On the existence of η-Einstein contact metric structures

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Conférence AFRIMath 2023:
Topologie et Géométrie, Sept. 11-15, Dangbo, Bénin

15 septembre 2023

Introduction

- The so-called Goldberg conjecture asserts that any compact *almost* Kähler-Einstein manifold is necessarily Kähler-Einstein.
- This conjecture was established by in the particular case of Einstein metrics having nonnegative scalar curvature [1].
- Odd-dimensional analogues of almost Kähler and Kähler structures are K-contact and Sasakian structures, respectively.
- A formulation of an analogue of Goldberg conjecture for odd-dimensional manifolds is that any compact K-contact manifold M having an Einstein metric is necessarily Sasakian-Einstein see [2]

1. K. Sekigawa, On some compact Einstein almost Kähler manifolds, J. Math. Soc. Japan 39 (1987), 677-684.
2. C. P. Boyer and K. Galicki, Sasakian geometry. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2008. xii+613 pp.

Introduction

- It is known that any K-contact structure on a closed manifold can be approximated by quasi-regular K-contact structures whose characteristic orbit spaces are symplectic orbifolds [3].
- Boyer and Galicki adapt Sekigawa's proof to symplectic orbifolds and use it to prove a contact metric analogue of Goldberg conjecture [4].
  They essentially prove that every compact K-contact-Einstein manifold is Sasakian-Einstein.

3. P. Rukimbira, Properties of K-contact manifolds. Journal de la Faculté des Sciences de l'U.C.A.D. 11(2015) no.3 23–33.
4. C. P. Boyer and K. Galicki, Sasakian geometry. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2008. xii+613 pp.

Introduction

- A naive extenson of Goldberg's conjecture would states that every compact Einstein contact metric manifold is Sasakian-Einstein.
  However, such a statement is false in general since **Blair and Sharma**[5] give a counter-example on the flat torus $\mathbb{T}^3$ by presenting a contact metric structure on $\mathbb{T}^3$ which is not K-contact (thus not Sasakian).
- Observe that this counter-example cannot be extended to higher dimension since no contact manifold of dimension ≥ 5 admits a flat contact metric structure[6].

5. Three-dimensional locally symmetric contact metric manifolds. Bolll. Un. Mat. Ital. A (7) 4 (1990), 385-390.
6. D.E. Blair, On the non-existence of flat contact metric structures. Tohoku Math. J. 28 (1976), 373-379.

Introduction

- Motivated by Blair and Sharma counter-example, we attempt to find criteria for the non-existence of Sasakian-Einstein structures on compact contact metric manifolds.
- In this talk, we give results that are analogous to those obtained by **Banyaga and Massamba**[7] in their work on the non-existence of Einstein metrics on some symplectic manifolds.
- First, from a Riemannian metric g_0 on a contact manifold (M, η), we construct a contact metric g called η-compatible metric.
- Secondly, we find a necessary and sufficient condition for the existence of a transverse Einstein metric on a contact manifold (M, η).
- Finally, we use this result to recover a result by **Boyer and Galicki** where they extended the Goldberg's conjecture to η-Einstein metrics on K-contact manifolds.

7. A. Banyaga and F. Massamba, Non-existence of certain Einstein metrics on some symplectic manifolds. Forum Math. 28 (2016), 527-537.

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Contact structures

Definition

A contact manifold is a $(2n+1)$-dimensional smooth manifold M endowed with a 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere in M.

- There exist a unique vector field ξ associated with η satisfying : $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$. Such a vector field ξ is called the Reeb vector field of (M, η).
- By the characteristic foliation of (M, η), we mean the 1-dimensional foliation $\mathcal{F}_\xi$ generated by ξ.

Quasi-regular contact form

Definition

η is called *quasi-regular* if there is a positive integer k such that each point $x \in M$ admits a coordinate chart $(U, x_1, \cdots, x_n, y_1, \cdots, y_n, t)$ such that each leaf of $\mathcal{F}_\xi$ passes through U at most k times.

When $k = 1$, η is simply called *regular. We say that η is irregular if it is not quasi-regular.*

Contact metric structure

Definition

A *contact metric structure* on M is defined by a contact form η with its Reeb vector field, a Riemannian metric g and a $(1,1)$-tensor field ϕ such that :

$$\phi^2 X + X = \eta(X)\xi, \quad (1)$$

$$d\eta(X,Y) = 2g(X,\phi Y), \quad (2)$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad (3)$$

for all vector fields X, Y on M, where $d\eta(X,Y) = X\eta(Y) - Y\eta(X) - \eta([X,Y])$.

