# A brief overview of tropical geometry

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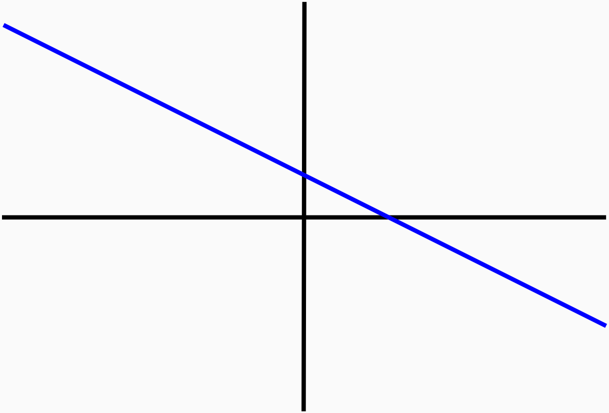
Nantes Université

# Algebra and geometry



$$x + 2y - 2 = 0$$

# Tropical algebra

- Tropical addition : $\text{``}a + b\text{''} = max(a, b)$

  Ex : $\text{``}3 + 2\text{''} = 3$

- Tropical mulitplication : $\text{``}ab\text{''} = a + b$

  Ex : $\text{``}3 \times 2\text{''} = 5$

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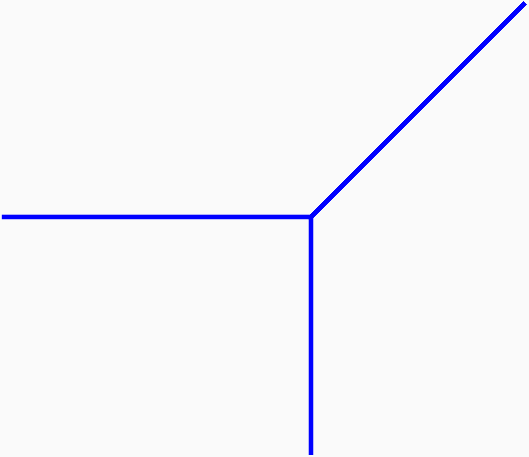

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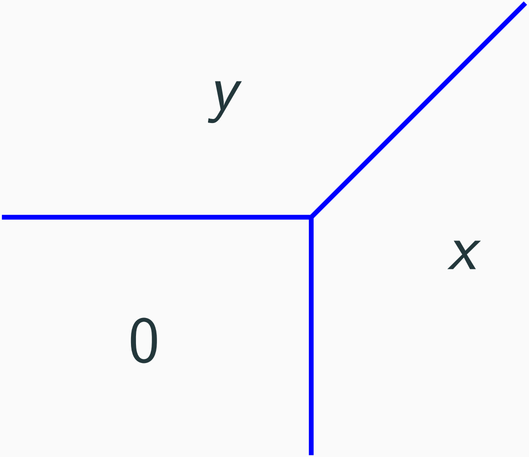

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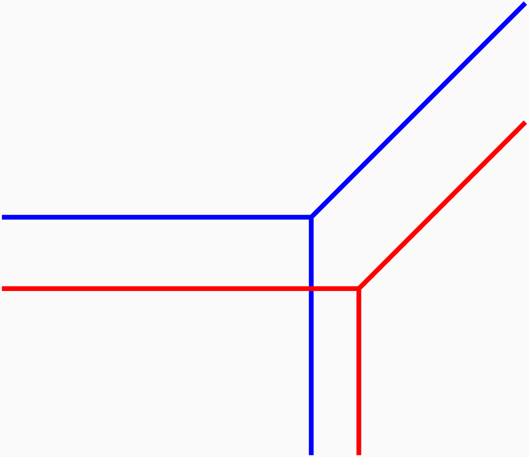

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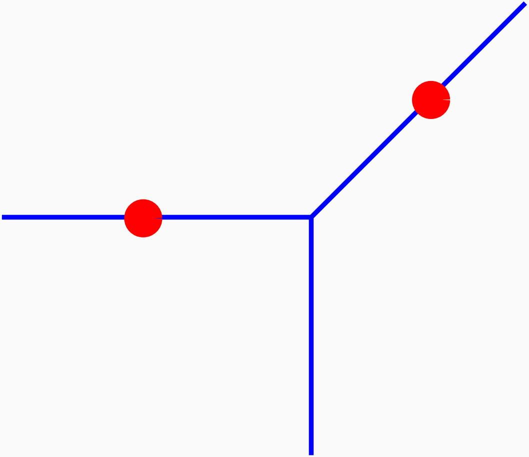

