Method for computing hodge diamonds of codimension 3 Calabi-Yau threefold in product of projective spaces.

Andre Saint Eudes Mialebama Bouesso
Université Marien Ngouabi
Faculte des Sciences et Techniques
Brazzaville,Congo
and
Institute of Mathematics of the Polish Academy of Science
Warsaw, Poland

September 15, 2023

Overview

Motivation

Pfaffian varieties

Investigation method

Computation of hodge diamonds

Motivation

What is a Calabi-Yau ?

Definition

By a Calabi-Yau threefold we mean a smooth projective complex variety X of dimension 3 with trivial canonical bundle (i.e $K_X = \wedge^3 T^*X = \mathcal{O}_X$) and $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$.

complete intersection

A smooth projective variety $X \subset \mathbb{P}^n$ is called complete intersection if $X = V(f_1, \cdots, f_{\mathrm{codim}(X)})$. The canonical bundle of a complete intersection is given by $K_X = \mathcal{O}_X(-n - 1 + \sum_{i=1}^{\mathrm{codim}(X)} d_i)$.

Example: complete intersection Calabi-Yau

1. A quintic hypersurface in $\mathbb{P}^4$ i.e $X = V(f)$ with $\deg(f) = 5$ and $K_X = \mathcal{O}_X(-4 - 1 + 5) = \mathcal{O}_X$;
2. In $\mathbb{P}^5$ we have :
   2.1 the intersection of a quadric and a quartic i.e $X = V(f^2, f^4)$ with $K_X = \mathcal{O}_X(-6 + 2 + 4)$;
   2.2 the intersection of two cubics i.e $X = V(f^3, f^3)$ with $K_X = \mathcal{O}_X(-6 + 3 + 3)$;
3. in $\mathbb{P}^6$ we have the intersection of two quadrics and a cubic i.e $X = V(f^2, g^2, f^3)$ with $K_X = \mathcal{O}_X(-7 + 2 + 2 + 3)$.
4. In product of projective spaces, complete intersections Calabi-Yau where studied in
   P. Candelas, A. M. Dale, C.A. lütken, R. Schimmrigk, Complete Intersection Calabi-Yau Manifolds. UTTG-10-87.

Main goal

Our main goal in this project is to investigate and discuss all codimension 3 non-complete intersection Calabi-Yau threefold in product of projective spaces. We are mostly interested by :

1. $\mathbb{P}^2 \times \mathbb{P}^4$;
2. $\mathbb{P}^1 \times \mathbb{P}^5$;
3. $\mathbb{P}^3 \times \mathbb{P}^3$;
4. $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$;
5. $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^3$;
6. $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3$;
7. $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$;
8. $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$;
9. $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$;

Motivation

Buschbaum and Eisenbud

1. Let $X \subset \mathbb{P}^n$ be a pfaffian variety associated to (t, E, N). X is then the degeneracy locus of the skew-symmetric map N and if N is generically of rank $2r$ it degenerates to rank $2r - 2$ in the expected codimension 3, in which case, the pfaffian complex gives the self-dual resolution of the ideal sheaf of X. Moreover, X is locally Gorenstein, subcanonical with $K_X = \mathcal{O}_X(t + 2s - n - 1)$.
2. Every codimension 3, Gorenstein varieties arise in this way.

Pfaffian varieties

Investigation

Given a vector $t = (t_1, t_2) \in \mathbb{Z}^2$ and a decomposable vector bundle F of odd rank $2u + 1$ on $Y = \mathbb{P}^2 \times \mathbb{P}^4$, a global section $N \in H^0(Y, \wedge^2 F(t))$ defines an alternating morphism $N : F^*(t) \to F$. The pfaffian complex associated to (t, F, N) is given by the exact sequence

$$0 \longrightarrow \mathcal{O}_Y(-t-2s) \xrightarrow{P^t} F^*(-t-s) \xrightarrow{N} F(-s) \xrightarrow{P} \mathcal{O}_Y$$