Notice that identity (1) implies that $\phi\xi = 0$ and $\eta \circ \phi = 0$.
We say that (M, η, ξ, g, ϕ) is K-contact manifod if the Reeb field ξ is Killing, i.e. $L_\xi g = 0$.

Contact metric structure

Proposition (Boyer-Galicki)

Let (M, η) be a compact quasi-regular contact manifold. Then there exists a Riemannian metric g and a $(1,1)$-tensor field ϕ such that (M, η, ξ, g, ϕ) is K-contact manifold.

- A *Sasakian* manifold is a K-contact manifold (M, η, ξ, g, ϕ) satisfying :

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{4}$$

  for arbitrary vector fields X and Y on M, where ∇ denotes the Levi-Civita connection of the metric g.

- In dimension 3, K-contact structures are the same as Sasakian ones but, in dimensions $d \geq 5$, K-contact structures can be viewed as bridges between contact metric and Sasakian structures.

η-Einstein contact metric 14

Definition

A contact metric manifold (M, η, ξ, ϕ, g) is called η-Einstein if its Ricci tensor Ric_g can be written as :

$$Ric_g(X, Y) = \alpha g(X, Y) + \beta\eta(X)\eta(Y), \tag{5}$$

for some real numbers α and β. When $\beta = 0$, g is simply called an Einstein metric.

Construction of contact metric structures

- Starting from an arbitrary metric g_0 a contact manifold (M, η), one can construct a contact metric g, i.e. g satisfies the conditions (1)–(3).
- Let (M, η) be a contact manifold with its Reeb vector field ξ and set $D = \mathrm{Ker}(\eta)$. Pick a Riemannian metric g_0 such that $g_0(\xi, X) = \eta(X)$, for all vector fields X on M. Notice that $g_0(\xi, .) = \eta$ is the only requirement on the metric g_0.
- Since $d\eta$ is nondegenerate on D, we can define a (1,1) tensor field $A : D \to D$ by setting :

$$d\eta(X, Y) = 2g_0(X, AY), \tag{6}$$

for all smooth sections $X,\ Y$ of D.

Construction of contact metric structures 16

By construction A is skew-symmetric, that is $A^* = -A$ with A^* being the adjoint of A with respect to g_0. Moreover, A^*A is symmetric and positive definite :

$$g_0(X,\ A^*AX) = g_0(AX, AX) > 0, \tag{7}$$

for any nonzero section X of D.

- We set

$$B := \sqrt{A^*A}, \qquad \text{and} \qquad \phi = B^{-1}A : D \to D.$$

On the one hand, B commutes with both A and A^*. On the other hand, ϕ is an isometry with respect to g_0 (by definition).
Moreover $\phi^*\phi = I$ since

$$\phi^* = A^*(B^{-1})^* = -AB^{-1} = -B^{-1}A = -\phi.$$

So we have $\phi^2 = -Id$ on D.

Construction of contact metric structures

Next, we extend ϕ and B on TM by setting $\phi(\xi) = 0$ and $B\xi = 0$. Then, we obtain :

$$\phi^2 X = \phi^2(X - \eta(X)\xi + \eta(X)\xi) = \phi^2(X - \eta(X)\xi) = -X + \eta(X)\xi, \qquad (8)$$

for all $X \in \mathfrak{X}(M)$.
We also get $\eta \circ B = 0$. Finally, define the metric g by setting :

$$g(X, Y) := g_0(X, BY) + \eta(X)\eta(Y), \qquad (9)$$

$\forall X, Y \in \mathfrak{X}(M)$.
In particular, for $X = \xi$ we get

$$g(\xi, Y) = \eta(BY) + \eta(Y) = \eta(Y),$$

for all $X \in \mathfrak{X}(M)$. Thus $g(\xi, \cdot) = \eta$.

Construction of contact metric structures

Proposition (2)

Under the above notations, (η, ξ, g, ϕ) is a contact metric structure on M.

Construction of contact metric structures

Démonstration.