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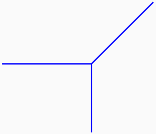
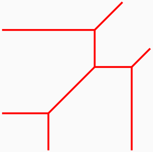
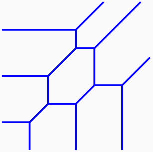


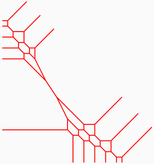
A line



A conic



A cubic



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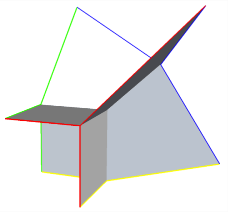
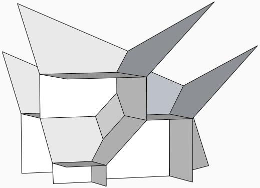


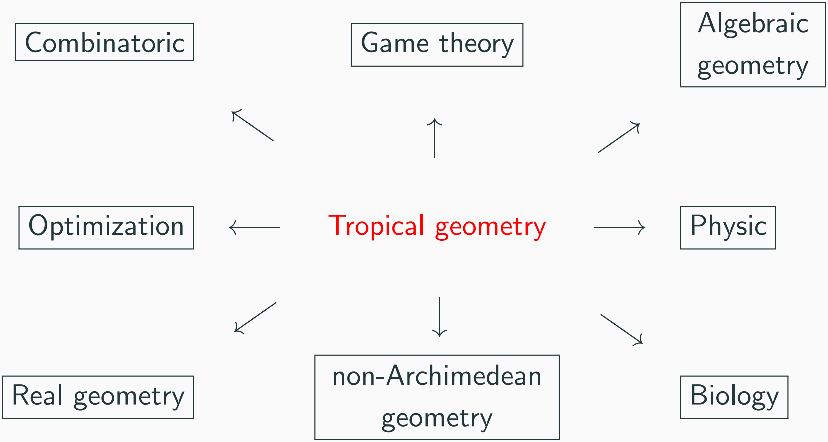
A plane



A quadric

# Applications

The tropical world sits at the frontier of the classical world



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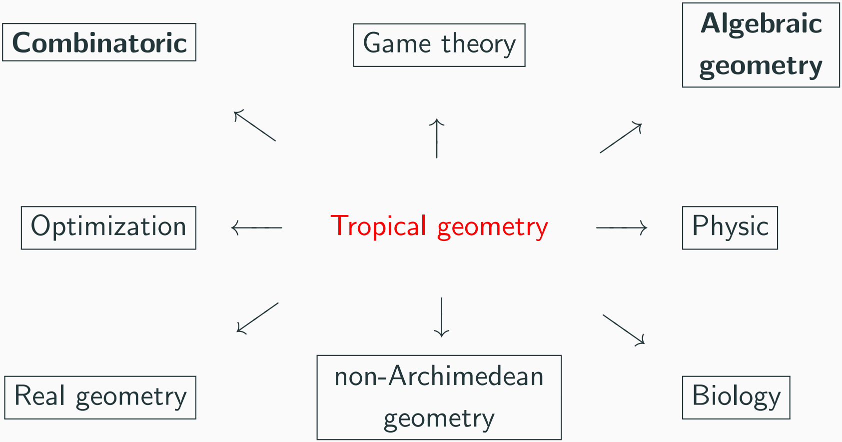

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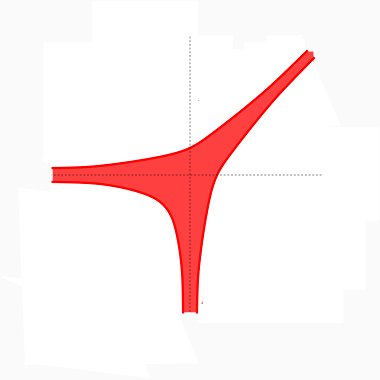
$$\text{“}x + y\text{”} = \lim_{t \to +\infty} \text{“}x +_t y\text{”}$$

# Amoeba of the line $L$ defined by $z - w + 1 = 0$



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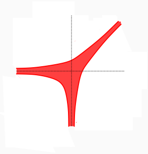

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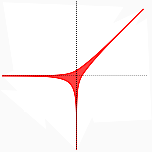

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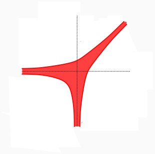

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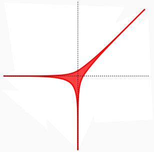

