On Cartan-Schouten metrics on Lie groups

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Parallel Transport and Connection

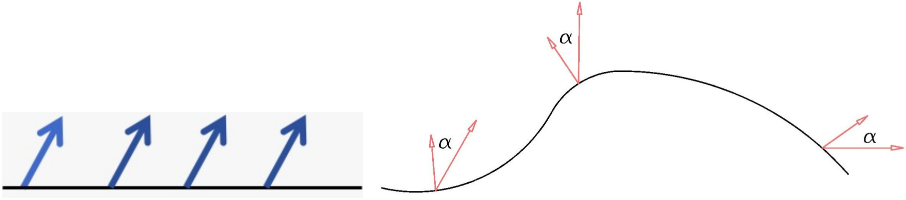


Figure 1: Parallel Transport

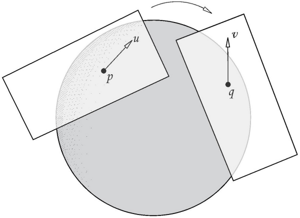


Figure 2: Connecting Tangent Vectors at different points

- Connection ⇒ Covariant Derivative ⇒ Parallel Transport

- Covariant derivative generalizes to tensors: if, for example, μ is a metric on a manifold M, we get

$$(\nabla\mu)(X, Y, Z) = X \cdot \mu(Y, Z) - \mu(\nabla_X Y, Z) - \mu(Y, \nabla_X Z) \quad (1)$$

for all vector fields X, Y, Z on M.

Definition 1

Given a connection ∇, an objet T is said to be compatible, parallel or covariantly constant with right the connection if

$$\nabla T = 0. \quad (2)$$

- A connection ∇ on M is compatible with μ if : $\forall X, Y, Z \in \mathfrak{X}(M)$,

$$X[\mu(Y, Z)] = \mu(\nabla_X Y, Z) + \mu(Y, \nabla_X Z). \quad (3)$$

Let (M, μ) be a (pseudo)-Riemannian manifold.

- There exists on (M, μ) a unique torsion free connection ∇ which is compatible with μ : Levi-Civita connection.
- It is given by the Koszul formula: $\forall X, Y, Z \in \mathfrak{X}(M)$,

$$\mu(\nabla_X Y, Z) = \frac{1}{2}\Big\{X[\mu(Y,Z)] + Y[\mu(Z,X)] - Z[\mu(X,Y)] + \mu(Z,[X,Y]) - \mu(Y,[X,Z]) - \mu(X,[Y,Z])\Big\} \quad (4)$$

- Question : given a connection on a manifold M, does it exist a metric which is parallel with right to it ?
- We aim to study such a problem for Lie groups with a particular connection : the so-called Cartan-Schouten connection.

Plan

1. Cartan-Schouten connection on a Lie group
   - Lie group, Lie algebra, exponential map, geodesics
   - Cartan-Schouten connections on a Lie group
2. Cartan-Schouten metrics on Lie groups
   - First examples in Low dimensions
   - Biinvariant metric
   - Fundamental results
   - Cartan-Schouten metrics on perfect Lie groups
3. Applications to dimension 3
   - Rigid motions, screw motion, and so on
   - The special affine Lie group

Let M be a smooth ùanifold.

- $T_x M$: tangent space of M at the point x of M;
- TM : tangent bundle of M,
- T^*M : cotangent bundle of M.
- Given a vector field X, we note $X_{|x} \in T_x M$ the value of X at the point $x \in M$.
- If $f : M \to N$ is a differentiable map between two manifolds, its linear tangent map will be noted by

$$f_* : TM \to TN \quad \text{ou } Tf : TM \to TN.$$

- Hence, for all $x \in M$ and any $X \in \mathfrak{X}(M)$,

$$(f_* X)_{|f(x)} = f_{*x}(X_{|x}) = T_x f \cdot X_{|x}. \tag{5}$$

Definition 2

A Lie group is smooth manifold together with a group structure such that

$$\begin{array}{rcl} \mu : G \times G & \longrightarrow & G \\ (g,h) & \longmapsto & gh \end{array} \qquad || \qquad \begin{array}{rcl} i : G & \longrightarrow & G \\ g & \longmapsto & g^{-1} \end{array} \tag{6}$$

are smooth maps.

Let G be a Lie group with unit element e.

