Spectrum of combinatorial $k-$systoles on Schottky surface of rank 2

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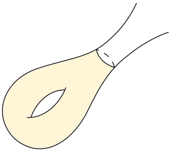
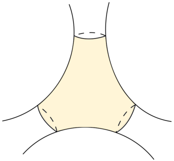
SCHOTTKY SURFACE OF RANK 2

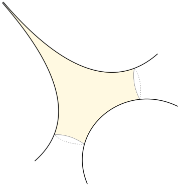
Let S be a Schottky surface of rank 2.



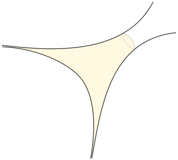
Hyperbolic torus



Pair of pants



Pair of pants with cusp



Pair of pants with two cusps

Combinatorial k−systole

Let $\mathcal{G}^c(S)$ be the set of closed geodesics on S and $k \in \mathbb{N}$.

$$s_k^c(S) := \inf\{L(\gamma) : \gamma \in \mathcal{G}^c(S);\;\; i(\gamma;\gamma) \geq k\}$$

- $L(\gamma)$ is the combinatorial length of γ
- $i(\gamma;\gamma)$ self-intersection numbers of γ.

Definition 1

A closed geodesic γ on S is a **combinatorial k-systole** if $L(\gamma) = s_k^c(S)$ and $i(\gamma;\gamma) \geq k$.

Example of combinatorial $k-$systole

The closed geodesic $a(a\bar{b})^2$ on a pair of pants is a combinatorial 5-systole with 6 self-intersections.

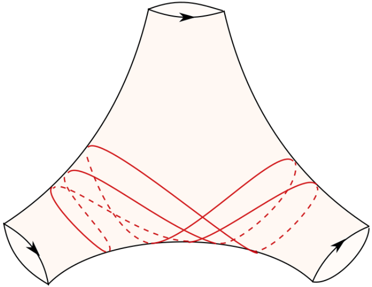


Figure – 5−systole with 6 self-intersections

Combinatorial $k-$systole

$$I_k^c(S) = \max\{i(\gamma;\gamma) : \ \gamma \text{ combinatorial } k\text{-systole}\}$$

Question

- **What is the asymptotic behaviour of $I_k(S)$?**

Answer

- $I_k^c(S) \sim k$ and $\limsup_{k\to+\infty} I_k^c(S) - k = +\infty.$

Question

- **Let m be an integer of $[k, I_k^c(S)]$, does it exist a combinatorial $k-$systole γ such that : $i(\gamma,\gamma) = m$?**

PLAN

Coding the limit set

Self-intersection and combinatorial length

Combinatorial k-systoles

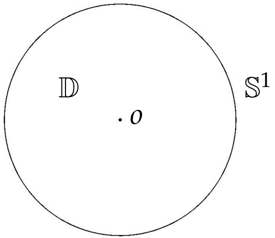
Coding the limit set

Self-intersection and combinatorial length

Combinatorial k-systoles

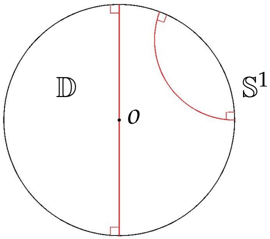
POINCARÉ DISK

$$\mathbb{D} = \{z = x + iy \in \mathbb{C}, |z| \leq 1\} \quad ds^2 = 4\frac{dx^2 + dy^2}{(1 - |z|^2)^2}$$



GEODESICS

$$\mathbb{D} = \{z = x + iy \in \mathbb{C}, \ \ |z| \leq 1\} \quad ds^2 = 4\frac{dx^2 + dy^2}{(1 - |z|^2)^2}$$



ISOMETRY 10

$g : \mathbb{D} \longrightarrow \mathbb{D}$ is an isometry of the Poincaré disk if and only if

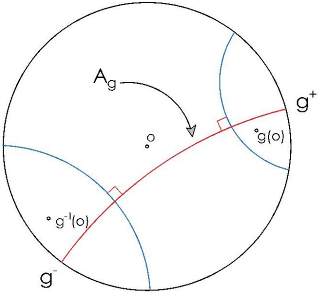
$$g(z) = \frac{az + \bar{c}}{cz + \bar{a}} \quad \text{or} \quad g(z) = \frac{a\bar{z} + \bar{c}}{c\bar{z} + \bar{a}} \ \forall\, z \in \mathbb{D}$$

where $a, c \in \mathbb{C}$ and $|a|^2 - |c|^2 = 1$.