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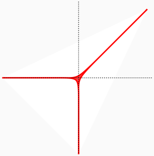

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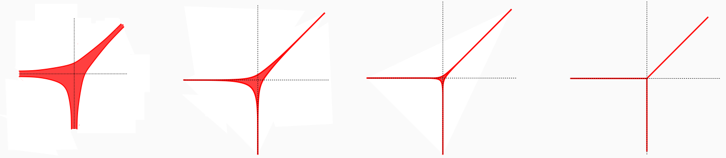

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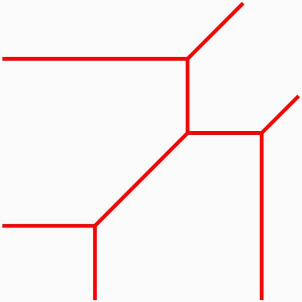
## Amoeba of the conic $C_t$ defined by $-1 + z + w - t^{-2}z^2 + t^{-1}zw - t^{-2}w^2 = 0$

$Log_t(C_t)$



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## Amoeba of a cubic $C_t$ defined by

$$-1 + z + w - t^{-2}z^2 + t^{-1}zw - t^{-2}y^2 + t^{-8}z^3 + t^{-5}z^2w + t^{-5}zw^2 + t^{-8}w^3 = 0$$

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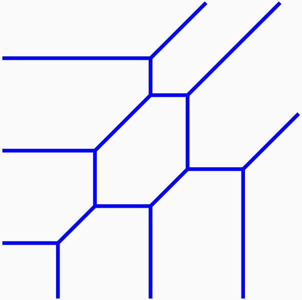

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# Tropical curves : combinatorial description

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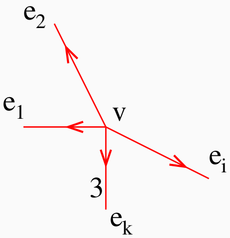
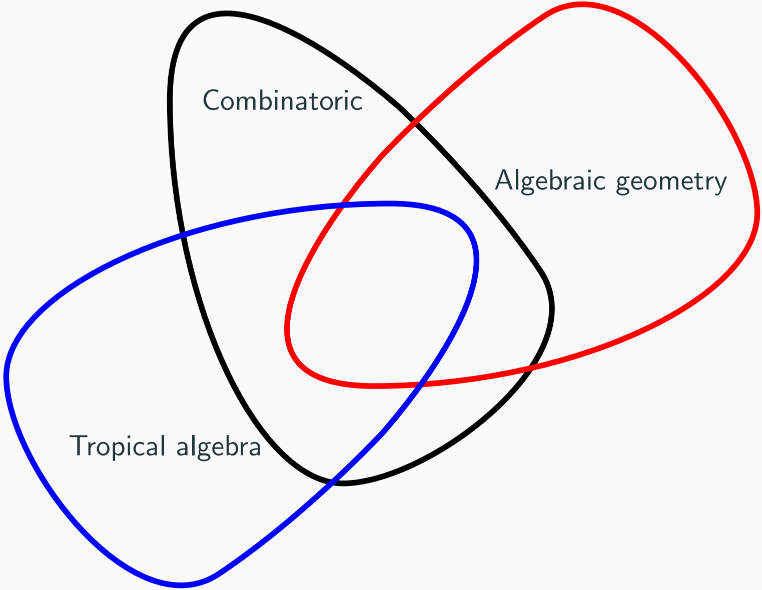



$\vec{v_{e_i}} \in \mathbb{Z}^n$ *primitive*

$$\sum w_{e_i} \vec{v_{e_i}} = 0$$

# What is tropical geometry ?



# Application to enumerative geometry

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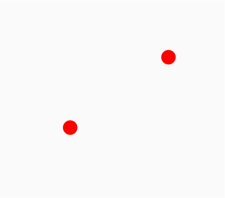
**Theorem (B-Mikhalkin, 2007)**
*Tropical curves enumeration using floor diagrams.*

# Enumerative geometry

How many lines through 2 points in the plane ?



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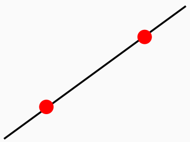

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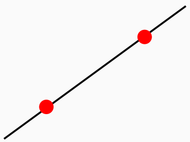

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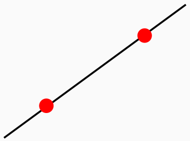

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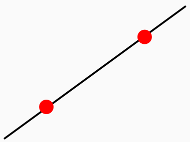

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# Rational curves

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$GW_{\mathbb{C}P^2}(d) =$ number of rational curves of degree $d$ passing through 3d-1 points.