- Left and right Translations:

$$\begin{array}{rcl} L_g : G & \longrightarrow & G \\ h & \longmapsto & gh \end{array} \quad \textbf{and} \quad \begin{array}{rcl} R_g : G & \longrightarrow & G \\ h & \longmapsto & hg. \end{array} \tag{7}$$

- Translations are diffeomorphisms.

Note $\mathfrak{X}(G)$ the Lie algebra of all smooth vector fields on a Lie group G.

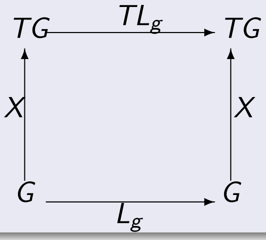
Definition 3

$X \in \mathfrak{X}(G)$ sur G is said to be left-invariant if for all $g \in G$,

$$L_{g*}X = X \qquad (8)$$

that is for all $g, h \in G$,

$$T_hL_g \cdot X_{|h} = X_{|gh}. \qquad (9)$$



- We note $\mathfrak{X}_L(G)$ the subset of $\mathfrak{X}(G)$ consisting of left-invariant vector fields on G.
- $\mathfrak{X}_L(G)$ is a Lie subalgebra of $\mathfrak{X}(G)$.

Definition 4

$\mathfrak{X}_L(G)$ is called the Lie algebra of the Lie group G.

Let $\xi \in T_eG$. One defines an element X^ξ of $\mathfrak{X}_L(G)$ by setting: $\forall g \in G$,

$$X^\xi_{|g} = T_eL_g \cdot \xi. \tag{10}$$

Proposition 1

The map

$$\begin{array}{rcl} f : T_eG & \longrightarrow & \mathfrak{X}_L(G) \\ \xi & \longmapsto & X^\xi \end{array} \tag{11}$$

is a linear isomorphism.

- One can then transport the Lie algebra structure of $\mathfrak{X}_L(G)$ on T_eG as follows: $\forall \xi, \eta \in T_eG$,

$$[\xi, \eta]_L := [X^\xi, X^\eta]_{|e}. \tag{12}$$

- Hence, the Lie algebra $\mathfrak{X}_L(G)$ can be identify with $(T_eG, [,]_L)$ and we will note it by $\mathcal{G}$.

- Let G be a Lie group with neutral element e and Lie algebra $\mathcal{G} = T_e G$.
- For any $\xi \in \mathcal{G} = T_e G$, there exists a unique integral curve $\gamma_\xi : \mathbb{R} \to G$ of X^ξ passing through e at $t = 0$:

$$\gamma_\xi(0) = e \quad \text{and} \quad \dot{\gamma}_\xi(t) = X^\xi_{|\gamma(t)}, \ \forall\, t \in \mathbb{R}. \tag{13}$$

Definition 5

The exponential map of the Lie group G is the map $\exp : \mathcal{G} \to G$ given by

$$\exp(\xi) = \gamma_\xi(1). \tag{14}$$

Definition 6

A connection ∇ on a Lie group G is left-invariant if for any left-invariant vector fields X and Y, the vector field $\nabla_X Y$ is also left-invariant.

Proposition 2

There exists a bijective correspondance between left-invariant connections on a Lie group G and bilinear maps $\alpha : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$.

The connection ∇ is related to α by:

$$\alpha(\xi, \eta) := (\nabla_{X^\xi} X^\eta)_{|e} \,. \tag{15}$$

Proposition 3

Let ∇ be a connection induced by a bilinear map $\alpha : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$. The following assertions are equivalent.

1. *For all $\xi \in \mathcal{G}$, the curve $t \mapsto \exp(t\xi)$ is a geodesic.*
2. *α is skew-symmetric.*

Definition 7

A left-invariant connection ∇ on a Lie group G such that for all $\xi \in \mathcal{G}$, the curve $t \mapsto \exp(t\xi)$ is a geodesic is called a **Cartan-Schouten connection**[a].

[a] Cartan, É. and Schouten, J. A. : *On the geometry of the group manifold of simple and semi-simple groups.* Proc. Amsterdam 29 (1926), 803-815.

- The bilinear map $\alpha : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ given by

$$\alpha(\xi, \eta) = \lambda[\xi, \eta], \tag{16}$$

for some $\lambda \in \mathbb{R}$, induces Cartan-Schouten connections.

- The torsion and the curvature read:

$$T(\xi, \eta) = (2\lambda - 1)[\xi, \eta], \quad R(\xi, \eta)\zeta = \lambda(\lambda - 1)\Big[[\xi, \eta], \zeta\Big]. \tag{17}$$

- For $\lambda = 0$ and $\lambda = 1$, we get $R \equiv 0$ and $T \not\equiv 0$.
- The connection is symmetric iff $\lambda = \dfrac{1}{2}$.