where $s = (s_1, s_2) \in \mathbb{N}^2$ and

$$s_i = c_1(F)_i + u \cdot t_i \text{ for } i = 1, 2.$$

$$P = \frac{1}{u!} \wedge^2 N$$

Pfaffian varieties

A projective variety $X \subset Y$ is called the pfaffian variety associated to (t, F, N) if the structure sheaf $\mathcal{O}_X$ is given by $\mathrm{Coker}(P)$. In this case we have the exact sequence

$$0 \longrightarrow \mathcal{O}_Y(-t-2s) \xrightarrow{P^t} F^*(-t-s) \xrightarrow{N} F(-s) \xrightarrow{P}$$

$$\mathcal{O}_Y \longrightarrow \mathcal{O}_X \longrightarrow 0$$

The sheaf $\mathrm{Im}(P) \subset \mathcal{O}_Y$ is called the pfaffian ideal sheaf of X and denoted by $\mathcal{I}_X$.

Pfaffian manifolds

The Pfaffian variety $X \subset Y$ satisfies then the exact sequence

$$0 \longrightarrow \mathcal{O}_Y(-t-2s) \xrightarrow{P^t} F^*(-t-s) \xrightarrow{N} F(-s) \xrightarrow{P} \mathcal{I}_X \longrightarrow 0$$

Pfaffian Calabi-Yau

A pfaffian variety $X \subset Y$ is called Calabi-Yau if its canonical bundle $K_X = \mathcal{O}_X(-3+t_1+2s_1, -5+t_2+2s_2)$ is trivial i.e $K_X = \mathcal{O}_X$ that is :

$$s_1 = \frac{3-t_1}{2}, s_2 = \frac{5+1-t_2}{2}.$$

For simplicity, choose $t_1 = t_2 = 1$ so that s_1, s_2 are integers. In this case $s_1 = 1, s_2 = 2$. Since $s_i = c_1(E)_i + u \cdot t_i$ then

$$c_1(E)_i = s_i - u \cdot t$$

Investigation in $\mathbb{P}^2 \times \mathbb{P}^4$

Assume that $F = \bigoplus_{i=1}^{2u+1} \mathcal{O}_X(a_i, b_i)$, then

$$\begin{cases} \sum_{i=1}^{2u+1} a_i = 1 - u \\ \sum_{i=1}^{2u+1} b_i = 2 - u \end{cases}$$

Investigation in $\mathbb{P}^2 \times \mathbb{P}^4$

For $u = 2$ we have $F = \bigoplus_{i=1}^{5} \mathcal{O}_X(a_i, b_i)$ and

$$\begin{cases} a_1 + \cdots + a_5 = -1 \\ b_1 + \cdots + b_5 = 0 \end{cases}$$

Theorem

Let $X \subset Y$ be a non-complete intersection pfaffian Calabi-Yau threefold associated to $((1,1), E, N)$. Then

$$-2 \leq a_i \leq 1 \ \forall \ i = 1, \ldots, 5$$

$$-3 \leq b_i \leq 2 \ \forall \ i = 1, \ldots, 5.$$

Investigation in $\mathbb{P}^2 \times \mathbb{P}^4$

The possible solutions are "

$\mathbb{P}^2$	$\mathbb{P}^4$
$(-2,-2,1,1,1)$	$(-3,-3,2,2,2)$
$(-2,-1,0,1,1)$	$(-3,-2,1,2,2)$
$(-2,0,0,0,1)$	$(-3,-1,0,2,2)$
$(-1,-1,0,0,1)$	$(-3,-1,1,1,2)$
$(-1,0,0,0,0)$	$(-3,0,0,1,2)$
	$(-2,-2,1,1,2)$
	$(-2,-1,1,1,1)$
	$(-2,-1,0,1,2)$
	$(-1,-1,0,1,1)$
	$(-1,-1,0,0,2)$
	$(-1,0,0,0,1)$
	$(0,0,0,0,0)$