# Enumeration of rational tropical curves

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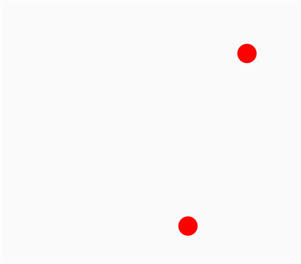

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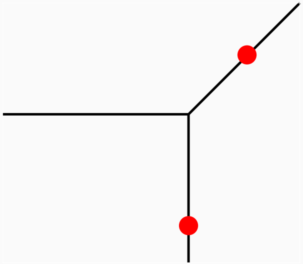

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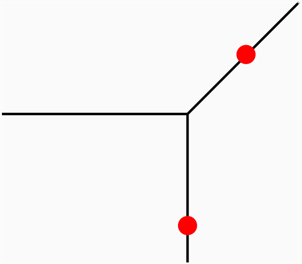

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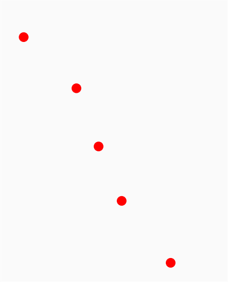

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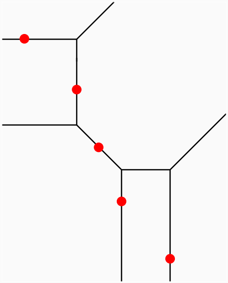

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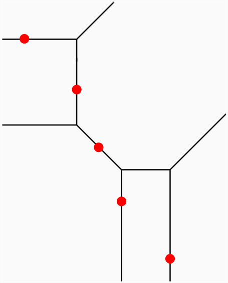

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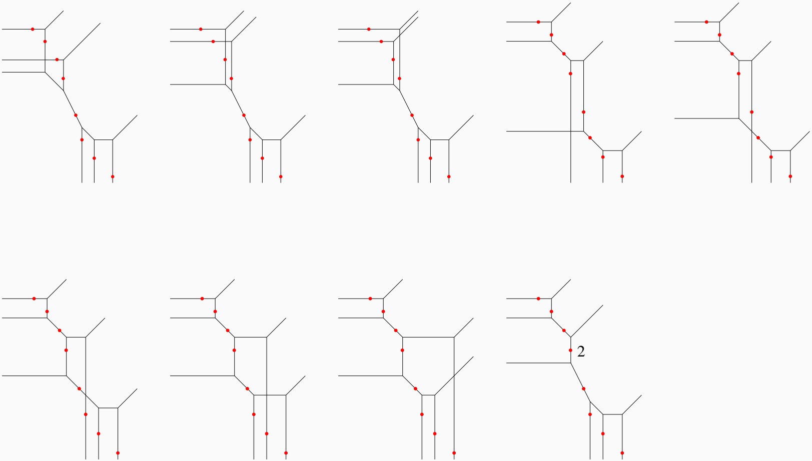

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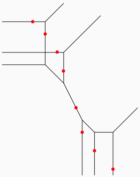

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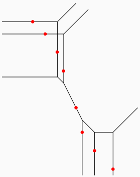

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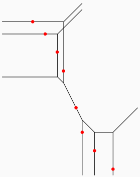

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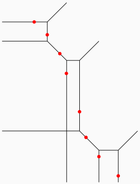

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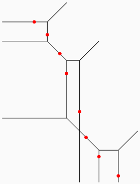

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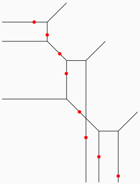

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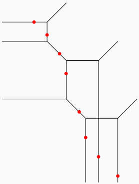

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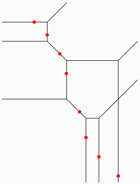

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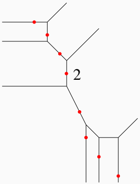


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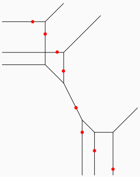
# Quantum tropical invariants (Block-Göttsche)

How many cubics through 8 points?