- The classical Cartan-Schouten connections on a Lie group.

Valeurs de λ	Courbures	Torsions
$\lambda = 0$	$R = 0$	$T(X, Y) = -[X, Y]$
$\lambda = \frac{1}{2}$	$R(X, Y)Z = -\frac{1}{4}[[X, Y], Z]$	$T = 0$
$\lambda = 1$	$R = 0$	$T(X, Y) = [X, Y]$

Table 1: Curvature and torsions of classical Cartan-Schouten connections

Classical Cartan-Schouten connections

Definition 8

On appelle

- Cartan-Schouten -1-connection:

$$\nabla^-_X Y = 0; \tag{18}$$

- Cartan-Schouten 0-connection or canonical Cartan-Schouten connection:

$$\nabla^0_X Y = \frac{1}{2}[X, Y]; \tag{19}$$

- Cartan-Schouten $+1$-connection:

$$\nabla^+_X Y = [X, Y]. \tag{20}$$

- Inverse problem : given a Lie group, does it exist a metric μ which is parallel with right to the 0-connection ?
- Study properties of Lie groups admitting such metrics.
- Study properties of metrics μ on Lie groups G which are compatible with the Cartan-Schouten 0-connection.
- Such metric μ satisfy

$$X \cdot \mu(Y, Z) = \frac{1}{2}\Big(\mu([X, Y], Z) + \mu(Y, [X, Z])\Big) \tag{21}$$

for any left invariant vector fields X, Y, Z on G.

Definition 9

A metric μ on a Lie group G which is compatible with the Cartan-Schouten canonical connection will be called a Cartan-Schouten metric.

- Consider the 3-dimensional Heisenberg group

$$\mathbb{H}_3 := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in \mathbb{R} \right\}. \tag{22}$$

- We identify $\mathbb{H}_3$ with $\mathbb{R}^3$, with the multiplication

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + xy'). \tag{23}$$

Proposition 4 (A. Diatta; B. M.; F. Sy)

Any Cartan-Schouten metric μ on $\mathbb{H}_3$ is of the form

$$\begin{aligned} \mu \;=\; & \left(\frac{1}{4}ay^2 - cy + m\right) dx^2 \\ & + \left(\frac{1}{4}ax^2 - bx + e\right) dy^2 + a\, dz^2 \\ & + \left(\frac{1}{4}axy - \frac{1}{2}cx - \frac{1}{2}by + d\right)\, dxdy \\ & - \left(\frac{1}{2}ay - c\right)\, dxdz - \left(\frac{1}{2}ax - b\right)\, dydz\,, \end{aligned} \tag{24}$$

where a, b, c, d, e, m are real constants such that

$$-ad^2 + aem - b^2 m + 2bcd - c^2 e \neq 0. \tag{25}$$

1. If we set $a = e = m = 1$ and $b = c = d = 0$, we recover the metric given by Thomson[1]

$$\mu = dx^2 + dy^2 + \left(dz - \frac{y}{2}dx - \frac{x}{2}dy\right)^2, \tag{26}$$

which is a Riemannian metric.

2. For $m = e = -a = 1$ and $b = c = d = 0$ we get

$$\begin{aligned} \mu &= \left(1 - \frac{1}{4}y^2\right) dx^2 + \left(1 - \frac{1}{4}x^2\right) dy^2 - dz^2 \\ &\quad - \frac{1}{4}xy\, dxdy + \frac{1}{2}y\, dxdz + \frac{1}{2}x\, dydz, \end{aligned} \tag{27}$$

which is a Lorentzian metric.

Remark 1

Both metrics have the same Levi-Civita connection, although one is Riemannian and the other Lorentzian.

[1] Thompson, G. : **Metrics Compatible with a Symmetric Connection in Dimension Three.** Journal of geometry and Physics, 19, (1996), 1-17.

- A metric μ on a Lie group G is said to be biinvariant if it is invariant under left and right translations.
- On the Lie algebra $\mathcal{G}$ of G, this is equivalent to : $\forall X, Y, Z \in \mathcal{G}$,

$$\mu_\epsilon\Big([X, Y], Z\Big) + \mu_\epsilon\Big(Y, [X, Z]\Big) = 0. \tag{28}$$

- Recall that a Cartan-Schouten metric on a Lie group satisfies:

$$\underbrace{X \cdot \mu(Y, Z)}_{=0 \text{ if } \mu \text{ is invariant}} = \frac{1}{2}\Big(\mu([X, Y], Z) + \mu(Y, [X, Z])\Big) \tag{29}$$

for any left invariant vector fields X, Y, Z on G.