Investigation in $\mathbb{P}^2 \times \mathbb{P}^4$

We want a bundle $F = \bigoplus_{i=1}^{5} \mathcal{O}_X(a_i, b_i)$ whose matrix $N : F^*(-1,-1) \to F$ is given by

$$N = \begin{pmatrix} 0 & (c_{12}, d_{12}) & (c_{13}, d_{13}) & (c_{14}, d_{14}) & (c_{15}, d_{15}) \\ (c_{12}, d_{12}) & 0 & (c_{23}, d_{23}) & (c_{24}, d_{24}) & (c_{25}, d_{25}) \\ (c_{13}, d_{13}) & (c_{23}, d_{23}) & 0 & (c_{34}, d_{34}) & (c_{35}, d_{35}) \\ (c_{14}, d_{14}) & (c_{24}, d_{24}) & (c_{34}, d_{34}) & 0 & (c_{45}, d_{45}) \\ (c_{15}, d_{15}) & (c_{25}, d_{25}) & (c_{35}, d_{35}) & (c_{45}, d_{45}) & 0 \end{pmatrix}$$

where $c_{ij} = a_i + a_j + 1,\ d_{ij} = b_i + b_j + 1,\ 1 \le i < j \le 5$ and $X = V(f_1, \ldots, f_5)$ is given by the maximal pfaffians of N.

Investigation in $\mathbb{P}^2 \times \mathbb{P}^4$

Bundle	Matrix	Pfaffian	Smoothness
$(-1,0),(-1,0),(0,0),(0,0),(1,0)$	$\begin{pmatrix} 0 & 0 & 01 & 01 & 11 \\ & 0 & 01 & 01 & 11 \\ & & 0 & 11 & 21 \\ & & & 0 & 21 \\ & & & & 0 \end{pmatrix}$	$V(22, 22, 12, 12, 02)$	Smooth
$(-1,2),(0,1),(0,1),(0,-2),(0,-2)$	$\begin{pmatrix} 0 & 04 & 04 & 01 & 01 \\ & 0 & 13 & 10 & 10 \\ & & 0 & 10 & 10 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}$	$V(20, 11, 11, 14, 14)$	Smooth
$(-1,-1),(0,1),(0,1),(0,1),(0,-2)$	$\begin{pmatrix} 0 & 01 & 01 & 01 & 0 \\ & 0 & 13 & 13 & 10 \\ & & 0 & 13 & 10 \\ & & & 0 & 10 \\ & & & & 0 \end{pmatrix}$	$V(23, 11, 11, 11, 14)$	Smooth
$(-1,0),(-1,0),(0,1),(0,0),(1,-1)$	$\begin{pmatrix} 0 & 0 & 02 & 01 & 10 \\ & 0 & 02 & 01 & 10 \\ & & 0 & 12 & 21 \\ & & & 0 & 20 \\ & & & & 0 \end{pmatrix}$	$V(22, 22, 11, 12, 03)$	Smooth

Investigation in $\mathbb{P}^2 \times \mathbb{P}^4$

Bundle	Matrix	Pfaffian	Smoothness
$(-1,2),(0,0),(0,0),(0,-1),(0,-1)$	$\begin{pmatrix} 0 & 03 & 03 & 02 & 02 \\ & 0 & 11 & 10 & 10 \\ & & 0 & 10 & 10 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}$	$V(20,12,12,13,13)$	Smooth
$(-1,1),(0,0),(0,0),(0,0),(0,-1)$	$\begin{pmatrix} 0 & 02 & 02 & 02 & 01 \\ & 0 & 11 & 11 & 10 \\ & & 0 & 11 & 10 \\ & & & 0 & 10 \\ & & & & 0 \end{pmatrix}$	$V(21,12,12,12,13)$	smooth
$(-1,0),(0,0),(0,0),(0,0),(0,0)$	$\begin{pmatrix} 0 & 01 & 01 & 01 & 01 \\ & 0 & 11 & 11 & 11 \\ & & 0 & 11 & 11 \\ & & & 0 & 11 \\ & & & & 0 \end{pmatrix}$	$V(22,12,12,12,12)$	smooth
$(-1,0),(-1,0),(0,1),(0,0),(1,-1)$	$\begin{pmatrix} 0 & 0 & 02 & 01 & 10 \\ & 0 & 02 & 01 & 10 \\ & & 0 & 12 & 21 \\ & & & 0 & 20 \\ & & & & 0 \end{pmatrix}$	$V(22,22,11,12,03)$	Smooth

