) + k x_4^3 \quad (11)$$

with $k \neq 0$ and $b_{044} = 0, b_{144} = 0$. Using Equation (11) we obtain many examples of smooth cubic threefolds with no Eckardt points but containing triple lines.

Triple lines and Eckardt points

Example:
The following table gives the list of some smooth complex cubic threefolds $X_i \subset \mathbb{P}^4$ obtained using Equation (11).

X_1	$x_0^2x_2 + x_1^2x_3 + x_1x_2^2 + x_0x_3^2 + x_1x_3^2 + x_4^3 = 0$
X_2	$x_0^2x_2 + x_1^2x_3 + x_0x_2^2 + x_1x_2^2 + x_0x_3^2 + 2x_1x_2x_4 + 2x_0x_3x_4 + x_4^3 = 0$
X_3	$x_0^2x_2 + x_1^2x_3 + x_1x_2^2 + x_0x_3^2 + 2x_0x_3x_4 + x_4^3 = 0$
X_4	$x_0^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_3^2 + x_0x_3^2 + x_0x_4^2 + x_1x_4^2 + x_4^3 = 0$

Table 1: Some smooth cubic threefolds.

Triple lines and Eckardt points

Table 2 gives the information about the number n_E of Eckardt points and the number n_T of triple lines of these cubics.

cubic threefold	X_1	X_2	X_3	X_4
n_E	0	0	0	0
n_T	27	9	2	1

Table 2: Eckardt points and triple lines.

- The main component.


Merci!