HYPERBOLIC ISOMETRY

g an isometry which does not fix 0.

$$C(g) = \{z \in \mathbb{D} : \; d(o,z) = d(g(o),z)\}$$

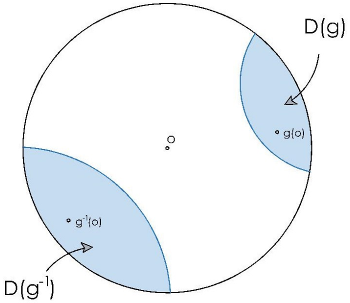


HYPERBOLIC ISOMETRY

g an isometry which does not fix o.

$$C(g) = \{z \in \mathbb{D} : \quad d(o, z) = d(g(o), z)\}$$

$$D(g) = \{z \in \mathbb{D} : \quad d(g(o), z) \leq d(o, z)\}$$



Dynamic of hyperbolic isometry

$$Ext(D(g)) = \mathbb{D} \setminus D(g) \quad g(D(g^{-1})) = Ext(D(g))$$

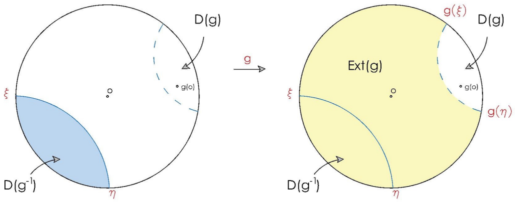
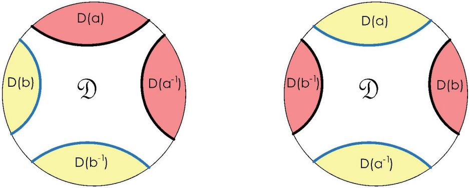


Figure – dynamic of hyperbolic isometry

SCHOTTKY GROUP



Definition 2

Let a et b be two hyperbolic isometries satisfying the following separation hypothesis :

$$\overline{D(a) \cup D(a^{-1})} \cap \overline{D(b) \cup D(b^{-1})} = \emptyset.$$

The group $\Gamma := \langle a, b \rangle$ is a Schottky group of rank 2.

Reduced words 15

$\overline{\Gamma} := \{a, \bar{a}, b, \bar{b}\}$ where $\bar{a} = a^{-1}$ and $\bar{b} = b^{-1}$.

Definition 3

- A finite word $e_1 e_2 \cdots e_n,\ \ e_j \in \overline{\Gamma}$ is said to be **reduced** if $e_j \neq \bar{e}_{j+1}$ for every $j = 1, \cdots n-1$ and **cyclically reduced** if it is reduced and $e_n \neq \bar{e}_1$.
- An infinite word $e_1 e_2 \cdots$ or a bi-infinite word $\cdots e_{-1} e_0 e_1 \cdots$ is reduced if each of its finite subword is reduced.

FIRST ORDER INTERVAL

For $e \in \overline{\Gamma} = \{a, \bar{a}, b, \bar{b}\}$, $A(e) = \overline{D(e)} \cap \mathbb{S}^1$: first order interval

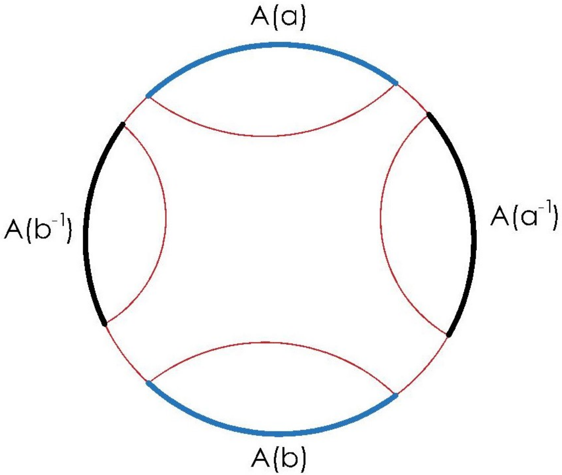


Figure – first order interval

M-TH ORDER INTERVAL

If $e_1e_2\cdots$ is a reduced word in $\overline{\Gamma}$, then
$A(e_1e_2\cdots e_n) = e_1e_2\cdots e_{n-1}A(e_n)$ is *m-th order interval*.