- So any biinvariant metric is Cartan-Schouten.

Right trivializations of the tangent and the cotangent bundle of a Lie group

- Right trivialization, of the cotangent bundle T^*G of a Lie group G:

$$f_1 : T^*G \to G \times \mathcal{G}^*, \quad (\sigma, \nu_\sigma) \mapsto (\sigma, \nu_\sigma \circ T_\epsilon R_\sigma). \tag{30}$$

- Likewise, the right trivialization of the tangent bundle TG is given by

$$f_2 : TG \to G \times \mathcal{G}, \quad (\sigma, X_\sigma) \mapsto (\sigma, T_\sigma R_{\sigma^{-1}} X_\sigma). \tag{31}$$

- We endow the manifolds $G \times \mathcal{G}$ and $G \times \mathcal{G}^*$ with the group structures:

$$(\sigma_1, x)(\sigma_2, y) \ := \ \Big(\sigma_1\sigma_2, x + Ad_{\sigma_1} y\Big) \tag{32}$$

$$(\sigma_1, f)(\sigma_2, g) := \Big(\sigma_1\sigma_2, f + Ad^*_{\sigma_1} g\Big). \tag{33}$$

Lie group structures on tangent and cotangent bundles of Lie groups

- T^*G inherits a Lie group structure obtained by pulling back (33):

$$(\sigma, \nu_\sigma)(\tau, \alpha_\tau) \ = \ \Big(\sigma\tau, \nu_\sigma \circ T_{\sigma\tau} R_{\tau^{-1}} \ + \ \alpha_\tau \circ T_{\sigma\tau} L_{\sigma^{-1}}\Big) \, . \quad (34)$$

- TG also inherits a Lie group structure, the pullback of (32):

$$(\sigma, X_\sigma)(\tau, Y_\tau) \ = \ \Big(\sigma\tau, T_\sigma R_\tau X_\sigma + T_\tau L_\sigma Y_\tau\Big) \, . \quad (35)$$

A characterization of Lie groups with a biinvariant metric 30

Theorem 10 (A. Diatta; B. M.; F. Sy)

If a Lie group G has a biinvariant metric, then the Lie groups T^*G and TG are isomorphic.

Proposition 5

Any left (or right) invariant Cartan-Schouten metric μ is biinvariant.

Indeed, the Koszul formula writes : for all vector fields X, Y, Z on G:

$$\begin{aligned}\mu([X,Y],Z) &= X\Big(\mu(Y,Z)\Big) + Y\Big(\mu(Z,X)\Big) - Z\Big(\mu(X,Y)\Big) \\ &\quad -\mu(X,[Y,Z]) - \mu(Y,[X,Z]) + \mu(Z,[X,Y]). \qquad (36)\end{aligned}$$

This also can be written as :

$$\mu(X,[Y,Z]) + \mu(Y,[X,Z]) = X(\mu(Y,Z)) + Y(\mu(Z,X)) - Z(\mu(X,Y)). \qquad (37)$$

If X, Y, Z are left-invariant each of the terms on the right hand side vanishes.

- We are then interested in non invariant Cartan-Schouten metrics.
- Generalization of biinvariant metrics.
- Larger family : strictly contains Lie groups with biinvariant metrics.
- Other statistical models.

Theorem 11 (A. Diatta; B. M.; F. Sy)

Let G be a Lie group with neutral element ε *and* $\mathcal{G}$ *its Lie algebra.*

1. *If* μ *is a Cartan-Schouten on G then the value* $\mu_\varepsilon := \bar{\mu}$ *is* $ad_{[\mathcal{G},\mathcal{G}]}$*-invariant; that is*

$$\bar{\mu}\Big(\big[[x,y],a\big],b\Big) + \bar{\mu}\Big(a,\big[[x,y],b\big]\Big) = 0, \tag{38}$$

   for all $x, y, a, b \in \mathcal{G}$.