First of all , since $g(\xi, X) = \eta(X)$ for all $X \in \mathfrak{X}(M)$ we get :

$$2g(\xi, \phi X) = 2\eta(\phi X) = 0 = d\eta(\xi, X). \tag{10}$$

Therefore,

$$\begin{aligned} 2g(X, \phi Y) &= 2g((X - \eta(X)\xi + \eta(X)\xi, \phi Y) \\ &= 2g(X - \eta(X)\xi, \phi Y) \end{aligned}$$

for all $X, Y \in \mathfrak{X}(M)$. We deduce :

$$\begin{aligned} 2g(X, \phi Y) &= 2g_0((X - \eta(X)\xi, B\phi Y) \\ &= 2g_0(X - \eta(X)\xi, AY) \\ &= d\eta(X - \eta(X)\xi, Y) \\ &= d\eta(X, Y), \end{aligned}$$

Construction of contact metric structures

Démonstration.

for all $X, Y \in \mathfrak{X}(M)$. Hence (2) is satisfied. Moreover, for all $X, Y \in \mathfrak{X}(M)$, we have :

$$g(\phi X, \phi Y) = g_0(\phi X, B\phi Y) = g_0(\phi^2 X,\ \phi B\phi Y)$$

since ϕ is an isometry with respect to g_0.
Furthermore,

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta \circ \phi = 0 \quad \text{and} \quad \phi B\phi Y = \phi^2 BY = -BY,$$

it follows

$$g(\phi X, \phi Y) = g_0(X, BY) = g(X, Y) - \eta(X)\eta(Y).$$

This completes the proof. □

Construction of contact metric structures

- Given a metric g_0 on a contact manifold (M, η), we call g_0^T, its associated transverse metric, that is, the restriction of g_0 to $D = \ker\eta$.
- Using the Levi Civita connection ∇^0 of g_0, we define a connection $(\nabla^0)^T$ on D as follows :

$$\begin{aligned} (\nabla^0)^T_X Y &= \nabla^0_X Y - \eta(\nabla^0_X Y)\xi, \quad \forall\ X, Y \in \Gamma(D) \\ (\nabla^0)^T_\xi Y &= [\xi, Y] - \eta([\xi, Y])\xi, \quad \forall\ Y \in \Gamma(D) \end{aligned}$$

- The connection $(\nabla^0)^T$ is called the *transverse connection* associated with g_0. Define its *transverse curvature tensor* $(R^0)^T$ *as follows :*

$$(R^0)^T(X, Y)Z = (\nabla^0)^T_X \circ (\nabla^0)^T_Y Z - (\nabla^0)^T_Y \circ (\nabla^0)^T_X Z - (\nabla^0)^T_{[X,Y]} Z,$$

for all $Z \in \Gamma(D)$.

Examples

In our second example, we take the 3-dimensional Lie group Sol_3, that is, the space $\mathbb{R}^3$ with coordinates (x, y, z) and the group multiplication :

$$(x, y, z) \cdot (a, b, c) = (x + e^{-z}a, y + e^{z}b, z + c).$$

It can also be viewed as the closed subgroup of $\mathrm{GL}(3, \mathbb{R})$ consisting of matrices of the form :

$$\begin{pmatrix} e^{-z} & 0 & x \\ 0 & e^{z} & y \\ 0 & 0 & 1 \end{pmatrix} \quad x, y, z \in \mathbb{R}.$$

Consider the metric of Sol_3 defined by :

$$g_0 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2. \tag{11}$$

Examples

The following left invariant vector fields form an orthonormal basis with respect to g_0.

$$e_1 = e^{-z}\partial_x, \;\; e_2 = e^{z}\partial_y, \;\; e_3 = \partial_z.$$

Define the skew symmetric operator A by setting :

$$g_0(X, AY) = \frac{1}{2} d\eta(X, Y)$$

where η is the left invariant contact form on Sol_3 defined by :

$$\eta = \frac{1}{\sqrt{2}}\left(e^{z}dx + e^{-z}dy\right). \tag{12}$$

We obtain that

$$A = \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

Examples

By using $B = \sqrt{A^t A}$ in our construction we get

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{13}$$

in the basis $\{e_1, e_2, e_3\}$. We have $\ker \eta = \mathrm{Vect}\{e_3, e_1 - e_2\}$ and the Reeb field is given by :

$$\xi = \frac{1}{\sqrt{2}} \left(e^{-z} \partial_x + e^z \partial_y \right) = \frac{1}{\sqrt{2}} \left(e_1 + e_2 \right). \tag{14}$$