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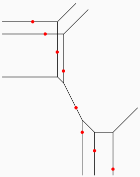

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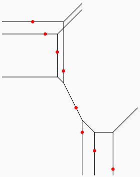

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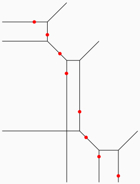

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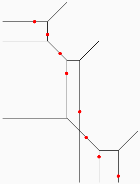

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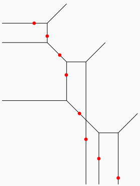

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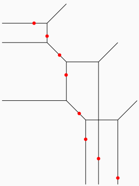

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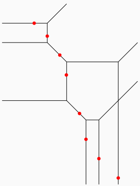

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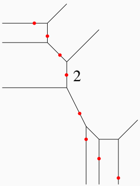
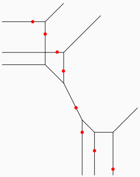



$q^{-1} + 2 + q$

# Quantum tropical invariants (Block-Göttsche)

How many cubics through 8 points? $q^{-1} + 10 + q$



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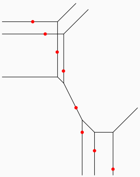

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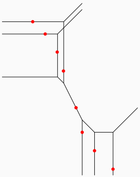

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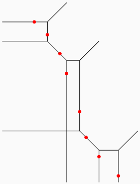

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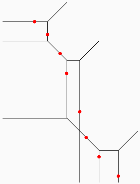

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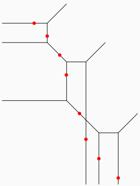

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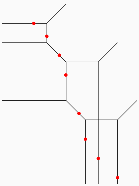

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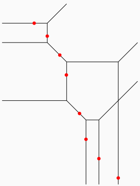

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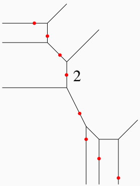


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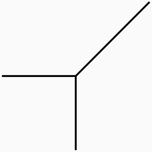
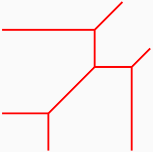
All matroid are realizable in tropical geometry.

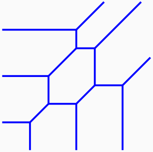
# Topology of tropical curves



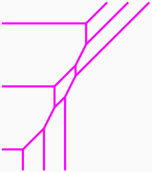
A line



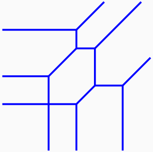
A conic



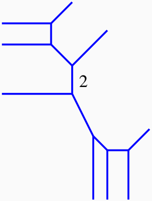
A cubic



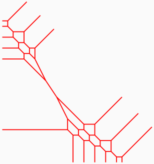
Another cubic



One more cubic




A cubic again



A sextic

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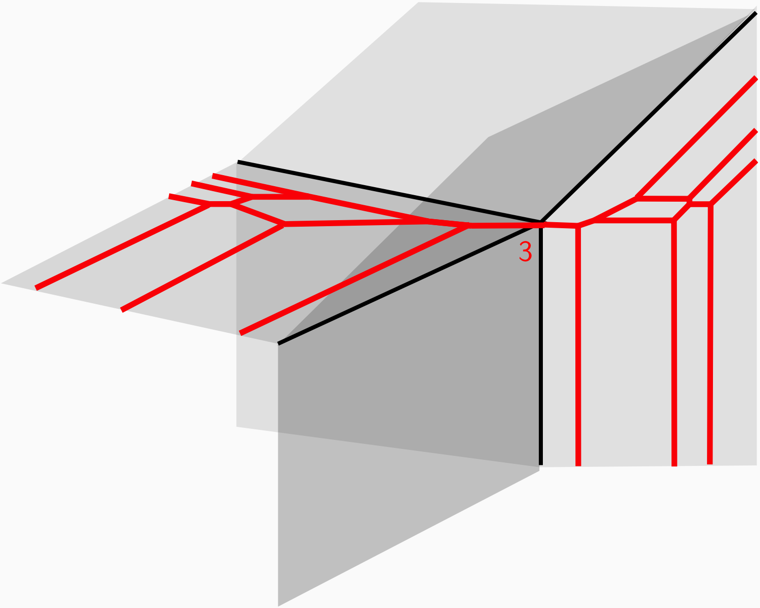
**Theorem (Mikhalkin-Sturmfels, $\sim$2000)**
*The maximal genus of a tropical curve of degree $d$ in $\mathbb{R}^2$ is*

$$\frac{(d-1)(d-2)}{2}.$$

# A cubic of genus 2



# Topology of tropical curves

**Theorem (Bertrand-B-López de Medrano)**
*There exists a tropical plane $L \subset \mathbb{R}^n$ such that for any $d \geq 1$, $L$ contains a tropical curve $C \subset L$ of degree $d$ with*

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$+$ Higher dimensional generalizations.