Hodge diamonds in $\mathbb{P}^2 \times \mathbb{P}^4$

Hodge diamonds

Let $X \subset Y$ be Calabi-Yau threefold and $0 \leq p, q \leq 3$, define

$$h^{p,q}(X) = \dim_{\mathbb{C}}(H^q(X, \Omega_X^p)).$$

Our goal is to compute

$$h^{1,1}(X) = \dim_{\mathbb{C}}(H^1(X, \Omega_X)) \text{ and } h^{1,2}(X) = \dim_{\mathbb{C}}(H^2(X, \Omega_X))$$

for each Calabi-Yau X.

Hodge diamonds

To compute $H^1(X, \Omega_X)$ and $H^2(X, \Omega_X)$, consider the inclusion $i : X \to Y$ leading to the short exact sequence

$$0 \to TX \to TY_{|X} \to \mathcal{N}_{X_{|Y}} \to 0$$

whose dual gives

$$0 \to \mathcal{N}^*_{X_{|Y}} \to \Omega_{Y_{|}X} \to \Omega_X \to 0$$

leading to the long exact sequence (1) on cohomology

$$0 \to H^0(X, \mathcal{N}^*_{X_{|Y}}) \to H^0(X, \Omega_{Y_{|}X}) \to \underbrace{H^0(X, \Omega_X)}_{0} \to$$

$$H^1(X, \mathcal{N}^*_{X_{|Y}}) \to H^1(X, \Omega_{Y_{|}X}) \to H^1(X, \Omega_X) \to$$

$$H^2(X, \mathcal{N}^*_{X_{|Y}}) \to H^2(X, \Omega_{Y_{|}X}) \to H^2(X, \Omega_X) \to$$

$$H^3(X, \mathcal{N}^*_{X_{|Y}}) \to H^3(X, \Omega_{Y_{|}X}) \to \underbrace{H^3(X, \Omega_X)}_{0} \to 0$$

Euler sequence

In order to compute $H^i(X, \Omega_{Y_{|X}})$, let us consider the dual version of the Euler sequence :

$$0 \longrightarrow \Omega_{Y_{|X}} \longrightarrow \underbrace{\mathcal{O}_X(-1,0)^3 \oplus \mathcal{O}_X(0,-1)^5}_{S} \longrightarrow \mathcal{O}_X^2 \longrightarrow 0$$

leading to the long exact sequence (2)

$$0 \longrightarrow H^0(X, \Omega_{Y_{|X}}) \longrightarrow \underbrace{H^0(X,S)}_{0} \longrightarrow \underbrace{H^0(X, \mathcal{O}_X \oplus \mathcal{O}_X)}_{\mathbb{C}^2} \longrightarrow$$

$$H^1(X, \Omega_{Y_{|X}}) \longrightarrow \underbrace{H^1(X,S)}_{0} \longrightarrow \underbrace{H^1(X, \mathcal{O}_X \oplus \mathcal{O}_X)}_{0} \longrightarrow$$

$$H^2(X, \Omega_{Y_{|X}}) \longrightarrow \underbrace{H^2(X,S)}_{0} \longrightarrow \underbrace{H^2(X, \mathcal{O}_X \oplus \mathcal{O}_X)}_{0} \longrightarrow$$

$$H^3(X, \Omega_{Y_{|X}}) \longrightarrow \underbrace{H^3(X,S)}_{34} \longrightarrow \underbrace{H^3(X, \mathcal{O}_X \oplus \mathcal{O}_X)}_{\mathbb{C}^2} \longrightarrow 0$$