2. *In particular, if* $\exp : \mathcal{G} \to G$ *is a diffeomorphism and* $\log := \exp^{-1}$, *we have: for any* $\sigma \in G$ *and any* $x, y \in \mathcal{G}$,

$$\Big(\mu(x^+,y^+)\Big)(\sigma) = \bar{\mu}\Big(Ad_{\exp(\frac{1}{2}\log\sigma)}x, Ad_{\exp(\frac{1}{2}\log\sigma)}y\Big) \tag{39}$$

$$= \sum_{p,q=0}^{\infty} \frac{1}{2^{p+q}p!q!}\bar{\mu}\Big(ad^p_{\log\sigma}x, ad^q_{\log\sigma}y\Big)\Big) \tag{40}$$

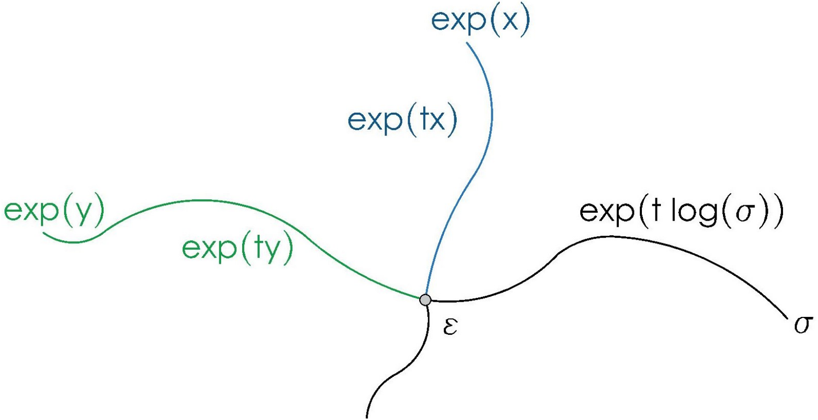


Figure 3: Construction of the metric by parallel transportation along geodesics

Theorem 12 (A. Diatta; B. M.; F. Sy)

If a perfect Lie group G possesses a Cartan-Schouten metric μ, then μ is necessarily a biinvariant metric.

- If X_1, X_2, Y, Z are all left invariant vector fields on G, we have

$$\mu\Big(\big[[X_1, X_2], Y\big], Z\Big) + \mu\Big(Y, \big[[X_1, X_2], Z\big]\Big) = 0. \tag{41}$$

- Since $[\mathcal{G}, \mathcal{G}] = \mathcal{G}$, the equality (41) linearly extends to

$$\mu\Big([X, Y], Z\Big) + \mu\Big(Y, [X, Z]\Big) = 0, \tag{42}$$

for any left invariant vector fields X, Y, Z on G.

Proposition 6 (A. Diatta; B. M.; F. Sy)

Suppose a perfect Lie group G has a Cartan-Schouten metric. Then

- T^*G *is a perfect Lie group;*
- *in particular, if G is semisimple, then T^*G is a perfect Lie group.*

Theorem 13 (A. Diatta; B. M.; F. Sy)

Let G be a n-dimensional simple Lie group. Every Cartan-Schouten metric μ on T^*G is biinvariant and has signature (n, n).

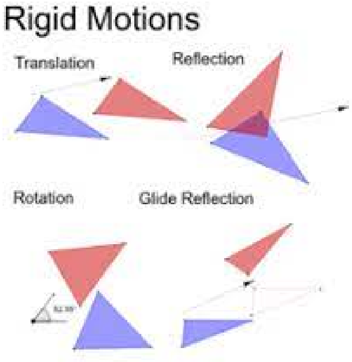
1. *If n is odd, then $\mu = sK_0 + t\langle , \rangle$, with $s, t \in \mathbb{R}$, $t \neq 0$; so*

$$[\mu] = \begin{pmatrix} s\mathbb{I}_{p,n} & t\mathbb{I}_n \\ t\mathbb{I}_n & \mathbf{0}_n \end{pmatrix} \tag{43}$$

   with $\mathbb{I}_{p,n} = diag(-1, \dots, -1, 1, \dots, 1)$, $p \geq 0$, $s, t \in \mathbb{R}$, $t \neq 0$ and $\mathbf{0}_n$ is the zero $n \times n$ matrix. So space of such metrics $= \mathbb{R} \times \mathbb{R}^*$,

2. *If n is even, then $\mu = s_1 K_0 + s_2 K_J + t_1\langle , \rangle + t_2\langle , \rangle_J$, with $s_1, s_2, t_1, t_2 \in \mathbb{R}$. That is, for any $x, y \in \mathcal{G}$, $f, g \in \mathcal{G}^*$,*

$$\begin{aligned} \mu\Big((x, f), (y, g)\Big) &= s_1 K_0(x, y) + s_2 K_J(x, y) \\ &\quad + t_1\langle (x, f), (y, g)\rangle + t_2\langle (x, f), (y, g)\rangle_J. \end{aligned} \tag{44}$$