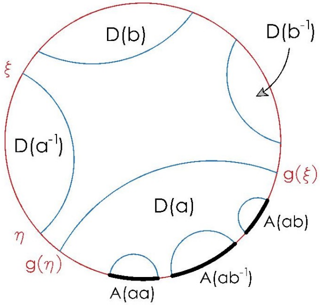


Figure – second order interval

$$A(e_1e_2\cdots e_n) \subset A(e_1e_2\cdots e_{n-1}) \subset \cdots \subset A(e_1).$$

LIMIT SET

Definition 4

The limit set of the group Γ is the set $\Lambda = \overline{\Gamma.o} \setminus \Gamma.o$

Proposition 5

$$\bigcap_{n=1}^{\infty} A(e_1 e_2 \cdots e_n) = \lim_{n \to +\infty} e_1 e_2 \cdots e_n.o \in \Lambda.$$

$$\Lambda = \bigcap_{n=1}^{\infty} \bigcup_{e_1 e_2 \cdots \in W_+} A(e_1 e_2 \cdots e_n).$$

$W_+ :=$ *set of infinite reduced words*

CODING THE LIMIT SET

- From now on we write $\eta = e_1 e_2 \cdots$ if $\eta = \lim_{n \to +\infty} e_1 e_2 \cdots e_n . o$.
- $\bar{e}_1 \eta = e_2 e_3 \cdots$ and if $f \in \overline{\Gamma}$, $f \neq \bar{e}_1$ then $f\eta = f e_1 e_2 \cdots$.
- If η is fixed by a hyperbolic isometry, then $\exists\, g \in \Gamma$ such that $g(\eta)$ is **periodic**. We write $g(\eta) = \overline{e_1 e_2 \cdots e_m}$ where $e_1 e_2 \cdots e_m$ is the period of the word .

Coding oriented geodesics

- Each reduced bi-infinite word $\mathbf{e} = \cdots e_{-1}e_0e_1\cdots$ determines a unique oriented geodesic with positive endpoint $e_1e_2\cdots$ and negative endpoint $\bar{e}_0\bar{e}_{-1}\cdots$. We denote this geodesic $\gamma(\mathbf{e})$.
- Let $\sigma^n(\mathbf{e}),\ n \in \mathbb{Z}$ be the word whose $j-th$ entry is in position $j+n$ in $\mathbf{e}$. Then $\gamma(\sigma^n(\mathbf{e})) = (e_1\cdots e_n)^{-1}\gamma(\mathbf{e})$.

Cyclic lexicographically ordering

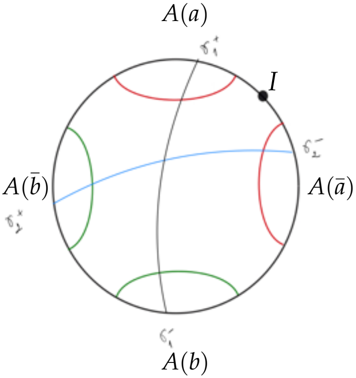


Figure – The order $<_I$ defined by a fixed point $I \in \partial\mathbb{D}$.

$$A_a = \{a, \bar{b}, b, \bar{a}\},\ A_{\bar{b}} = \{\bar{b}, b, \bar{a}, a\},\ A_b = \{b, \bar{a}, a, \bar{b}\} \text{ and } A_{\bar{a}} = \{\bar{a}, a, \bar{b}, b\}.$$

Birman-Series Theorem

Let $\eta = e_1 e_2 \cdots$ and $\zeta = f_1 f_2 \cdots$ be distinct points of the limit set Λ. Then η precedes ζ in anticlockwise order around $\mathbb{S}^1$ starting from the point I (see figure 5) if and only if either :

1. e_1 precedes f_1 in the alphabet A_a, or
2. $e_i = f_i$ for each $i = 1, \cdots, m$ and e_{m+1} precedes f_{m+1} in the alphabet $A_{\bar{e}_m}$.