Examples

From the relation :

$$g(X,Y) = g_0(X,BY) + \eta(X)\eta(Y),$$

for all $X, Y \in \mathfrak{X}(Sol_3)$, it follows

$$g(e_3,e_3) = g_0(e_3,Be_3) = \sqrt{2}.$$

By similar computations, we get :

$$g(e_1,e_1) = g(e_2,e_2) = \frac{2+\sqrt{2}}{2\sqrt{2}}, \qquad g(e_1,e_2) = g(e_2,e_1) = \frac{-2+\sqrt{2}}{2\sqrt{2}},$$

$$g(e_1,e_3) = g(e_3,e_1) = g(e_2,e_3) = g(e_3,e_2) = 0.$$

Then the metric g is given by the matrix

$$g = \begin{pmatrix} \frac{2+\sqrt{2}}{2\sqrt{2}} & \frac{-2+\sqrt{2}}{2\sqrt{2}} & 0 \\ \frac{-2+\sqrt{2}}{2\sqrt{2}} & \frac{2+\sqrt{2}}{2\sqrt{2}} & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}. \qquad (15)$$

Examples

The almost complex structure ϕ on $\mathrm{Ker}(\eta)$, which is extended on $\mathfrak{X}(\mathrm{Sol}_3)$, is defined by :

$$\phi(e_1) = -e_3, \quad \phi(e_2) = e_3, \quad 2\phi(e_3) = e_2 - e_1.$$

Proposition 6 ensures that (η, ξ, g, ϕ) is a contact metric structure on Sol_3.

Main Results

Proposition (3)

Let g_0 be a Riemannian metric on a contact manifold (M, η) with $g_0(\xi, \cdot) = \eta$. Let B be its $(1,1)$-tensor field defined as above and let g the η-compatible metric derived from g_0, i.e.

$$g(X, Y) := g_0(X, BY) + \eta(X)\eta(Y),$$

$\forall$ X and Y. Denote by ∇^0 and ∇ the Levi Civita connections of g_0 and g, respectively and R^0, R the curvature tensors of g_0, g, respectively. If $\nabla^0 B = 0$ then we have

$$\nabla^T_X Y = (\nabla^0)^T_X Y, \qquad (16)$$

$$B(R^T(X, Y)Z) = (R^0)^T(X, Y)BZ \qquad (17)$$

Main Results

To prove Relation (16), it is enough to show that :

$$g(\overset{T}{\nabla}_X Y, Z) = g((\overset{T}{\nabla^0})_X Y, Z) \tag{18}$$

for all $X, Y \in \Gamma(D)$ and for all $Z \in \mathfrak{X}(M)$.
Since every $V \in \mathfrak{X}(M)$ can be written as $V = \eta(V)\xi + Z$ and $Z \in \Gamma(D)$. We will distinguish two cases :

- We ssume $V = \xi$.

$$g((\overset{T}{\nabla^0})_X Y, \xi) = \eta((\nabla^0)_X Y) - \eta((\nabla^0)_X Y) = 0. \tag{19}$$

and we show that

$$g(\overset{T}{\nabla}_X Y, \xi) = g((\nabla^0)_X Y, \xi) = 0. \tag{20}$$

- Now we assume $X, Y, Z \in \Gamma(D)$. Notice that, in this case,

$$g(\overset{T}{\nabla}_X Y, Z) = g((\overset{T}{\nabla^0})_X Y, Z) \iff g(\nabla_X Y, Z) = g_0(B\nabla^0_X Y, Z). \tag{21}$$

Main Results

We only have to show that

$$g(\nabla_X Y, Z) = g_0(B\nabla^0_X Y, Z). \tag{22}$$

To do so, we start by writing a Koszul type formula for both g_0 and g, and we subtract $2g(\nabla_X Y, Z)$ from $2g(\nabla^0_X Y, Z)$ with the use of the hypothesis $\nabla^0 B = 0$ we get the result.

Main Results

For the second relation we obtain :

$$B(R^{T}(X,Y)Z) = B\left(\nabla^{T}_{X}\circ\nabla^{T}_{Y}Z - \nabla^{T}_{Y}\circ\nabla^{T}_{X}Z - \nabla^{T}_{[X,Y]}Z\right)$$
$$= B\left((\nabla^{0})^{T}_{X}\circ(\nabla^{0})^{T}_{Y}Z\right) - B\left((\nabla^{0})^{T}_{Y}\circ(\nabla^{0})^{T}_{X}Z\right) - B\left((\nabla^{0})^{T}_{[X,Y]}Z\right).$$