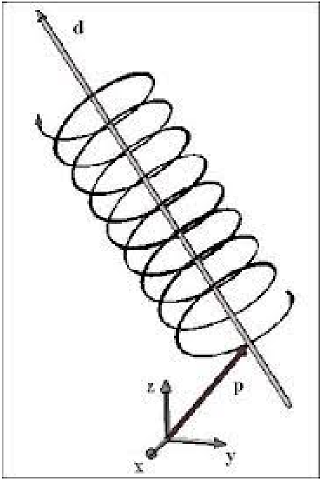
- A rigid motion of an object is a continuous movement of the particles in the object such that the distance between any two particles remains fixed at all times.
- The net movement of a rigid body from one location to another via a rigid motion is called a rigid displacement.

- A rigid displacement may consist of both translation and rotation of the object.
- A rigid displacement is represented by an element of the special affine group

$$SE(3) := SO(3) \ltimes \mathbb{R}^3 = \left\{ \begin{pmatrix} R & d \\ 0 & 1 \end{pmatrix}, R \in SO(3), d \in \mathbb{R}^3 \right\}, \quad (45)$$

- That is, A rigid motion is represented by a curve on $SE(3)$[2].

[2] Zefran,M.; Kumar, V. and Croke, C.; Metrics and Connections for Rigid-Body Kinematics. The International Journal of Robotics Research, 18 (2) 242 (1999).



- A screw displacement is a rigid displacement consisting of a rotation at constant angular velocity around an axis (the screw or twist axis) followed by a translation with constant (translationnal) velocity along the same axis.
- From the famous Chasles theorem, any rigid motion can be realized as a screw motion.

Theorem 14

The Lie group $SE(3)$ of rigid motions of the Euclidean space $\mathbb{R}^3$, is isomorphic to both $T^*SO(3)$ and $TSO(3)$, endowed with their Lie group structures induced by the right trivializations.

Theorem 15

Let μ be a pseudo-Riemannian metric on $SE(3)$ such that every screw motion is a geodesic.

- *Then μ is biinvariant and furthermore, its matrix is of the form*

$$[\mu] = \begin{pmatrix} s\mathbb{I}_3 & t\mathbb{I}_3 \\ t\mathbb{I}_3 & \mathbf{0}_3 \end{pmatrix} \tag{46}$$

  in some basis of $SE(3)$.
- *There is no Riemannian metric on SE(3) for which every screw motion is a geodesic.*

- We consider the Lie group $SE(2,1) := SO(2,1) \ltimes \mathbb{R}^3$ made of invertible affine displacements of $\mathbb{R}^3$ whose linear parts are (oriented and) preserve the Lorentz metric in $\mathbb{R}^3_1$.
- A natural way to represent the Lie group $SE(2,1)$ is as the following group of 4×4 real matrices

$$SE(2,1) := SO(2,1) \ltimes \mathbb{R}^3 = \left\{ \begin{pmatrix} R & v \\ 0 & 1 \end{pmatrix}, R \in SO(2,1), v \in \mathbb{R}^3 \right\}, \tag{47}$$

- By analogy with the group $SE(3) := SO(3) \ltimes \mathbb{R}^3$ of rigid displacement of the Euclidean 3-space, we call $SE(2,1)$ the group of rigid displacements of the Minkowski 3-space.
- By rigid motion in the Minkowski 3-space, one means a curve in $SE(2,1)$.

Theorem 16

The special affine Lie group $SE(2,1) = SO(2,1) \ltimes \mathbb{R}^3$ *is isomorphic to both* $TSO(2,1)$ *and* $T^*SO(2,1)$ *endowed with their Lie group structure induced by the right trivializations.*

Theorem 17

Let μ be a pseudo-Riemannian metric on $SE(2,1)$ such that every screw motion on R^3_1 is a geodesic.

- *Then μ is biinvariant and furthermore, its matrix is of the form*

$$[\mu] = \begin{pmatrix} s\mathbb{I}_{1,3} & t\mathbb{I}_3 \\ t\mathbb{I}_3 & \mathbf{0}_3 \end{pmatrix} \tag{48}$$

  in some basis of $SE(2,1)$.
- *There is no Riemannian metric on $SE(2,1)$ for which every screw motion is a geodesic.*

The End !

Thank You For Kind Attention !