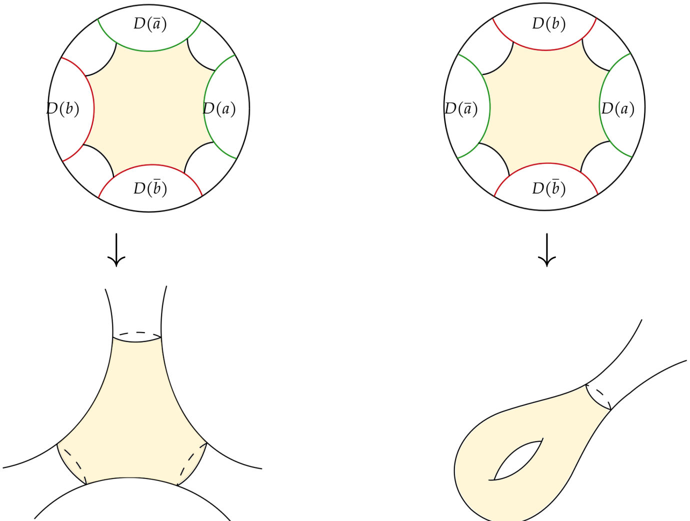
Coding the limit set

Self-intersection and combinatorial length

Combinatorial k-systoles

SCHOTTKY SURFACE OF RANK 2

$\Gamma := \langle a, b\rangle$ a Schottky group of rank 2. $S = \mathbb{D}/\Gamma$ is a **Schottky surface of rank 2.**



Coding of closed geodesics

Combinatorial length

- Let $w = e_1 e_2 \cdots e_n$ be a cyclically reduced word and $W = \cdots www \cdots$.
  The endpoints of $\gamma(W)$ is fixed by the isometries w and w^{-1}. Thus these isometries fix the geodesic $\gamma(W)$.
- The projection of the geodesic $\gamma(w)$ to S is a closed geodesic of S.
- This closed geodesic is associated to the word w.

Definition 6

The **combinatorial length** of a closed geodesic γ of S is the length of the word associated to γ. We denote it by $L(\gamma)$.

SELF-INTERSECTION

$$\gamma \in \mathcal{G}^c(S) \longrightarrow w = e_1 e_2 \cdots e_m.$$

$$\gamma_i(e_i e_{i+1} \cdots e_{i-1}; \quad \bar{e}_{i-1} \cdots \bar{e}_{i+1} \bar{e}_i), \quad i = 1, \cdots, m.$$

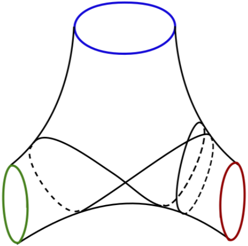
Proposition 7

The geodesics $\gamma_i,\ 1 \leq i \leq m$, are the unique lifts of γ on $\mathbb{D}$ which intersect the fundamental domain of Γ.

The number of self-intersections of the closed geodesic γ is given by :

$$i(\gamma;\gamma) = \#\{\gamma_i \cap \gamma_j \cap \mathcal{D} \neq \emptyset;\ 1 \leq i < j \leq L(\gamma)\}.$$

EXAMPLE



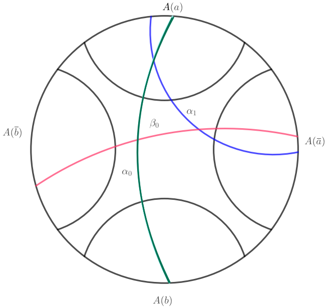


Figure – self-intersections of the geodesic $a^2\bar{b}$ on a pair of pants

Equivalence relation

$$A_\gamma = \{\gamma_i \cap \gamma_j : \quad 1 \leq i < j \leq L(\gamma)\}$$

$$i(\gamma;\gamma) = \#(A_\gamma \cap \mathcal{D}).$$

We define the following equivalence relation on A_γ :

$$\gamma_i \cap \gamma_j \sim \gamma_k \cap \gamma_l \iff \exists g \in \Gamma \mid g(\gamma_i) = \gamma_k \text{ and } g(\gamma_j) = \gamma_l.$$