Since $\nabla^{0}B = 0$, Equation $(\nabla^{0}_{X}BY = B\nabla^{0}_{X}Y + (\nabla^{0}_{X}B)Y = B\nabla^{0}_{X}Y)$ implies,

$$\begin{aligned} B(R^{T}(X,Y)Z) &= (\nabla^{0})^{T}_{X}\circ(\nabla^{0})^{T}_{Y}BZ - (\nabla^{0})^{T}_{Y}\circ(\nabla^{0})^{T}_{X}BZ - (\nabla^{0})^{T}_{[X,Y]}BZ \\ &= (R^{0})^{T}(X,Y)BZ. \end{aligned}$$

Main Results

Proposition (4)

Let g_0 be a Riemannian metric on a contact manifold (M, η) such that $g_0(\xi, \cdot) = \eta$, where ξ is the Reeb field. Let (η, ξ, g, ϕ) be the associated contact metric defined as in Proposition 2, i.e.,

$$g(X, Y) := g_0(X, BY) + \eta(X)\eta(Y),$$

for all vector fields X and Y on M. Assume $(\nabla^0)^T B = 0$ and there is a smooth function α such that $\mathrm{Ric}_{g_0^T} = \alpha g_0^T$ then $\mathrm{Ric}_{g^T} = \alpha g^T$.
If in addition (η, ξ, g, ϕ) is K-contact then g is η-Einstein.

Where $\mathrm{Ric}_{g_0^T}$ be the Ricci curvature of the transverse metric associated with g_0.

Main Results

Proof :

For each $x \in M$, there is a coordinate neighborhood U around x and a local orthonormal basis $(e_1, \ldots, e_{2n}, \xi)_x$ of $T_x M$ with respect to both g and g_0.

$$\begin{aligned} Ric_{g^T}(X,Y) &= \left(\sum_{i=1}^{2n} g^T \left(R^T(e_i, X)Y, e_i \right) \right) \\ &= \left(\sum_{j=1}^{2n} g_0^T \left(BR^T(e_i, X)Y, e_i \right) \right) \\ &= \left(\sum_{j=1}^{2n} g_0^T \left((R^0)^T(e_i, X)BY, e_i \right) \right) \\ &= Ric_{g_0^T}(X, BY), \end{aligned}$$

for any X, Y vector fields tangent to D.

So using the assumtion we have,

$$\begin{aligned}\mathrm{Ric}_{g^T}(X,Y) &= \mathrm{Ric}_{g_0^T}(X,BY)\\ &= \alpha g_0^T(X,BY))\\ &= \alpha g^T(X,Y),\end{aligned}$$

for all $X, Y \in \Gamma(D)$.
Now, assume (η, ξ, g, ϕ) is K-contact then, by a result of **Boyer-Galicki**[8] (see Theorem 7.3.12), the Ricci curvature Ric_g of g satisfies :

$$\mathrm{Ric}_g(X,Y) = \mathrm{Ric}_{g^T}(X,Y) - 2g(X,Y) \quad \text{and} \quad \mathrm{Ric}_g(Z,\xi) = 2n\eta(Z), \tag{23}$$

for all $X, Y \in \Gamma(D)$ and for all $Z \in \mathfrak{X}(M)$.
Relations (23) imply that g is η-Einstein if and only if g^T is Einstein.

8. C. P. Boyer and K. Galicki, Sasakian geometry. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2008. xii+613 pp.

Main Results

Theorem (1)

Consider a contact manifold (M, η). The existence of a contact metric structure g on (M, η) whose associated transverse metric g^T is Einstein is equivalent to the existence of a Riemannian metric g_0 such that $\mathrm{Ric}_{g_0^T} = \alpha g_0^T$ for some constant α and

$$(\nabla^0)^T B(g_0) = 0 \qquad (24)$$

where ∇^0 is the Levi Civita connection of g_0.

Main Results

Proof :

Suppose (η, ξ, g, ϕ) is a contact metric structure on M, where g^T satisfies $\mathrm{Ric}_{g^T} = \alpha g^T$ for some $\alpha \in \mathbb{R}$.

Then we can take $g = g_0$, i.e. g coincides with the metric g_0 derived from it.

In this case, A would be the identity operator $\mathbb{I}$ on D and $B(g_0) = \mathbb{I} - \eta \otimes \xi$ so that $B\xi = \xi - \xi = 0$.