Hodge diamaonds

Sequences (1) and (2) lead to the following observation :

$$(3)\begin{cases} H^0(X,\mathcal{N}^*_{X_{|Y}}) \cong H^0(X,\Omega_{Y_{|}X}) = 0 \\ H^2(X,\Omega_{Y_{|}X}) = 0 \\ H^1(X,\Omega_{Y_{|}X}) \cong \mathbb{C}^2 \\ 0 \to H^1(X,\mathcal{N}^*_{X_{|Y}}) \to \mathbb{C}^2 \to H^1(X,\Omega_X) \to H^2(X,\mathcal{N}^*_{X_{|Y}}) \to 0 \\ 0 \to H^3(X,\Omega_{Y_{|X}}) \to H^3(X,S) \to \mathbb{C}^2 \to 0 \\ 0 \to H^2(X,\Omega_Y) \to H^3(X,\mathcal{N}^*_{X_{|Y}}) \to H^3(X,\Omega_{Y_{|X}}) \to 0 \end{cases}$$

and therefore

$$\dim(H^3(X,\Omega_{Y_{|X}})) = \dim(H^3(X,S)) - 2 = 32 \quad (4)$$

and

$$h^{1,2}(X) = \dim(H^3(X,\mathcal{N}^*_{X_{|Y}})) - \dim(H^3(X,\Omega_{Y_{|X}}))$$

$$h^{1,2}(X) = \dim(H^3(X,\mathcal{N}^*_{X_{|Y}})) - \dim(H^3(X,S)) + 2$$

$$= \dim(H^3(X,\mathcal{N}^*_{X_{|Y}})) - 32 \quad (5)$$

Conormal bundle

The second conormal bundle short exact sequence is given by :

$$0 \longrightarrow \mathcal{I}_X^2 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{N}^*_{X/Y} \longrightarrow 0$$

which leads to the sequence (6)

$$0 \longrightarrow H^0(Y, \mathcal{I}_X^2) \longrightarrow H^0(Y, \mathcal{I}_X) \longrightarrow H^0(Y, \mathcal{N}^*_{X/Y}) \longrightarrow$$

$$H^1(Y, \mathcal{I}_X^2) \longrightarrow H^1(Y, \mathcal{I}_X) \longrightarrow H^1(Y, \mathcal{N}^*_{X/Y}) \longrightarrow$$

$$H^2(Y, \mathcal{I}_X^2) \longrightarrow H^2(Y, \mathcal{I}_X) \longrightarrow H^2(Y, \mathcal{N}^*_{X/Y}) \longrightarrow$$

$$H^3(Y, \mathcal{I}_X^2) \longrightarrow H^3(Y, \mathcal{I}_X) \longrightarrow H^3(Y, \mathcal{N}^*_{X/Y}) \longrightarrow$$

$$H^4(Y, \mathcal{I}_X^2) \longrightarrow H^4(Y, \mathcal{I}_X) \longrightarrow H^4(Y, \mathcal{N}^*_{X/Y}) \longrightarrow$$

$$H^5(Y, \mathcal{I}_X^2) \longrightarrow H^5(Y, \mathcal{I}_X) \longrightarrow H^5(Y, \mathcal{N}^*_{X/Y}) \longrightarrow$$

$$H^6(Y, \mathcal{I}_X^2) \longrightarrow H^6(Y, \mathcal{I}_X) \longrightarrow H^6(Y, \mathcal{N}^*_{X/Y}) \longrightarrow 0$$

we see that the computation of $H^i(Y, \mathcal{N}^*_{X/Y})$ requires $H^i(Y, \mathcal{I}_X^2)$ and $H^i(Y, \mathcal{I}_X)$.