Thus the number of self-intersections of the closed geodesic γ is given by :

$$i(\gamma;\gamma) = \#(A_\gamma / \sim).$$

SELF-INTERSECTIONS ON PAIR OF PANTS

Theorem 8 (E.A.A.D- M.Gaye)

Let γ be a closed geodesic on **a pair of pants** *associated to the word $w = s_1^{i_1} r_1^{j_1} \cdots s_n^{i_n} r_n^{j_n}$, then*

$$i(\gamma;\gamma) = nL(\gamma) - 2n^2 + H(w) - \sum_{k=1}^{n} \sum_{l=k+1}^{n} |i_k - i_l| + |j_k - j_l|.$$

Self-intersections on pair of pants

Question 1

- **What is the maximal number of self-intersections of a closed geodesic γ of combinatorial length $L(\gamma)$?**

PAIR OF PANTS

Lemma 9

For any word $w = s_1^{i_1} r_1^{j_1} \cdots s_n^{i_n} r_n^{j_n}$, there exists a permutation σ of $\{i_1, i_2, \cdots, i_n\}$ and a permutation τ of $\{j_1, j_2, \cdots, j_n\}$ such that :

$$H(w) \leq H\left(a^{\sigma(i_1)} \bar{b}_1^{\tau(j_1)} \cdots a^{\sigma(i_n)} \bar{b}^{\tau(j_n)}\right).$$

$$i(\gamma;\gamma) \leq nL(\gamma) - n^2.$$

PAIR OF PANTS

Theorem 10 (E.A.A.D- Masseye Gaye 2021)

Let γ be a closed geodesic on **a pair of pants** *and $L(\gamma)$ the combinatorial length of γ. Then*

$$i(\gamma;\gamma) \leq \begin{cases} \dfrac{L^2(\gamma)}{4} - 1 & \text{if } L(\gamma) \text{ is even} \\[2ex] \dfrac{L^2(\gamma) - 1}{4} & \text{if } L(\gamma) \text{ is odd} \end{cases}$$

HYPERBOLIC TORUS

Theorem 11 (M.Chas-A.Phillips 2008)

Let γ be a closed geodesic on **a hyperbolic torus** *and $L(\gamma)$ the combinatorial length of γ. Then*

$$i(\gamma;\gamma) \leq \begin{cases} \dfrac{(L(\gamma)-2)^2}{4} & \text{if } L(\gamma) \text{ is even} \\[2ex] \dfrac{(L(\gamma)-1)\,(L(\gamma)-3)}{4} & \text{if } L(\gamma) \text{ is odd} \end{cases}$$

Coding the limit set

Self-intersection and combinatorial length

Combinatorial k-systoles

Definition of a combinatorial k-systole

S : Schottky surface of rank 2.
$\mathcal{G}^c(S)$: closed geodesics on S.

$$s_k^c(S) := \inf\{L(\gamma) : \gamma \in \mathcal{G}^c(S);\ \ i(\gamma;\gamma) \geq k\}$$

Definition 12

A **combinatorial k-systole** is a closed geodesic γ on S such that $L(\gamma) = s_k^c(S)$ and $i(\gamma;\gamma) \geq k$.

Example of combinatorial $k-$systole

The closed geodesic $a(a\overline{b})^2$ on a pair of pants is a combinatorial 5-systole and $i(a(a\overline{b})^2, a(a\overline{b})^2) = 6$.

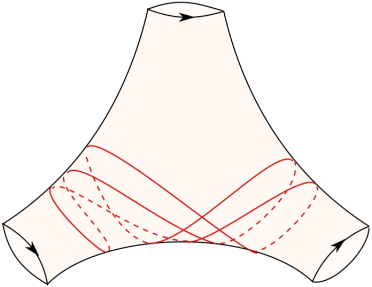


Figure – 5−systole with 6 self-intersections

COMBINATORIAL $k-$SYSTOLE

Definition 13

Let k be a positive integer. A **combinatorial** $k-$**systole** γ_k is **exact** if $i(\gamma_k;\gamma_k) = k$.