Then

$$(\nabla^0)^T B = 0, \quad \text{i.e.,} \quad B\Big((\nabla^0)^T_X Y\Big) = (\nabla^0)^T_X BY,$$

for all $X \in \mathfrak{X}(M)$, $Y \in \Gamma(D)$. Indeed, $(\nabla^0)^T_X Y \in \Gamma(D)$ and since $B(g_0) = \mathbb{I}$ on $\Gamma(D)$,

we get

$$B\Big((\nabla^0)^T_X Y\Big) = (\nabla^0)^T_X Y \quad \text{and} \quad (\nabla^0)^T_X BY = (\nabla^0)^T_X Y.$$

Main Results

Conversely, assume that (M, η) is a contact manifold that admits a metric g_0 satisfying Equation (24) and $\mathrm{Ric}_{g_0^T} = \alpha g_0^T$, for some constant α then the associated quadruple (η, ξ, g, ϕ) described by Proposition 2 defines a contact metric structure on M.
As in the proof of Proposition 4, we see that the derived metric g satisfies $\mathrm{Ric}_{g^T} = \alpha g^T$.

Application to Goldberg's conjecture

In **Boyer-Galicki**[9], Boyer and Galicki extend the study of an odd-dimensional analogue of Goldberg's conjecture in the more general framework of η-Einstein metrics. They prove the following result :

Theorem (Boyer-Galicki)

If (M, η, g) be a compact K-contact manifold, where g is η-Einstein with $\alpha > -2$ then (M, η, g) is Sasakian. Moreover, if we take $g' = \lambda g + \lambda(\lambda - 1)\eta \otimes \eta$, where $\lambda = \frac{\alpha+2}{2n+2}$ then (M, η, g') is Sasakian-Einstein.

9. C. P. Boyer and K. Galicki, Sasakian geometry. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2008. xii+613 pp.

Application to Goldberg's conjecture

Before stating our next result, we recall the structure theorem for K-contact manifolds

Theorem (Boyer-Galicki book)

Let (M, η, ξ, g, ϕ) be a compact quasi-regular K-contact manifold. Then the following properties hold :

1. The leaf space $\mathcal{Z} := M/\mathcal{F}_\xi$ is an almost Kähler orbifold whose induced almost Kähler metric will be denoted by h
2. (M, η, ξ, g, ϕ) is Sasakian if and only if $(\mathcal{Z}, \omega, h)$ is Kähler.
3. The metric g is η-Einstein if and only if and only if h is Einstein.

Application to Goldberg's conjecture

Now, we will give another proof of Boyer and Galicki's result where the K-contact assumption is replaced by a condition of quasi-regularity. We have :

Theorem (2)

Let (M, η) be a compact quasi-regular contact manifold of dimension $2n+1>3$. If (M, η) admits an η-Einstein metric $\widetilde{g}$ with $\alpha > -2$ then M is Sasakian.

Application to Goldberg's conjecture

For the proof

- Consider a compact quasi-regular contact manifold (M, η) of dimension $2n+1>3$ that admits an η-Einstein metric $\widetilde{g}$ with $\alpha > -2$.
- Then Proposition 1 ensures that there is a compatible K-contact structure (M, η, ξ, g).
- Apply the previous Theorem to obtain its associated compact almost Kähler orbifold $(\mathcal{Z}, \omega, h)$.
- We know that any η-Einstein metric $\widetilde{g}$ on (M, η) satisfying $\alpha > -2$ induces an Einstein metric $\widetilde{h}$ on $\mathcal{Z}$ whose Ricci curvature $\mathrm{Ric}_{\widetilde{h}}$ is positive.

- Now assume there is no Sasakian structure on M or equivalently $(\mathcal{Z}, \omega)$ has no Kähler structure.
- Since the proof of Theorem 5.4 in **Banyaga-Massamba** [10] uses only local connection computations, it can be carried over to the orbifold case.
- We deduce that there is no Einstein $\widetilde{h}$ on $(\mathcal{Z}, \omega)$ with $\mathrm{Ric}_{\widetilde{h}} > 0$.
- This contradicts our assumption that (M, η) admits an η-Einstein metric $\widetilde{g}$ such that $\alpha > -2$ because, in this case, the transverse metric $\widetilde{g}^T$ is Einstein and the almost Kähler orbifold $(\mathcal{Z}, \widetilde{h})$ (with $\pi^*\widetilde{h} = \widetilde{g}$) is Kähler (where $\pi : (M, g) \to (\mathcal{Z}, h)$ is the canonical projection map). So M must be Sasakian.

10. Non-existence of certain Einstein metrics on some symplectic manifolds. Forum Math. 28

Thanks

Merci pour votre attention ! ! !