Cohomology of the ideal sheaf

From the resolution of the sheaf ideal $\mathcal{I}_X$ we obtain the short exact sequences

$$\begin{cases} 0 \longrightarrow \mathcal{O}_Y(-3,-5) \longrightarrow F^*(-2,-3) \longrightarrow M \longrightarrow 0 \\ 0 \longrightarrow M \longrightarrow F(-1,-2) \longrightarrow \mathcal{I}_X \longrightarrow 0 \end{cases}$$

leading to

$$H^i(Y,\mathcal{I}_X) = \begin{cases} \mathbb{C} \text{ if } i = 4 \\ 0 \text{ otherwise} \end{cases}$$

Theorem

Let $X \subset Y$ be a codimension 3 Calabi-Yau variety. Then

$$H^i(Y,\mathcal{I}_X) = \begin{cases} \mathbb{C} \text{ if } i = 4 \\ 0 \text{ otherwise} \end{cases}$$

Cohomology of the square of the ideal sheaf

The ideal $\mathcal{I}_X^2$ satisfies the resolution

$$0 \longrightarrow \wedge^3 E \xrightarrow{v_3} L_{(4,1)}(E) \otimes \mathcal{O}_Y(3,5) \xrightarrow{v_2} S^2(E) \xrightarrow{v_1} \mathcal{I}_X^2 \longrightarrow 0$$

where $E = F(-1,-2)$. This can be separated in two shorts exact sequences :

$$0 \longrightarrow \wedge^3(E) \xrightarrow{v_3} L_{(4,1)}(E) \otimes \mathcal{O}_Y(3,5) \longrightarrow M \longrightarrow 0$$

$$0 \longrightarrow M \longrightarrow S^2(E) \xrightarrow{v_1} \mathcal{I}_X^2 \longrightarrow 0$$

Leading to to $H^i(Y, \mathcal{I}_X^2) = 0 \ \forall \ i = 0,1,2,3,6$ and

$$0 \to H^4(Y, \mathcal{I}_X^2) \to H^6(Y, \wedge^3 E) \to H^6(Y, L_{(4,1)}(E) \otimes \mathcal{O}_Y(3,5))$$

$$\to H^5(Y, \mathcal{I}_X^2) \to 0$$

and this case happen when $H^6(Y, S^2 E) = 0$ in each case we have :

$$E = \mathcal{O}_Y(-2,-2)^{\oplus 2} \oplus \mathcal{O}_Y(-1,-2)^{\oplus 2} \oplus \mathcal{O}_Y(0,-2)$$

$$0 \to H^4(Y,\mathcal{I}_X^2) \to \underbrace{H^6(Y,\wedge^3 E)}_{125} \to \underbrace{H^6(Y,L_{(4,1)}(E)\otimes \mathcal{O}_Y(3,5))}_{38} \to H^5(Y,\mathcal{I}_X^2) \to 0.$$

The map in the middle is surjective, leading to $H^5(Y,\mathcal{I}_X^2) = 0$ and

$$H^4(Y,\mathcal{I}_X^2) = \ker(H^6(Y,\wedge^3 E) \to H^6(Y,L_{(4,1)}(E)\otimes \mathcal{O}_Y(3,5)))$$

that is

$$\dim(H^4(Y,\mathcal{I}_X^2)) = 87.$$

The sequence (6) leads to the exact sequence

$$0 \to H^3(Y,\mathcal{N}^*_{X/Y}) \to \underbrace{H^4(Y,\mathcal{I}_X^2)}_{87} \to \underbrace{H^4(Y,\mathcal{I}_X)}_{\mathbb{C}} \to \underbrace{H^4(Y,\mathcal{N}^*_{X/Y})}_{0} \to 0$$

and

$$\dim((H^3(Y,\mathcal{N}^*_{X/Y})) = 86$$

and since $H^3(X,\mathcal{N}^*_{X/Y}) = H^3(Y,\mathcal{N}^*_{X/Y})$ and $h^{1,2}_X = \dim((H^3(Y,\mathcal{N}^*_{X/Y})) - 32$ then

$$h^{1,2}_X = 86 - 32 = 54.$$

$$E = \mathcal{O}_Y(-2,0) \oplus \mathcal{O}_Y(-1,-1)^{\oplus 2} \oplus \mathcal{O}_Y(-1,-4)^{\oplus 2}$$

$$0 \to H^4(Y, \mathcal{I}_X^2) \to \underbrace{H^6(Y, \wedge^3 E)}_{267} \to \underbrace{H^6(Y, L_{(4,1)}(E) \otimes \mathcal{O}_Y(3,5))}_{148} \to H^5(Y, \mathcal{I}_X^2) \to 0.$$