For $k \in \mathbb{N}$,

$$I_k^c(S) = \max\{i(\gamma;\gamma) : \gamma \text{ combinarial k-systole}\}.$$

Question 2

- **What is the asymptotic behaviour of $I_k(S)$?**

PAIR OF PANTS

Let S be a pair of pants and n an integer.

- $n^2 < k \leq n^2 + n \implies s_k^c(S) = 2n + 1$. In particular, the closed geodesic $a(a\bar{b})^n$ is a combinatorial k-systole and $n^2 \leq k \leq I_k^c(S) = n^2 + n$.
- $n^2 - n < k \leq n^2 - 1 \implies s_k^c(S) = 2n$. In particular, the closed geodesic $a\bar{b}(a\bar{b})^{n-1}$ is a combinatorial k-systole and $n^2 - n \leq k \leq I_k^c(S) = n^2 - 1$.

Hyperbolic torus

Let S be a hyperbolic torus and n an integer.

1. $n^2 < k \leq n^2 + n \implies s_k^c(S) = 2n + 3$. In particular, the closed geodesic $a^{n+1}b^{n+2}$ is a combinatorial k-systole and $n^2 < k \leq I_k^c(S) = n^2 + n$.
2. Si $n^2 - n < k \leq n^2 \implies s_k^c(S) = 2n + 2$. In particular, the closed geodesic $a^{n+1}b^{n+1}$ is a combinatorial k-systole and $n^2 - n < k \leq I_k^c(S) = n^2$.

COMBINATORIAL k−SYSTOLE

Theorem 14 (E.A.A.D-M.Gaye-A.K.Sané)

Let S be a Schottky surface of rank 2.Then

$$\lim_{k\to+\infty}\frac{I_k^c(S)}{k}=1.$$

$$\liminf_{k\to+\infty}\ I_k^c(S)-k=0\ \ et\ \ \limsup_{k\to+\infty}\ I_k^c(S)-k=+\infty.$$

SPECTRUM OF COMBINATORIAL $k-$SYSTOLES

Question 3

- **Let m be an integer of $[k, I_k^c(S)]$, does it exist a combinatorial $k-$systole γ such that : $i(\gamma, \gamma) = m$?**

Spectrum of combinatorial $k-$systoles

Hyperbolic surface

Theorem 15 (E.A.A.D-M.Gaye-A.K.Sané)

Let S be a hyperbolic torus and $k \in \mathbb{N}$. Then for any integer m of $[k, I_k^c(S)]$, there exists a combinatorial $k-$systole γ such that $i(\gamma, \gamma) = m$.

Spectrum of combinatorial $k-$systoles

Pair of pants

Theorem 16 (E.A.A.D-M.Gaye-A.K.Sané)

Let S be a pair of pants and $k \in \mathbb{N}$.

1. *If $s_k^c(S)$ is odd, then for any integer m of $[k, I_k^c(S)]$, there exists a combinatorial $k-$systole γ such that $i(\gamma;\gamma) = m$.*
2. *If $s_k^c(S)$ is even, then for any odd integer m of $[k, I_k^c(S)]$, there exists a combinatorial $k-$systole γ such that $i(\gamma;\gamma) = m$.*

GEOMETRIC LENGTH

ERLANDSONN-PARLIER CONJECTURE

- V.Erlandsson- H.Parlier- T.H.Vo :
  Σ hyperbolic surface with at least one cusp, $\exists K_0 \in \mathbb{N}$ such that for any $k \geq K_0$, **the $k-$systoles are exacts**

$$\lim_{k \to +\infty} I_k(\Sigma) - k = 0.$$

Conjecture (Erlandsson-Parlier)

If Σ is a compact hyperbolic surface, then

$$\limsup_{k \to +\infty} I_k(\Sigma) - k = +\infty.$$

COUNTING CLOSED GEODESICS

$$\#\mathcal{G}(L,K) = \{\gamma \in \mathcal{G}^c(S); \quad L(\gamma) \leq L; \;\; i(\gamma,\gamma) \leq k\}$$

Question 4

- **What is the asymptotic behaviour of $\#\mathcal{G}(L,K)$?**

Merci !!!!