The map in the middle is surjective, leading to $H^5(Y, \mathcal{I}_X^2) = 0$ and

$$H^4(Y, \mathcal{I}_X^2) = \ker(H^6(Y, \wedge^3 E) \to H^6(Y, L_{(4,1)}(E) \otimes \mathcal{O}_Y(3,5)))$$

that is

$$\dim(H^4(Y, \mathcal{I}_X^2)) = 119.$$

The sequence (6) leads to the exact sequence

$$0 \to H^3(Y, \mathcal{N}^*_{X/Y}) \to \underbrace{H^4(Y, \mathcal{I}_X^2)}_{119} \to \underbrace{H^4(Y, \mathcal{I}_X)}_{\mathbb{C}} \to \underbrace{H^4(Y, \mathcal{N}^*_{X/Y})}_{0} \to 0$$

and

$$\dim((H^3(Y, \mathcal{N}^*_{X/Y})) = 118$$

$h_X^{1,2} = \dim((H^3(Y, \mathcal{N}^*_{X/Y})) - 32$ then

$$h_X^{1,2} = 118 - 32 = 86.$$

$$E = \mathcal{O}_Y(-2,-3) \oplus \mathcal{O}_Y(-1,-1)^{\oplus 3} \oplus \mathcal{O}_Y(-1,-4)$$

$$0 \to H^4(Y, \mathcal{I}_X^2) \to \underbrace{H^6(Y, \wedge^3 E)}_{339} \to \underbrace{H^6(Y, L_{(4,1)}(E) \otimes \mathcal{O}_Y(3,5))}_{250} \to 0$$

$$H^i(Y, \mathcal{I}_X^2) = 0 \ \forall \ i = 0,1,2,3,5$$

$$\dim(H^6(Y, \mathcal{I}_X^2)) = \dim(H^6(Y, S^2 E)) = 30$$

$$H^4(Y, \mathcal{I}_X^2) = \ker(H^6(Y, \wedge^3 E) \to H^6(Y, L_{(4,1)}(E) \otimes \mathcal{O}_Y(3,5)))$$

that is

$$\dim(H^4(Y, \mathcal{I}_X^2)) = 89.$$

$$0 \to H^3(Y, \mathcal{N}^*_{X/Y}) \to \underbrace{H^4(Y, \mathcal{I}_X^2)}_{89} \to \underbrace{H^4(Y, \mathcal{I}_X)}_{\mathbb{C}} \to \underbrace{H^4(Y, \mathcal{N}^*_{X/Y})}_{0} \to 0$$

and

$$\dim((H^3(Y, \mathcal{N}^*_{X/Y})) = 88$$

$h^{1,2}_X = \dim((H^3(Y, \mathcal{N}^*_{X/Y})) - 32$ then

$$h^{1,2}_X = 88 - 32 = 56.$$

$$E = \mathcal{O}_Y(-2,0) \oplus \mathcal{O}_Y(-1,-2)^{\oplus 2} \oplus \mathcal{O}_Y(-1,-3)^{\oplus 2}$$

$$0 \to H^4(Y, \mathcal{I}_X^2) \to \underbrace{H^6(Y, \wedge^3 E)}_{127} \to \underbrace{H^6(Y, L_{(4,1)}(E) \otimes \mathcal{O}_Y(3,5))}_{28} \to H^5(Y, \mathcal{I}_X^2) \to 0.$$

The map in the middle is surjective, leading to $H^5(Y, \mathcal{I}_X^2) = 0$ and

$$H^4(Y, \mathcal{I}_X^2) = \ker(H^6(Y, \wedge^3 E) \to H^6(Y, L_{(4,1)}(E) \otimes \mathcal{O}_Y(3,5)))$$

that is

$$\dim(H^4(Y, \mathcal{I}_X^2)) = 99.$$

The sequence (6) leads to the exact sequence

$$0 \to H^3(Y, \mathcal{N}^*_{X/Y}) \to \underbrace{H^4(Y, \mathcal{I}_X^2)}_{99} \to \underbrace{H^4(Y, \mathcal{I}_X)}_{\mathbb{C}} \to \underbrace{H^4(Y, \mathcal{N}^*_{X/Y})}_{0} \to 0$$

and

$$\dim((H^3(Y, \mathcal{N}^*_{X/Y})) = 98$$

$h_X^{1,2} = \dim((H^3(Y, \mathcal{N}^*_{X/Y})) - 32$ then

$$h_X^{1,2} = 98 - 32 = 66.$$

$$E = \mathcal{O}_Y(-2,-1) \oplus \mathcal{O}_Y(-1,-2)^{\oplus 3} \oplus \mathcal{O}_Y(-1,-3)$$

$$0 \to H^4(Y,\mathcal{I}_X^2) \to \underbrace{H^6(Y,\wedge^3 E)}_{104} \to \underbrace{H^6(Y,L_{(4,1)}(E)\otimes\mathcal{O}_Y(3,5))}_{25} \to H^5(Y,\mathcal{I}_X^2) \to 0.$$

The map in the middle is surjective, leading to $H^5(Y,\mathcal{I}_X^2) = 0$ and

$$H^4(Y,\mathcal{I}_X^2) = \ker(H^6(Y,\wedge^3 E) \to H^6(Y,L_{(4,1)}(E)\otimes\mathcal{O}_Y(3,5)))$$

that is

$$\dim(H^4(Y,\mathcal{I}_X^2)) = 79.$$

The sequence (6) leads to the exact sequence

$$0 \to H^3(Y,\mathcal{N}^*_{X/Y}) \to \underbrace{H^4(Y,\mathcal{I}_X^2)}_{79} \to \underbrace{H^4(Y,\mathcal{I}_X)}_{\mathbb{C}} \to \underbrace{H^4(Y,\mathcal{N}^*_{X/Y})}_{0} \to 0$$

and

$$\dim((H^3(Y,\mathcal{N}^*_{X/Y})) = 78$$

and since $H^3(X,\mathcal{N}^*_{X/Y}) = H^3(Y,\mathcal{N}^*_{X/Y})$ and $h^{1,2}_X = \dim((H^3(Y,\mathcal{N}^*_{X/Y})) - 32$ then

$$h^{1,2}_X = 78 - 32 = 46.$$

$$E = \mathcal{O}_Y(-2,-2) \oplus \mathcal{O}_Y(-1,-2)^{\oplus 4}$$

$$0 \to H^4(Y, \mathcal{I}_X^2) \to \underbrace{H^6(Y, \wedge^3 E)}_{110} \to \underbrace{H^6(Y, L_{(4,1)}(E) \otimes \mathcal{O}_Y(3,5))}_{28} \to H^5(Y, \mathcal{I}_X^2) \to 0.$$

The map in the middle is surjective, leading to $H^5(Y, \mathcal{I}_X^2) = 0$ and

$$H^4(Y, \mathcal{I}_X^2) = \ker(H^6(Y, \wedge^3 E) \to H^6(Y, L_{(4,1)}(E) \otimes \mathcal{O}_Y(3,5)))$$

that is

$$\dim(H^4(Y, \mathcal{I}_X^2)) = 82.$$

The sequence (6) leads to the exact sequence

$$0 \to H^3(Y, \mathcal{N}^*_{X/Y}) \to \underbrace{H^4(Y, \mathcal{I}_X^2)}_{82} \to \underbrace{H^4(Y, \mathcal{I}_X)}_{\mathbb{C}} \to \underbrace{H^4(Y, \mathcal{N}^*_{X/Y})}_{0} \to 0$$

and

$$\dim((H^3(Y, \mathcal{N}^*_{X/Y})) = 81$$

$h_X^{1,2} = \dim((H^3(Y, \mathcal{N}^*_{X/Y})) - 32$ then

$$h_X^{1,2} = 81 - 32 = 49.$$

Thank you for your kind attention!.